ELEVENTH EDITION

Elementary and Middle School MATHEMATICS

TEACHING DEVELOPMENTALLY

John A. Van De Walle ▪ Karen S. Karp ▪ Jennifer M. Bay-Williams

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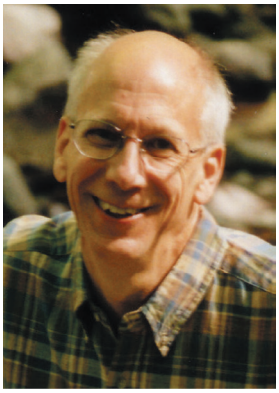


John A. Van de Walle

The late **Dr. John A. Van de Walle** was a professor emeritus at Virginia Commonwealth University. He was a leader in mathematics education who regularly gave professional development workshops focused on mathematics instruction that engage students in mathematical reasoning and problem solving for K–8 teachers in the United States and Canada. He visited and taught in many classrooms and worked with teachers to implement student-centered mathematics lessons. He co-authored the *Scott Foresman-Addison Wesley Mathematics K–6* series and contributed to the original Pearson School mathematics program *enVisionMATH*. Additionally, John was very active in the National Council of Teachers of Mathematics (NCTM), writing book chapters and journal articles, serving on the board of directors, chairing the educational materials committee, and speaking at national and regional meetings.

Karen S. Karp

Dr. Karen S. Karp is a professor at Johns Hopkins University (Maryland). Previously, she was a professor of mathematics education at the University of Louisville for more than twenty years. Prior to entering the field of teacher education, she was an elementary school teacher in New York. She is the coauthor of *Strengths-Based Teaching and Learning in Mathematics: 5 Teaching Turnarounds for Grades K-6*; the three book series, *The Math Pact: Achieving Instructional Cohesion within and across Grades*; and the *What Works Clearinghouse Practice Guide on Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades.* She is a former member of the board of directors for the National Council of Teachers of Mathematics (NCTM) and a former president of the Association of Mathematics Teacher Educators. In 2020, she received the NCTM Lifetime Achievement Award and in 2021 became a member of the United States National Committee on Mathematics Instruction. She continues to work in classrooms to support teachers in ways to instruct students with special needs in math.

Jennifer M. Bay-Williams

Dr. Jennifer Bay-Williams is a professor at the University of Louisville. She is a leader in mathematics education, regularly working to support mathematics instruction. Beyond her work on this book and the related *Teaching Student Centered Mathematics* book series, Jennifer has written other books that focus on effective mathematics teaching. Most recently she has written a book series called *Figuring out Fluency in Mathematics K-12* and *Math Fact Fluency: 60+ Games and Assessment Tools to Help Learning and Retention*. For teacher leaders, she has authored *Everything You Need for Mathematics Coaching*, and in the area of financial literacy, she authored a three-book series titled *On the Money* (a financial literacy book series). Over the years, Jennifer has taught elementary, middle, and high school in the United States and in Peru. She continues work with teachers around the world and in her local school districts to support their work. Jennifer has been actively involved in NCTM, NCSM, AMTE, and TODOS: Mathematics for All. Currently, she is serving as an associate editor for *Mathematics Teacher: Learning and Teaching in preK-12.*

About the Contributor

Jonathan Wray

Dr. Jonathan Wray is the technology contributor to *Elementary and Middle School Mathematics, Teaching Developmentally* (sixth to eleventh editions). He is Coordinator for Secondary Mathematics in the Howard County Public School System. He has served as the president of the Association of Maryland Mathematics Teacher Educators (AMMTE), the Maryland Council of Teachers of Mathematics (MCTM), and the manager of the Elementary Mathematics Specialists and Teacher Leaders (ems&tl) Project. Jon also served as an elected member of the National Council of Teachers of Mathematics (NCTM) Board of Directors (2012–2015). He has been recognized for his expertise in infusing technology in mathematics teaching and was named an outstanding technology leader in education by the Maryland Society for Educational Technology (MSET). Jon is the recipient of the 2020 Ross Taylor/Glen Gilbert National Mathematics Leadership Award given by the National Council of Supervisors of Mathematics.

Preface

All students can learn mathematics with understanding. It is through the teacher's actions that every student can have this experience. We believe that teachers must create a classroom environment in which students are given opportunities to solve problems and work together, using their ideas and strategies to solve them. Effective mathematics instruction involves posing tasks that engage students in the mathematics they are expected to learn. Then, by allowing students to interact with and productively struggle with *their own mathematical ideas* and *their own strategies*, they will learn to see the connections among mathematical topics and the real world. Students value mathematics and feel empowered to use it.

Creating a classroom in which students design solution pathways, engage in productive struggle, and connect one mathematical idea to another is complex. Questions arise, such as, "How do I get students to wrestle with problems if they just want me to show them how to do it? What kinds of tasks lend themselves to this type of engagement? Where can I learn the mathematics content I need to be able to teach in this way?" With these and other questions firmly in mind, the goals of this resource are to:

1. Illustrate what it means to teach mathematics by using a problem-based approach.
2. Focus attention on student thinking, including the ways students might reason about numbers and possible challenges and misconceptions they might have.
3. Serve as a go-to reference for teaching pre-K–8 mathematics content, with useful learning progressions and research-based instructional strategies to guide instructional decision-making.
4. Present a robust collection of high-quality tasks and problem-based activities that can engage students in learning significant mathematical concepts and skills while developing positive mathematics identities.

New to This Edition

The following are highlights of the most significant changes in the eleventh edition.

Learning Progressions

This book has always focused on teaching mathematics developmentally. In this edition, every chapter has learning progressions. In many cases, these are visuals at the start of the chapter to serve as an advanced organizer of how we approach the content in the book. In other cases, specific ideas within a chapter have a specific learning progression. These progressions can be useful for planning units, understanding prior knowledge and future directions, and can inform assessment development.

Focus on Reasoning Strategy Instruction

As described in Chapters 2 and 3, developing procedural fluency is important and requires that students learn reasoning strategies. We have increased attention to what real fluency looks like in these early chapters, and then increased attention to teaching reasoning strategies to students across the number chapters, beginning with basic fact reasoning strategies in Chapter 9, whole-number reasoning strategies in Chapters 11 and 12, and reasoning strategies for fractions in Chapter 15.

Less Talk, More Action

In every revision, we receive wonderful suggestions on what we can add and rarely a suggestion on what to cut. We have continued the tradition of having this book serve as a reference for teaching all mathematics concepts in K–8. But it is a lot, so in this edition, we worked hard to pare down on narrative, in particular in Part II. The overall pages may not be too different, because we have added more activities and tasks, new figures, and reflection questions. In the end, we believe these changes make this edition more accessible to our readers, whether they are just getting started, looking for ideas for their 20th year of teaching, or preparing a professional learning workshop for their colleagues.

Technology Turnover

Over four years' time, in particular time that included significant online learning, the technology used to support mathematics instruction has changed, so many new apps and more graphing tools are available. Throughout Part II of the book, the suggestions in the activities and the Technology Notes have been significantly revised.

Writing to Learn

Back by popular demand! Each chapter closes with discussion questions focused on the big ideas of the chapter. These prompts can be used for discussions, assignments, or break-out rooms.

Children's Literature

Children's literature that is perfect for integrating into mathematics lessons is peppered throughout the book. For example, in Chapter 6 you will find a collection of children's books that can form the beginning of a culturally diverse mathematics literature library. Additionally, every chapter in Part II includes an updated selection of children's literature. This includes two new extensive listings of children's books at different reading levels on a variety of measurement (103 choices) and geometry topics (62 books).

New Activities and Activity Pages

We have responded to the many requests to have more student recording pages to go with the activities in the book. This edition has more than 40 new activities and many new and revised activity pages, blackline masters, and teacher resource pages.

Major Changes to Chapters

Every chapter in the eleventh edition has been revised to reflect the most current research, standards, and exemplars. This is evident in the approximately *360* new references in the eleventh edition! This represents our ongoing commitment to synthesize and present the most current evidence of effective mathematics teaching. First, we set out to make each chapter more accessible to all teachers. That means we cut some discussion, reorganized a number of chapters, and revised many figures. As we noted in the acknowledgements, we are grateful to reviewers and colleagues for their ideas in making these improvements. Here we share changes *not* already described earlier.

Chapter 1: Developing Confident and Competent Mathematics Learners

The introduction to teaching developmentally has dramatically shifted to emphasize the purpose and focus of teaching mathematics well. Added to this chapter is a strengths-based approach (readers identify their strengths too!), the addition of ideas from the *Catalyzing Change: Initiating Critical Conversations* series, and myths in teaching mathematics.

Chapter 2: Exploring What It Means to Know and Do Mathematics

This chapter is now sequenced like the title; the chapter begins with a focus on mathematical proficiency, now addressing each of the five strands. Then readers have the opportunity to explore three tasks. (The probability task has been moved to the probability chapter.) And this chapter is 1000 words shorter, providing the reader more time to really do the math!

Chapter 3: Teaching Problem-Based Mathematics

Although the teaching practices were an important part of Chapter 3 in the 10th edition, they now provide the organizational structure for this completely revamped chapter, based on the *Mathematics Teaching Framework* (Huinker & Bill, 2017). And there is a new section describing learning progressions, because that was a priority to include in all Part II chapters. Much of the discussions have been updated, and a new table lists all of the problem-solving strategies.

Chapter 4: Planning in the Problem-Based Classroom

This much leaner chapter (3000 fewer words!) has revised, concise sections on families and differentiation. The family section includes a family newsletter, bookmarks to support homework, and ideas for developing a strong relationship with families. The section on differentiation zooms in on tasks and how to make them accessible and challenging to a range of learners. The elements within the before-during-after lesson model were revised.

What grew, in this chapter, was the collection of instructional routines, which now includes Three Reads, Number Strings, Splat!, WODB, and Estimation 180.

Chapter 5: Creating Assessments *for* Learning

Newly revised to reflect the most effective assessment approaches, Chapter 5 has a focus on peer and self-assessments, including a new self-assessment tool to capture students' dispositions toward math. A mathematical practices rubric is also shared as are many new figures and updated resources.

Chapter 6: Teaching Mathematics Equitably to All Students

Chapter 6 is completely redesigned, with a new beginning that connects the Mathematics Teaching Framework (Huinker & Bill, 2017) to equitable mathematics teaching, attending to three domains. Included in this new discussion is attention to developing mathematics identity, developing agency, avoiding implicit bias, and recognizing structural barriers. Each section on specific groups of learners (e.g., students with special needs in math, multilingual learners, etc.) was revised significantly, based on major work that has emerged in the past four years (more than 50 new references). Also, there is a new table providing a list of wonderful culturally diverse children's literature.

Chapter 7: Developing Early Number Concepts and Number Sense

This important first chapter in Part II begins with a look at mathematics content for the youngest learners by including a new learning progression for early numeracy. Added too are discussions of both perceptual and conceptual subitizing and more explicit conversations about linking counting on and counting back to addition and subtraction. There are also more virtual manipulatives and other web-based materials to respond to shifts to online learning options.

Chapter 8: Developing Meaning for the Operations

Every student can develop an understanding of each operation when addition, subtraction, multiplication, and division are learned through a concrete, semi-concrete, abstract (CSA) approach. This chapter has been reorganized to emphasize the importance of CSA. New features in this chapter include a list of attributes of effective contexts, a table of interpretations for remainders, new activities with related activity pages, and more children's literature.

Chapter 9: Developing Basic Fact Fluency

This chapter has a lot of changes. There are two new learning progressions that focus on fact sets and strategies, along with expanded attention to assessment, addressing the problem with timed tests and providing more guidance on effective assessment tools. Enjoy the many new games and activities that are there not for the fun of them (though they are fun), but to provide better ways for students to learn their facts meaningfully. There are also several new figures and tables.

Chapter 10: Developing Whole-Number Place-Value Concepts

There is a major reorganization of the chapter into progressions, including two from Roger Howe on the five stages of place-value notation and the historical evolution of notation. There is also a structuring of the content to reflect the CSA representations. One representation has been enhanced through the addition of the bottom-up hundred chart! Notice the addition of new children's literature for possible integration.

Chapter 11: Developing Strategies for Addition and Subtraction Computation

In this chapter, we added a progression on general computational strategy development that aligns with other computation discussed throughout the book. Importantly, we changed the language from invented strategies to reasoning strategies to reflect current literature in the field. Logically, we also increased attention to teaching reasoning strategies. Many figures were added or reconstructed, and the organization is based again on CSA.

Chapter 12: Developing Strategies for Multiplication and Division Computation

Chapter 12 is reorganized around CSA and focused on strategy use. As such, there are two new tables for reasoning strategy progressions, one for multiplication and one for division. There is also a new table that shows the four steps in the process of using an open array with partial products. The computational estimation section includes a revised table and new activities.

Chapter 13: Algebraic Thinking

Beyond the title change, this chapter has much new content as well as a redesign. Overall, this increases the focus on algebraic thinking in elementary school, building the strong foundation for algebraic thinking and mathematical modeling across K–8. There is an increased focus on generalizations, in particular an expanded section on the properties of the operations. More activity pages are provided to facilitate the presentation of the activities.

Chapter 14: Developing Fraction Concepts

New additions to this chapter include an expanded section on equal-sharing that connects to the related learning progression, an increased use of fraction strips and number lines throughout, and more attention to the unit. The "Fractions as Numbers" section was also adapted to be more concise.

Chapter 15: Developing Fraction Operations

Just as Chapters 11 and 12 increased attention to reasoning strategies, so has this chapter on operations with fractions. Furthermore, there is an expanded focus on estimating sums/differences (based on research findings that this improves student success). This chapter also has many new figures to support understanding.

Chapter 16: Developing Decimal and Percent Concepts and Decimal Computation

It is sometimes challenging to come up with realistic contexts for tenths, hundredths, and thousandths, so we've added a table that shares real-world options. Similar to the other chapters on computation, we are focusing on reasoning strategies for addition and subtraction of decimals through a new table. There is also a new progression for percent problem types.

Chapter 17: Ratios and Proportional Reasoning

This chapter had several major changes. First, we increased attention to social justice and real-life contexts throughout the chapter (6 new activities in this chapter alone). Additionally, the proportional reasoning representations and strategies were completely revised. The chapter was reorganized to focus on comparing ratios early in the chapter and to emphasize making connections to other mathematics at the end of the chapter.

Chapter 18: Developing Measurement Concepts

Because it spans K–8, measurement is a large chapter filled with new figures, activities, and activity pages. To begin, you will find a new progression for the measurement process and, later in the chapter, a new progression for learning about money. As in several other chapters, we have added new online manipulatives as options for instruction. There is also an exciting addition of a children's literature resource of 103 measurement books sorted by topic, such as length, area, volume, and time.

Chapter 19: Developing Geometric Thinking and Geometric Concepts

Similar to Chapter 18, this chapter covers grades K–8 and includes a new table with the van Hieles' Levels of Geometric Thought. The van Hiele progression is one of the most important progressions in mathematics, and we've added research about the importance of teachers studying these levels. Not surprisingly, the chapter is reorganized to reflect the van Hiele levels. A new table shares sites for virtual geometry tools so they can be integrated with online learning if needed. In response to a reviewer's suggestion, there is increased attention to the Pythagorean theorem.

Chapter 20: Developing Concepts of Data and Statistics

This leaner chapter has numerous changes throughout the chapter to reflect the new *GAISE II* recommendations. In addition, there is an increased focus on culturally responsive mathematics instruction and many new and updated figures.

Chapter 21: Exploring Concepts of Probability

One of the significant changes in this chapter was connecting qualitative and quantitative probability descriptions to support student understanding. The probability ladder is new and provides a strong visual to support this connection and explore the probability continuum. There is also more attention to selecting models (tree diagrams or area models) to determine probabilities.

Chapter 22: Exploring Exponents, Negative Numbers, and Real Numbers

This chapter has an expanded, clearer discussion of the challenges of using the mnemonic PEMDAS. Adding and subtracting negative numbers has been revised in various ways, including in a

hot-air balloon context. Like other chapters focusing on the operations, this chapter also turns more attention to fluency and flexibility with rational numbers.

An Introduction to *Teaching Developmentally*

If you look at the table of contents, you will see that the chapters are separated into two distinct sections. Part I (Chapters 1–6) addresses important ideas about mathematics teaching that apply across concepts. Among the ideas in Part I, we focus mathematics learning on students' strengths, knowing what it means to do mathematics, determining effective ways to get your students *actively engaged* through intentional planning and implementing effective teaching and assessing strategies. Throughout Part I and especially in Chapter 6, you learn how to ensure that each and every student develops mathematical proficiency and a positive mathematics identity.

Part II (Chapters 7–22) offers specific guidance for teaching every major mathematics topic in the pre-K–8 curriculum. Each chapter begins with big ideas for that content, along with learning outcomes. Each section provides guidance on how to teach that content through and illustrative examples through many figures and problem-based activities. Each of these chapters concludes with a list of challenges (and ways to support students with those challenges). The support found in each chapter can guide your planning and assessing as you work to ensure that every student develops competence and confidence in mathematics.

Hundreds of tasks and activities are embedded in the text. Take out a pencil and paper, or use technology, and try the problems, thinking about how you might solve them *and* how students at the intended grades might solve them. This is one way to engage actively in *your learning* about how *students learn* mathematics. In so doing, this book will increase your own understanding of mathematics, the students you teach, and how to teach them effectively.

Pedagogical Features

By flipping through the book, you will notice many section headings, a large number of figures, and various special features. All are designed to support your teaching, now and into the future. Here are a few features to enhance your reading and support your teaching.

Learning Outcomes

To help readers know what they should expect to learn, each chapter begins with learning outcomes. Numbered self-checks are interspersed along the way to align with each learning outcome.

Developing Fraction Concepts

CHAPTER 14

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **14.1** Describe and give examples for fraction constructs and fraction representations.
- **14.2** Describe three types of fraction representations, and what situations and manipulatives fit within each.
- **14.3** Explain foundational concepts of fractional parts, including equal sharing, iterating, and partitioning.
- **14.4** Illustrate the concept of equivalence across fraction models.
- **14.5** Describe strategies for comparing fractions and ways to teach this topic conceptually.

Big Ideas

Much of the research and literature on effective teaching of mathematics suggests that teachers plan their instruction around big ideas rather than on isolated skills or concepts. At the beginning of each chapter in Part II, you will find a list of the big mathematical ideas associated with the chapter. Teachers find these lists helpful to envision quickly the mathematics they are to teach.

BIG IDEAS

- Fractions can and should be represented across different interpretations (e.g., part-whole and division) and different representations: area (e.g., $\frac{1}{3}$ of a garden), length (e.g., $\frac{3}{4}$ of an inch), and set (e.g., $\frac{1}{2}$ of the marbles).
- The number line is a critical representation in developing fraction concepts.
- Fractions are equal shares of a whole or a unit. Therefore, equal sharing activities (e.g., 2 sandwiches shared with 4 friends) build on whole-number knowledge to introduce fractional quantities.
- Partitioning and iterating are strategies students can use to understand the meaning of fractions. Partitioning can be thought of as splitting the whole equally (e.g., splitting a whole into fourths), and iterating can be thought of as making a copy of each piece and counting them all (e.g., one-fourth, two-fourths, etc.).

Activities

The numerous activities found in every chapter of Part II have always been rated by readers as one of the most valuable parts of the book. Some activity ideas are described directly in the text and in the illustrations. Others are presented in the numbered activity boxes. Every activity is a problem-based task (as described in Chapter 3) and is designed to engage students in *doing* mathematics.

Activity 21.9 Fair or Unfair Coin Games

CCSS-M: 7.SP.C.6; 7.SP.C.7a

For each of these games, ask students first to predict whether they believe the game is fair and then play the game, considering whether the game is fair or unfair (and why). You can play these games within one lesson or on different days.

Coin Game 1: Three students form a group and are given two like coins (e.g., two pennies). For each flip of both coins, one player gets a point, based on the following rules:

Player A: Two heads; Player B: Two tails; Player C: One of each

The game is over after 20 tosses. The player who has the most points wins. Have students play the game two or three times.

Coin Game 2: Partners compete in this game, flipping two like coins (e.g., two pennies). For each flip of both coins, one player gets a point, based on the following rules:

Player A: Same face (e.g., both heads); Player B: One of each

After playing the game, ask students to create data displays of their group's data. Pool the class data and examine their results. Two-coin experiments lend themselves to many representations (e.g., tables, lists, various versions of tree diagrams), and these representations help students understand theoretical probability (English & Watson, 2016). When the full class has played the game several times, conduct a discussion on the fairness of each game. Challenge students to make an argument based on the data and game rules as to whether the game is fair. For multilingual learners, discuss the meaning of *fair* prior to beginning the game and review the term when asking students to create an argument. As another challenge, have students design their own fair game (see Expanded Lesson: Design a Fair Game for details).

Expanded Lesson: Design a Fair Game

Adaptations to Challenge and Support Students

Many Part II Activities have one or more icons (challenging students, supporting multilingual learners, supporting students with special needs in math). When you see these icons, it means that suggestions are embedded in the activity to differentiate the task so that it is accessible and meaningful to every student.

Activity 9.2 How Many Feet in the Bed?

CCSS-M: 1.OA.A.1; 1.OA.C.6; 2.OA.B.2

Read *How Many Feet in the Bed?* by Diane Johnston Hamm (1991). On the second time reading the book, ask students how many more feet are in the bed each time a new person gets in. Ask students to record the equation (e.g., $6 + 2$) and tell how many feet in total. Two less can be considered as family members get out of the bed. Find opportunities to make the connection between counting on and adding using a number line. For multilingual learners, be sure that they know what the phrases "two more" and "two less" mean (and clarify the meaning of *foot,* which is also used for measuring). Acting out this situation with students in the classroom can be a great illustration for both multilingual learners and students with special needs in math.

Formative Assessment Notes

Assessment is an integral process within instruction. Similarly, it makes sense to think about what to be listening for (assessing) as you read about different areas of content development. Throughout the Part II chapters, there are formative assessment notes with brief descriptions of ways to assess a topic in that section. Reading these assessment notes as you read the text can help you understand how best to assist students to grow in their mathematics understanding.

FORMATIVE ASSESSMENT Notes. Any of the preceding problems can be used as a formative assessment with an observation look-for tool. On the look-for tool would be such concepts as: (1) determines a reasonable estimate; (2) models the problem with manipulative or picture; (3) solves using an efficient method; (4) writes an accurate equation. Students may be able to illustrate but then not find fraction equivalences to solve the problem; or they may represent and solve but struggle to write an equation. A look-for tool provides guidance for what future instruction is needed. ■

Technology Notes

Like assessment, technology is infused into instruction. Thus, also in Part II, we include technology notes (updated significantly in this edition!). Descriptions include open-source (free) software, apps, and other Web-based resources.

***TECHNOLOGY* Note.** Dreambox (www.dreambox.com) Teacher Tools offers a bar (length) model for exploring addition and subtraction. Students solve problems visually, trading out equivalent fractions and recording their answers. Other sites offer various virtual manipulatives by which a student can build a problem and solve it. Those sites include Toy Theater Teacher Tools (toytheatre.com), Math Learning Center free apps (mathlearningcenter.org), and Brainingcamp (brainingcamp.com). On each site, you or the student can select the manipulative you will use to represent the problems. ■

Standards for Mathematical Practice Margin Notes

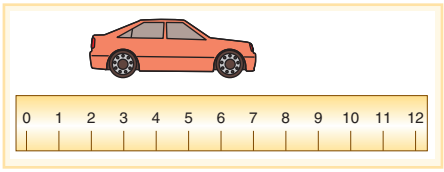
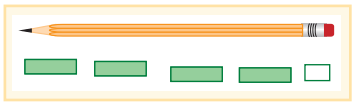
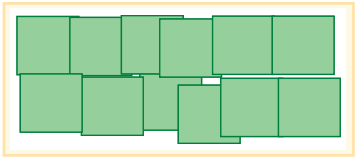
The Standards for Mathematical Practice continue to be an important component of many state standards. They describe what it means to do math! There are eight of them, as described in Chapter 1. In Part II, we use margin notes to showcase where an instructional suggestion connects to a particular mathematical practice.

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

Tables of Common Challenges

Found in every chapter of Part II are tables of common challenges that students might experience related to each topic. Each challenge has been identified in the research literature. Notice that suggestions for every case list how to support students as they navigate through these potentially difficult areas.

TABLE 18.2 COMMON CHALLENGES IN MEASUREMENT AND HOW TO HELP

Common Challenge	What It Looks Like	How to Help
1. Focus on the ending point of a ruler rather than the length.	Students read the number on the ruler that aligns with the object's right edge. 	• Move items along a ruler and ask students if they believe the item has changed size. • Iterate units to show how long the item is and then show how those units match the units on the ruler. • Give students experiences with the Broken Ruler.
2. Place multiple individual units with gaps between the units or with some of the units overlapping; or students use unequal units.	Students have gaps between units or use unequal units when measuring length:  Students overlap units when measuring area: 	• Ask students to estimate the measure before measuring the object and explain why it is a reasonable estimate. • If students use unequal units, ask what the number refers to (e.g., the number of white rods or brown rods). • If students overlap length units, have them use linking cubes to measure length. • If students overlap area units, use a unit with some thickness, making it difficult to overlap the pieces (e.g., square tiles or pattern blocks).

End-of-Chapter Resources

At the end of each chapter are *Resources*, which include "Writing to Learn" assessments, "Literature Connections" (found in all Part II chapters), and "Recommended Readings."

Writing to Learn. These exercises provide teacher candidates or teachers with a variety of opportunities to articulate, in their own words, the concepts they are learning. Prompts and questions have been designed to emphasize analysis and comprehension.

Literature Connections. Here you will find examples of children's literature for launching into the mathematics concepts in the chapter just read. For each title suggested, there is a brief description of how the mathematics concepts in the chapter can be connected to the story.

Recommended Readings. In this section, you will find an annotated list of articles and books to augment the information found in the chapter. These recommendations include NCTM articles and books and other professional resources designed for the classroom teacher.

RESOURCES FOR CHAPTER 14

WRITING TO LEARN

1. Describe the benefits of using equal sharing tasks.
2. Give examples of representations and contexts that fall into each of the three categories of fraction models (area, length, and set).
3. What does partitioning mean? Explain and illustrate.
4. What does iterating mean? Explain and illustrate.
5. What are two ways to build the conceptual relationship between $\frac{11}{4}$ and $2\frac{3}{4}$?
6. Explain why the number line is so important to developing fraction concepts.
7. Describe two ways to compare $\frac{5}{12}$ and $\frac{5}{8}$ mentally.

LITERATURE CONNECTIONS

Apple Fractions

Pallotta (2002)

Interesting facts about apples are connected to fractions as fair shares. Words for fractions are used and connected to fraction symbols, making it a good connection for fractions in grades 1–3.

Halfsy: A Math Adventure

Deitelbaum (2020)

Halfsy is on a journey from 0 to 1 (a length context for fractions), and on the way notices half of other things: the moon (area), birds (quantity).

The Doorbell Rang

Hutchins (1986)

The story is about sharing cookies, a changing situation as more and more children arrive. A great connection to sharing tasks.

The Lion's Share

McElligott (2009)

The story involves halving a rectangular cake over and over again, and provides a visual opportunity to explore unit fractions.

The Man Who Counted: A Collection of Mathematical Adventures

Tahan (1993)

This book contains a story, "Beasts of Burden," about a wise mathematician, Beremiz, and the narrator, who are traveling together on one camel. They are asked to figure out how to give each brother his share of 35 camels: *one-half to one*, *one-third to one*, *and one-ninth to one* (a set context). See Bresser (1995) for three days of activities with this book.

RECOMMENDED READINGS

Articles

Clarke, D. M., Roche, A., & Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School*, *13*(7), 373–380.

Ten excellent tips for teaching fractions are discussed and favorite activities are shared. An excellent overview of teaching fractions.

Monson, D., Cramer, K., & Ahrendt, S. (2020). Using models to build fraction understanding. *Mathematics Teacher: Learning & Teaching PreK-12*, *113*(2), 117–123.

This article provides excellent tasks and visuals to develop a strong understanding of fractions, including fractions on the number line.

Freeman, D. W., & Jorgensen, T. A. (2015). Moving beyond brownies and pizzas. *Teaching Children Mathematics*, *21*(7), 412–420.

This article describes students' thinking as they compare fractions. In the more4U pages, they offer excellent sets of tasks with a range of contexts, each set focusing on a different reasoning strategy for comparing fractions.

Books

Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals.* Portsmouth, NH: Heinemann.

With many tasks and examples of student work, this book is a must-have for supporting students' emerging concepts of fractions and decimals.

Pearson eText, Learning Management System (LMS)–Compatible Assessment Bank, and Other Instructor Resources

Pearson eText

The Pearson eText is a simple-to-use, mobile-optimized, personalized reading experience. It allows you to highlight, take notes, and review key vocabulary easily all in one place–even when offline. Seamlessly integrated videos and other rich media will engage you and give you access to the help you need, when you need it. To gain access or sign in to your Pearson eText, visit https://www.pearson.com/pearson-etext.

- **Video examples** Embedded throughout all chapters, these examples allow you to see key concepts and instructional strategies in action through authentic classroom video, as well as clips of children solving math problems. Additional videos feature your authors and other experts introducing and briefly explaining strategies for teaching important topics.
- **Blackline masters, activity pages, teacher resources, and expanded lessons** These documents are linked throughout each chapter and make it easy for instructors and students to download and print classroom-ready handouts that can be used in a methods class or school settings.
- **Access to BLMs, activity pages, teacher resources and expanded lessons** If you are not using this book as part of a University course, but as part of your professional work, we have created an open link to access the print resources used throughout the book: https://media.pearsoncmg.com/ab/vandewalle/vdw.html

LMS-Compatible Assessment Bank

With this new edition, all assessment types—quizzes and application exercises—are included in LMS-compatible banks for the following learning management systems: Blackboard (**9780136818380**), Canvas (**9780136818281**), D2L (**9780136818540**), and Moodle (**9780136818151**). These packaged files allow maximum flexibility for instructors when it comes to importing, assigning, and grading. Assessment types include:

- **Learning Outcome Quizzes:** Each chapter *Learning Outcome* is the focus of a *Learning Outcome Quiz* that is available for instructors to assign through their Learning Management System. The *Learning Outcomes* identify chapter content that is most important for learners and serve as the organizational framework for each chapter.
- **Application Exercises:** Each chapter provides opportunities for pre-service teachers or teachers to apply what they have learned through *Application Exercises*. Two to three *Application Exercises* are available for each chapter, aligned with the stated *Learning Outcomes*. The exercises require pre-service teachers to watch short videos, read scenarios, or think about situations and then answer open-ended questions. When used in the LMS environment, a model response written by experts is provided after teachers submit the exercise. This feedback helps guide teachers' learning and can assist the instructor in grading.
- **Chapter Tests:** Suggested test items are provided for each chapter and include questions in multiple choice and short answer/essay. When used in the LMS environment, the multiple-choice questions are automatically graded, and model responses are provided for short-answer and essay questions.

Instructor's Resource Manual (9780136818144)

The Instructor's Resource Manual for the eleventh edition includes a wealth of resources designed to help instructors teach the course, including chapter notes, activity suggestions, and suggested assessment questions.

PowerPoint® Slides (9780136818335)

Ideal for instructors to use for lecture presentations or student handouts, the PowerPoint presentation slides provide ready-to-use graphics and text images tied to the individual chapters and content development of the text.

Note: All instructor resources—LMS-compatible assessment bank, instructor's manual, and PowerPoint slides—are available for download at www.pearsonhighered.com. Use one of the following methods:

- From the main page, use the search function to look up either the lead author (Van de Walle) or the title (*Elementary and Middle School Mathematics: Teaching Developmentally*). Select the desired search result and then click the Resources tab to view and download all available resources.
- From the main page, use the search function to look up the ISBN (provided earlier) of the specific instructor resource you would like to download. When the product page loads, click the Downloadable Resources tab.

Acknowledgments

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Additionally, we are very grateful for the ideas we receive via email. You can email one of us directly or use the book email: teachingdevelopmentally@gmail.com.

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PART I Teaching Mathematics Developmentally: Big Ideas and Research-Based Practices

The fundamental core of effective teaching of mathematics combines an understanding of how students learn, how to promote that learning by teaching through problem solving, and how to plan for and assess that learning daily. That is the focus of these first six chapters, providing discussion, examples, and activities that develop the core ideas of learning, teaching, planning, and assessment for each and every student.

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CHAPTER 1

Developing Confident and Competent Mathematics Learners

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **1.1** Summarize the factors that influence the teaching of mathematics.
- **1.2** Describe the importance of content standards, process standards, and standards of mathematical practice.
- **1.3** Explore the qualities needed to learn and grow as a professional teacher of mathematics.

Did you know that the most important factor in a student's learning of mathematics is their teacher? You can ensure that every student becomes confident and competent in mathematics. You likely have many questions about how to make this vision a reality: What direction is there for what mathematics you will teach? What guidance is there for how you should teach? What can you achieve in your teaching to help students feel like they are good at math? How will you know whether students are learning? These questions and many more are addressed in this book. These three things—your knowledge of mathematics, how to teach mathematics, and how to focus on each learner—are the most important tools you can acquire to be a successful teacher of mathematics. We are here to help you hone those skills!

In this chapter, we begin this journey by addressing the big picture of teaching mathematics in K–8. We begin with some big ideas that are important about teaching mathematics developmentally, briefly describe the most significant influences on mathematics learning, and, finally, offer suggestions to you as a *teacher* of mathematics.

Learning How to Teach Mathematics Developmentally

As part of your desire to build successful learners of mathematics, you might recognize the challenge that mathematics is sometimes seen as the subject people love to hate. At social events—even at parent–teacher conferences—other adults may hear that you are a teacher of mathematics and share comments such as, "I could never do math," or, "I can't calculate the tip at a restaurant—I just hope they have suggestions for tips at the bottom of my receipt."

Instead of ignoring these disclosures, consider what positive action you can take. Would people confide that they don't read and hadn't read a book in years? Not likely. Families' and teachers' attitudes toward mathematics may enhance or detract from students' ability to think they can do math. Teaching mathematics developmentally can transform this positioning of mathematics into a positive experience for all students. Here we share some global ways to address misunderstandings and negative feelings about mathematics.

Math Is Important

Learning mathematics is essential to being productive at work and in our communities. It also facilitates curiosity in science and technology and enhances personal activities—not to mention that it's fun! While all students have many reasons to learn mathematics, the National Council of Teachers of Mathematics (NCTM) *Catalyzing Change* series (2018-2020) identifies several key purposes for early childhood and elementary and middle school students' learning mathematics. They are:

- Develop deep mathematical understanding as confident and capable learners.
- Understand and critique the world through mathematics.
- Experience the wonder, joy, and beauty of mathematics (p. 11 and p. 7).

These purposes of learning mathematics express what underlies the most effective ways to teach mathematics K–8. In alignment, the goal of this book is to help you learn these effective instructional methods that are often unique to mathematics. We base this book on the collective wisdom of an organization of mathematics teacher educators and mathematicians from across the United States who developed a set of standards that established what knowledge, skills, and dispositions are important in cultivating a well-prepared beginning teacher of mathematics, the *Standards for Preparing Teachers of Mathematics* (AMTE, 2020). As you dig into the information in the chapters ahead, your vision of what is possible for all your students, and your confidence to explore and teach mathematics, will grow in ways endorsed by experts in the field.

Ultimately, your students need to think of themselves as mathematicians in the same way as they think of themselves as readers. As students interact with our increasingly mathematical and technological world, they need to construct, modify, communicate, or integrate new mathematics-related information in many forms. Solving novel problems and approaching new situations with a mathematical perspective should come as naturally as using reading to comprehend facts, insights, or the latest news. Particularly because this century is a quantitative one (Hacker, 2016), we must prepare students to interpret the language and power of numeracy. Hacker states that "decimals and ratios are now as crucial as nouns and verbs" (p. 2). So, for your future students, consider how important mathematics is to interpreting and successfully surviving in our complex economy and changing environment and find ways to guide them toward these goals. Learning mathematics opens up a world of important ideas.

Mathematics Is about Effort, Not Genetics and Other Myths

Pearson eText
Video Example 1.1
In this video late author John Van de Walle provides his definition of mathematics.

Mathematics ability is not inherited—anyone can learn mathematics. Some of the previously mentioned quotations hint at a false assumption prevalent in the United States: that mathematics ability is hereditary. This just isn't true. What leads a person to be successful is their perseverance and support, both in and out of the classroom. In fact, engaging students in productive struggle is one of most effective ways to support their understanding (Baker, Jessup, Jacobs, Empson & Case, 2020). This speaks to the need to do a better job in schools to ensure that every child has access to a strong mathematics program!

The second myth that stands in the way of learning is that learning mathematics is about speed and memorizing facts and procedures. Have you observed a mathematician at work? They grapple with problems for hours, even years. Yes, they know facts and procedures to support their thinking, but the heart of mathematics is mathematical thinking, not being fast. Further, mathematics is not a bunch of isolated skills; it is a collective, interconnected whole.

We must help students see these connections to make sense of mathematics. There are also ways we were taught that didn't help us make such connections, such as talking about borrowing and carrying numbers, or that multiplication "makes numbers bigger," which is not so when you consider fractions, decimals—or zero, for that matter (Karp, Bush & Dougherty, 2014).

A third myth that interferes with learning mathematics is that people believe there is just one way to solve a problem. That is almost never the case. Even for a problem like $29 + 33$, there are numerous ways to solve it. Many adults use the standard algorithm, because that is the way they were taught. Algorithms focus on performing an operation on one place value at a time, moving from ones to tens in this example, and the standard algorithm is not necessarily the best way. (Can you think of an alternative that is more efficient?) Maybe even doing this problem in your head?

We begin thinking about teaching developmentally by tackling the first myth. You must believe that all children can learn mathematics, and that happens when you focus on their strengths!

Strengths-Based Approach

Part of becoming an effective teacher of mathematics is recognizing mathematical strengths—in students *and* in yourself. When you exude confidence and a can-do, problem-solving approach, you play an important role in developing the same qualities in your students. We know that when there is a focus on what a student doesn't do well—such as "you didn't pass that fractions test" or "I think you need more practice with your multiplication facts"—it affects a learner's confidence negatively. As Skinner, Louie, and Baldinger suggest, some students "are conditioned to see themselves as the dust of the mathematical universe and see others as the stars" (2019, p. 339). Rather than focus on deficits, effective mathematics teaching shifts attention to strengths (Kobett & Karp, 2020). All students need to know they are brilliant and capable of learning mathematical concepts and procedures. When you acknowledge their strengths, you can watch them flourish before your eyes.

This is true for teaching as well. Consider how you might react if you were told that "that math lesson you taught today was just not that effective." When you connect to your own humanity and explore your own mathematics identity (Aguirre, Mayfield-Ingram, & Martin 2013), the bond between you and your students grows in depth and breadth. For example, you can use what you know from your own experiences to add your empathy and compassion to new learners' initial experiences with mathematical topics. Here are two examples:

Rather than say, "Oh, this problem is easy," one of your strengths might be remembering how you heard those words as a learner and it made you feel uneasy. Instead, say, "Let's try this challenge together and explore some new ideas."

From your own past experiences, you know you have the strength of using multiple approaches to try to solve a problem. When you see a student who is stumped, you say, "Here's something I found helpful when I was stuck on a problem." Then you walk students down the path of possible solution strategies, including using manipulative materials, making drawings, creating tables of numbers, or even acting out a problem to find a successful solution strategy.

It's time to identify your strengths and acknowledge that you have the knowledge and resources to build and celebrate your students' superpowers (Kobett & Karp, 2020, p. 17.) We will help you in that process, starting right now.

Teaching Developmentally

With your strengths and your students' strengths in mind, you are ready to focus on the way in which mathematics is taught. The aforementioned second myth matters. Mathematics is connected, and that means new knowledge builds on students' prior knowledge. For example,

if you are working on sums like $29 + 34$, you can explore what strategies students use to add $9 + 4$. A student might use a Make 10 Strategy, rethinking the problem as $10 + 3$. You explicitly connect their understanding by asking questions like, "Can you use the Make 10 Strategy for $29 + 34$? Try it!" One part of teaching developmentally is to be sure you are purposeful in connecting the topic you are teaching now to related topics students have already learned.

So, let's find ways of countering negative statements about mathematics, especially if they are declared in the presence of students. Point out that, actually, it is a myth that only some people can be successful at learning mathematics. We all can! Only in that way can the chain of passing apprehension from adult to child be broken. There is much joy in solving mathematical problems, and it is essential that you demonstrate an excitement for learning and nurture a passion for mathematics in your students.

What Influences the Mathematics We Teach?

Learning Outcome 1.1 **Summarize the factors that influence the teaching of mathematics.**

How you are teaching mathematics should not look anything like the teaching of mathematics 50 years ago or even 20 years ago. Many factors influence these changes. Here we briefly discuss four major reasons.

The World Is Changing

In *The World Is Flat* (2007), Thomas Friedman discusses how globalization has created the need for people to have skills that are long-lasting and will survive the ever-changing landscape of available jobs. Friedman emphasizes that in a world that is digitized and surrounded by algorithms, math lovers will always have career opportunities and choices. Yet, there is a skills gap of qualified people; reports reveal that the United States will need to fill 3.5 million science, technology, engineering, and mathematics (STEM) jobs by 2025 (Radu, 2018).

Every teacher of mathematics has the job of preparing students with career skills while developing a love of math in students. The mathematician Stefan Banach stated, "Mathematics is the most beautiful and most powerful creation of the human spirit" (as cited in Kaluźa, 1996, n.p.). So, as you can see, there is an array of compelling reasons why children will benefit from the study of mathematics, using the approaches you will learn in this book. Help your students acquire the mental tools to make sense of mathematics—in some cases for mathematical applications that might not yet be known! This knowledge serves as a lens for interpreting the world.

Our changing world influences what should be taught in pre-K–8 mathematics classrooms, because there is a relationship between early mathematics performance and success in middle school (Bailey, Siegler, & Geary, 2014) and high school mathematics (Watts, Duncan, Siegler, & Davis-Kean, 2014), with third-grade mathematics scores pinpointed as a strong predictor of high school AP pass rates (Clark, Yu, Yi & Shi, 2018). As we prepare pre-K–8 students for jobs that possibly do not currently exist, we can predict that there will be few jobs in which just knowing simple computation is enough to be successful. We can also predict that many jobs and much of life will require interpreting complex data, designing algorithms to make predictions, and using multiple strategies to approach new problems.

As you prepare to help students learn mathematics for the future, you will need some perspective on the forces that effect change in the mathematics classroom. This chapter addresses the leadership that you, the teacher, will develop as you shape mathematics experiences for your students. Your beliefs about what it means to know and do mathematics and about how students make sense of mathematics will affect how you approach instruction and the understandings and skills your students take from the classroom. The enthusiasm you demonstrate about mathematical ideas will transform your students' interest in this amazing and beautiful discipline.

International Comparisons

One factor shaping what mathematics education looks like is the public or political pressure for change, due largely to student performance results in national and international studies. These large-scale comparisons of student performance continue to make headlines, provoke public opinion, and pressure legislatures to call for tougher standards backed by testing. These assessments provide strong evidence that mathematics teaching *must* change if U.S. students are to be competitive in the global market and understand the complex issues they will confront as responsible citizens of the world (Green, 2014).

National Assessment of Education Progress (NAEP). Since the 1960s, the United States has used the NAEP test to gather data on how fourth-, eighth-, and twelfth-grade students are doing in mathematics in what is known as the Nation's Report Card (https://nces.ed.gov/nationsreportcard). These data provide a tool for policy makers and educators to measure students' overall improvement over time, using four achievement levels: below basic, basic, proficient, and advanced, with proficient and advanced representing the desired grade-level achievement. In the most recent NAEP mathematics assessment in 2019, 41 percent of all U.S. students in grade 4 and 34 percent of students in grade 8 performed at the desirable levels of proficient and advanced (National Center for Education Statistics, 2019). Despite encouraging gains over the past 30 years, due to shifts in instructional practices (particularly at the elementary level) (Kloosterman, Rutledge, & Kenney, 2009a), students' performance in 2019 still shows room for improvement. We have work to do!

Trends in International Mathematics and Science Study (TIMSS). In 2019, 64 nations, at fourth grade and 46 at eighth grade, participated in TIMSS (TIMSS, 2019), the largest international comparative study of students' mathematics and science achievement—given seven times since it started in 1995. Data were gathered from a randomly selected group of 580,000 students, with approximately 20,000 from the United States. American students performed above the international average but were outperformed at fourth grade by 14 countries, headed by Singapore and including other countries such as Chinese Taipei, Japan, the Russian Federation, Northern Ireland, and Norway, and at eighth grade by 15 countries, also headed by Singapore and including the Republic of Korea, Hong Kong, Ireland, Israel, and Hungary. These data provide valuable benchmarks that allow the United States to reflect on our teaching practices and our overall competitiveness in preparing students for a global economy. If you've heard people talk about how mathematics is taught in Singapore—these rankings are why. We learn a common theme from these examples: These high-performing nations focus on teaching that emphasizes conceptual understanding and procedural fluency, both of which are critically important to the long-term growth of problem-solving skills (OECD, 2016; Rittle-Johnson, Schneider, & Star, 2015). In fact, teaching in these high-performing countries more closely resembles the long-standing recommendations of the National Council of Teachers of Mathematics, the major professional organization for mathematics teachers, discussed in the following section.

Research on Learning and Teaching

Over the years, there have been significant reforms in mathematics education that reflect research on how students learn mathematics. Just as we would not expect doctors to be using exactly the same techniques and medicines that were prevalent when you were a child, teachers' methods are evolving and transforming via a powerful collection of expert knowledge about how the mind functions and how to design effective instruction (Wiggins, 2013). The field of mathematics education actually knows a lot about how students learn (e.g., Cai, 2017; English & Kirshner, 2015; NRC, 2001). Earlier, we shared some of those ideas, and each chapter reflects the most current research in the field (hence all the citations!)

National Council of Teachers of Mathematics (NCTM). One transformative factor in the teaching of mathematics is the leadership of NCTM. The NCTM, with more than 60,000 members, is the world's largest organization of mathematics

educators and, as such, holds an influential role in both supporting teachers and emphasizing what's best for learners. NCTM's guidance in the creation and dissemination of standards for curriculum, assessment, and teaching led the way for other disciplines to create standards and for the eventual development of the mathematics standards used in your state today. For example, *Principles to Actions* (NCTM, 2014b) identifies eight effective teaching practices. This book, along with the *Catalyzing Change* series (2019–2020), provides strong support to all teachers. The premiere research journal is housed within NCTM, and the teacher journals, including the *Mathematics Teacher: Learning and Teaching PK-12*, is full of high-quality teaching ideas. Visit the NCTM website at www.nctm.org.

How Do We Use the Standards to Teach?

Learning Outcome 1.2 **Describe the importance of content standards, process standards, and standards of mathematical practice.**

What you teach is based on standards, but where did the standards come from? What we know about mathematics is the result of many thinkers of all races, ethnicities, and gender identities, beginning well before modern times. The heritage of these powerful ideas is diverse, and the legacy of many giants comes before what your state or school chose as the standards for your classroom. For that reason, it is helpful to think about how we came to teach what we teach and why.

In 1989, NCTM published the very first standards document of any discipline: "Curriculum and Evaluation Standards for School Mathematics." In 2010, the National Governors Association (NGA) Center for Best Practices and the Council of Chief State School Officers (CCSSO) presented the *Common Core State Standards* (http://www.corestandards.org/math), which incorporated ideas from many documents from the NCTM, including *Principles and Standards for School Mathematics* (2000) and *Curriculum Focal Points* (2006), as well as international curriculum documents. A large majority of U.S. states adopted these standards, and other states used these standards for reference while creating their own, similar version. (Check your state's website.) At this time, approximately 42 states, Washington, D.C., four territories, and the Department of Defense Schools have adopted standards similar to the CCSS-M. The movement that NCTM started in 1989 transformed the country from having little to no coherent vision on what school mathematics should be taught and when to a widely shared understanding of what students should know and be able to do across the grades.

Mathematics Content Standards

Standards articulate the *critical*, *essential*, or *emphasized areas* of mathematics content for each grade to provide a coherent curriculum built around big mathematical ideas. These larger groups of related standards are commonly called *domains*. These domains range from Counting and Cardinality, to Number and Operations in Base Ten, to Measurement and Data. Topics are sequenced based on what is known from research and practice about learning progressions, which are sometimes referred to as *learning trajectories* (Clements & Sarama, 2021; Clements, Sarama, Baroody, & Joswick, 2020).

Deconstructing and Understanding Standards. One important task you need to think about is learning about the standards you will teach. The actual statements are not often as easily communicated as, "Third graders will learn perimeter." Instead, you may find something like, "Represent a fraction $\frac{1}{b}$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $\frac{1}{b}$ and that the endpoint of the part based at 0 locates the number $\frac{1}{b}$ on the number line." You can reread that multiple times, but the best way to learn more about the standards incorporates what we know about brain-targeted teaching (Hardiman, 2012). First, consider drawing what each standard means as a way to delve into the topic (Strand & Bailey, 2020).

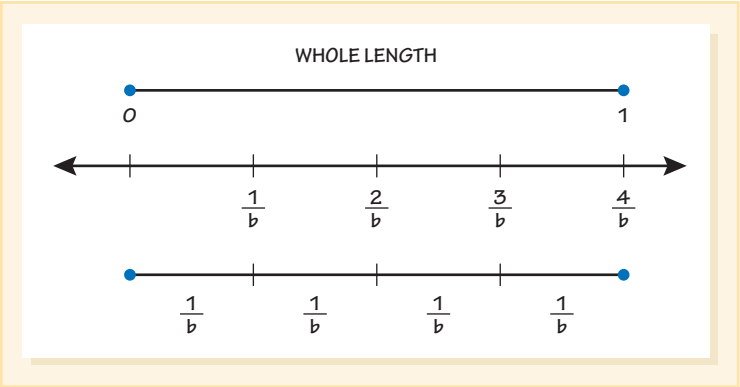


FIGURE 1.1 Decomposing a fraction standard in a drawing

Work with a colleague and divide and conquer as you unpack the ideas. See Figure 1.1 for a way to think about the preceding standard. As you draw, you will focus on the meaning and how you can link a group of standards through their connections and relationships.

Vertical Alignment of Standards. In addition to making sense of the standard at your grade, you need to know what students were taught the previous year (so you can assess whether they currently have that prior knowledge and then build on it), and you need to know where they are headed next year. Learning progressions help teachers know the sequence of what came before a particular concept as well as what to expect next as students reach key points along a pathway to desired learning targets (Corcoran, Mosher, & Rogat, 2009). Although these paths are not identical for all students, they can inform the order of instructional experiences that supports movement toward understanding and application of mathematics concepts. In other words, these progressions help teachers to teach mathematics developmentally! Bookmark the website for Achieve the Core (https://achievethecore.org/page/254/progressions-documents-for-the-common-core-state-standards-for-mathematics), where many helpful resources, including the Progression Documents, can be found. We also share numerous learning progressions in Part II!

Remember that standards set what students need to learn at a given grade but not how to teach that content. Instead, districts and schools make the final decisions about which curriculum to use or if they want to develop their own curriculum. Standards are not a curriculum. In fact, teachers are the essential piece in selecting and presenting the content standards into a coherent and engaging package.

Mathematical Practices

Beyond *what* mathematics is to be learned, standards describe the skills students need to have to do mathematics. For example, students need to be able to justify or explain their strategy. The vision for such actions was first described by NCTM as Process Standards (see, for example, *Principles and Standards for School* NCTM, 2000). The five process standards are: problem solving, reasoning and proof, communication, connections, and representations. Mathematical Practices were also captured in *Adding It Up* (National Research Council, 2001) (see Chapter 2 in this book).

The NCTM Process Standards (https://www.nctm.org/Standards-and-Positions/Principles-and-Standards/Process/) and the Strands of Mathematical Proficiency (https://www.nap.edu/read/9822/chapter/6) were blended into the *Standards for Mathematical Practice*. These are described in Table 1.1 and in Appendix A. Teachers must attend to these practices in the same way they attend to content, identifying particular practices to develop within lessons and units. These practices are blended with content. You will see that in Part II, we place Mathematical Practice notes in the margin to illustrate how these practices can emerge when you are engaging students in doing mathematics.

TABLE 1.1 THE STANDARDS FOR MATHEMATICAL PRACTICE

Mathematical Practice	K–8 Students Should Be Able To:
Make sense of problems and persevere in solving them.	• Explain what the problem is asking. • Describe possible approaches to a solution. • Consider similar problems to gain insights. • Use concrete objects or drawings to think about and solve problems. • Monitor and evaluate their progress and change strategies if needed. • Check their answers by using a different method. • Try again with another approach if one attempt is not successful or when they feel stuck.
Reason abstractly and quantitatively.	• Explain the relationship between quantities in problem situations. • Represent situations using symbols (e.g., writing expressions or equations). • Create representations that fit the word problem. • Use flexibly the different properties of operations and objects.
Construct viable arguments and critique the reasoning of others.	• Understand and use assumptions, definitions, and previous results to explain or justify solutions. • Make conjectures by building a logical set of statements. • Analyze situations and use examples and counterexamples. • Explain their thinking and justify conclusions in ways that are understandable to teachers and peers. • Compare two possible arguments for strengths and weaknesses to enhance the final argument.
Model with mathematics.	• Apply mathematics to solve problems in everyday life. • Make assumptions and approximations to simplify a problem. • Identify important quantities and use tools or representations to connect their relationships. • Reflect on the reasonableness of their answer based on the context of the problem.
Use appropriate tools strategically.	• Consider a variety of tools, choose the most appropriate tool, and use the tool correctly (e.g., manipulative, ruler, technology) to support their problem solving. • Use estimation to detect possible errors and establish a reasonable range of answers. • Use technology to help visualize, explore, and compare information.
Attend to precision.	• Communicate precisely, using clear definitions and appropriate mathematical language. • State accurately the meanings of symbols. • Specify appropriate units of measure and labels of axes. • Use a level of precision suitable for the problem context.
Look for and make use of structure.	• Identify and explain mathematical patterns or structures. • Shift viewpoints and see things as single objects or as composed of multiple objects or see expressions in many equivalent forms. • Explain why and when properties of operations are true in a particular context.
Look for and express regularity in repeated reasoning.	• Notice if patterns in calculations are repeated and use that information to solve other problems. • Use and justify the use of general methods or shortcuts by identifying generalizations. • Self-assess as they work to see whether a strategy makes sense, checking for reasonableness prior to finalizing their answer.

Source: Based on Council of Chief State School Officers. (2010). Common Core State Standards. Copyright © 2010 National Governors Association Center for Best Practices and Council of Chief State School Officers. All rights reserved.

How to Teach the Standards Effectively

NCTM also developed a publication that explores the specific learning conditions, school structures, and teaching practices that will be important for a high-quality education for all students. The book *Principles to Actions* (NCTM, 2014b) uses classroom stories and samples of student work to illustrate the careful, reflective work required of effective teachers of mathematics through six guiding principles (see Table 1.2 and Appendix B). A series of presentations

TABLE 1.2 THE SIX GUIDING PRINCIPLES FROM *PRINCIPLES TO ACTIONS*

Guiding Principle	Suggestions for Classroom Actions That Align with the Principles
Teaching and learning	• Select focused mathematics goals. • Use meaningful instructional tasks that develop reasoning, sense making, and problem-solving strategies. • Present and encourage a variety of mathematical representations that connect the same ideas or concepts. • Facilitate student discussions and conversations about important mathematical ideas. • Ask essential questions that are planned to be a catalyst for deeper levels of thinking. • Use a strong foundation of conceptual understanding as a foundation for procedural fluency. • Encourage productive struggle—it is a way to deepen understanding and move toward student application of their learning. • Generate ways for students to provide evidence of their thinking through discussions, illustrations, and written responses.
Access and equity	• Establish high expectations for all students. • Provide supports targeted to student needs (equity not equality). • Provide instructional opportunities for students to demonstrate their competence in different ways—creating tasks with easy entry points for students who struggle and extension options for those who finish quickly. • Identify obstacles to students' success and find ways to bridge or eliminate those barriers. • Develop all students' confidence that they can do mathematics. • Enhance the learning of all by celebrating students' diversity.
Curriculum	• Build connections across mathematics topics to capitalize on broad themes and big ideas. • Look for both horizontal and vertical alignment to build coherence. • Avoid thinking of a curriculum as a look-for tool or disconnected set of daily lessons.
Tools and technology	• Include an array of technological tools and manipulatives to support the exploration of mathematical concepts, structures, and relationships. • Think beyond computation when considering the integration of technology. • Explore connections to how technology use for problem solving links to career readiness.
Assessment	• Incorporate a continuous assessment plan to follow how students are performing and how instruction can be modified and thereby improved. • Move beyond test results that just look at overall increases and decreases to pinpoint specific student needs. • Consider the use of multiple assessments to capture a variety of student performance. • Encourage students to self-assess sometimes by evaluating the work of others to enhance their own performance. • Teach students how to check their work.
Professionalism	• Develop a long-term plan for building your expertise. • Build collaborations that will enhance the work of the group of collaborators as you enhance the performance of the students in the school. • Take advantage of all coaching, mentoring, and professional development opportunities and be a life-long learner. • Structure in time to reflect on and analyze your instructional practices.

(webcasts), led by the authors of the publication, explore several of the guiding principles and are available on the *Principles to Actions* portion of NCTM's website (www.nctm.org).

Pause & Reflect

Take a moment to think about which of these six guiding principles seem the biggest change since you were in school. Why do you think these changes to teaching were necessary? ●

An Invitation to Learn and Grow

Learning Outcome 1.3 **Explore the qualities needed to learn and grow as a professional teacher of mathematics.**

Think back to when you were a student in pre-K–8 classrooms. What are your remembrances of learning mathematics? Here are some thoughts from in-service and pre-service teachers of whom we asked the same question. Which description resonates with your thinking?

> I was really good at math in lower elementary grades, but because I never understood why math works, it made it very difficult to embrace the concepts as I moved into higher grades. I started believing I wasn't good at math so I didn't get too upset when my grades reflected that. *Kathryn*
>
> As a student, I always felt lost during mathematics instruction. It was as if everyone around me had a magic key or code that I missed out on getting. *Tracy*
>
> I consider myself to be really good at math and I enjoy mathematics-related activities, but I often wonder if I would have been GREAT at math and had a completely different career if I cared about math as much as I do now. Sometimes I feel robbed. *April*
>
> Math went from engaging, interactive instruction that I excelled at and loved, to lecture-style instruction that I struggled with. I could not seek outside help, even though I tried, because the teacher's way was so different from the way of the people trying to help me. I went from getting all As to getting low Bs and Cs without knowing how the change happened. *Janelle*
>
> Math class was full of elimination games where students were pitted against each other to see who could answer a math fact the fastest. Because I have a good memory I did well, but I hated every moment. It was such a nerve-wracking experience and for the longest time that is what I thought math was. *Lawrence*
>
> Math was never a problem because it was logical; everything made sense. *Tova*

As you can see, these memories run the gamut with an array of emotions and experiences. The question now becomes: What do you hope your former students will say as they think back to your mathematics instruction? The challenge is for each and every student to learn mathematics with understanding and enthusiasm. Would you relish hearing your students, 15 years after leaving your classroom, state that you encouraged them to be mathematically minded, curious about solving new problems, self-motivated, able to think critically about both correct and incorrect strategies, and that you nurtured them to be risk takers willing to persevere on challenging tasks? That is our expectation!

The mathematics education described in this book may not be the same as the mathematics content and the mathematics teaching you experienced in your grade K–8 experience. As a practicing or prospective teacher preparing to teach mathematics using a problem-solving approach, this book may require you to confront some of your personal beliefs—beliefs about what it means to *do mathematics*, how one goes about *learning mathematics*, how to *teach mathematics*, and what it means to *assess mathematics*. Success in mathematics isn't merely about speed or the notion that there is one right answer. Thinking and talking about mathematics as a means to sense making is a strategy that will serve us well in becoming a society where all individuals are confident in their ability to do math.

Becoming a Teacher of Mathematics

As we said at the start of this chapter, you bring many strengths to the teaching of mathematics, including your willingness to try new things, a fresh perspective on how technology can be integrated into instruction, and your stance as a lifelong learner. Here we describe characteristics, habits of thought, skills, and dispositions you will continue to cultivate to reach success as an effective teacher of mathematics and, we hope, will embrace as you read this book. This book is about increasing the quality of instruction. We are not about fixing children; we are instead engaged in fixing structures.

Knowledgeable. You will need to have a profound, flexible, and adaptive knowledge of mathematics content (Ma, 1999). This statement is not intended to intimidate if you feel that mathematics is not your strong suit, but it is meant to help you prepare for a serious semester of learning about mathematics and how to teach it. You cannot teach what you do not know. Teachers are the driving factors in students' school performance (Colvin & Edwards, 2018). An absence of high-quality opportunities for students to gain mathematics knowledge can result in students having economic challenges and little potential for social mobility (OECD, 2016). These findings add to the gravity of your responsibility, because a student's learning for the year in mathematics will likely come from you. If you are not sure of a fractional concept or other aspect of mathematics content knowledge, now is the time to make changes in your depth of content understanding to prepare best for your role as an instructional leader. You don't want to work on the brink of your knowledge base—instead you need to soak up the knowledge so you will feel more confident and can speak with added passion and enthusiasm. This book and your instructor will help you in that process.

Persistent. You need the ability to stave off frustration and demonstrate persistence. Dweck (2007) describes the brain as similar to a muscle—one that can be strengthened with a good workout! People are not just wired for learning mathematics; they must perform hard work and persevere to understand new ideas. As you move through this book and work the problems yourself, you will learn methods and strategies that will help you anticipate the experiences and strategies to get students past any stumbling blocks. It is likely that what works for you as a learner might work for some of your students. As you conduct these mental workouts, if you ponder, struggle, talk about your thinking, and reflect on how these new ideas fit or don't fit with your prior knowledge, you will enhance your teaching repertoire. Remember as you model these characteristics for your students, they too will value perseverance more than speed. In fact, Einstein did not describe himself as intelligent. Instead, he stated that he was just someone who continued to work on problems longer than others did. Creating opportunities for your students to engage in productive struggle is part of the learning process (Stigler & Hiebert, 2009; Warshauer, 2015) and will be discussed in greater depth in Chapter 2.

Positive Disposition. Prepare yourself by developing a positive attitude toward the subject of mathematics. Research shows that teachers with positive attitudes teach math in more successful ways that result in their students liking math more (Karp, 1991), performing at higher levels (Palardy & Rumberger, 2008), and developing confidence and skill that are personally meaningful and support the community in which they live. If you have ever thought, "I don't like math," that mindset will be evident in your instruction (Ramirez, Hooper, Kersting, Ferguson, & Yeager, 2018). The good news is that research shows that changing attitudes toward mathematics is relatively easy (Tobias, 1995) and that attitude changes are long-lasting (Dweck, 2007). In addition, math methods courses are found to be effective in increasing positive attitudes, more so than student teaching experiences (Jong & Hodges, 2015). Also, pre-service teachers who studied key concepts in mathematics methods classes were more effective in planning lessons on those big ideas—even years after taking the course (Morris & Hiebert, 2017). Not only can you acquire a more positive attitude toward mathematics, as a professional educator it is essential that you do.

To explore your students' attitudes toward mathematics, consider using this interview protocol. Here you can explore how the classroom environment may affect their attitudes.

Pearson eText Teacher Resource: Interview Protocol

Ready for Change. Demonstrate a readiness for change, even for change so radical that it may cause you disequilibrium. You may find that what is familiar will become unfamiliar and, conversely, what is unfamiliar will become familiar. For example, you may have always referred to "reducing fractions" as the process of changing $\frac{2}{4}$ to $\frac{1}{2}$, but this language is misleading, because the fractions are not getting smaller. Such terminology can lead children to mistaken connections when they ask, "Did the reduced fraction go on a diet?" A careful look will point out that *reducing* is not the term to use; rather, you are writing an equivalent fraction that is *simplified* or in *lowest terms*. Even though you may have used the language *reducing* for years, you need to become familiar with more precise language, such as *simplifying* fractions.

On the other hand, what is unfamiliar will become more comfortable. It may initially feel uncomfortable for you to ask students, "Did anyone solve it differently?" especially if you are worried that you might not understand their approach. Yet this question is essential to effective mathematics teaching. As you bravely use this strategy, it will become comfortable (and you will learn many new strategies!).

Another potentially new shift in practice is toward a mixed emphasis on teaching concepts and procedures. What happens in a procedure-focused classroom when a student doesn't understand division of fractions? A teacher with only procedural knowledge may say, "Just change the division sign to multiplication, flip over the second fraction, and multiply." We know the use of a memorized approach doesn't work well if we want students to understand fully the process of dividing fractions, so let's consider an example using $3\frac{1}{2} \div \frac{1}{2} =$ _____. Start by relating this division problem to students' prior knowledge of a whole-number division problem such as $25 \div 5 =$ _____, using a corresponding situation: "How many servings of 5 pizza slices are there in 25 slices?" Then ask students to put these same words to use in the original fraction division problem, such as, "You plan to serve each guest $\frac{1}{2}$ a pizza. If you have $3\frac{1}{2}$ pizzas, how many guests can you serve?" Yes, there are seven halves in $3\frac{1}{2}$; therefore, seven guests can be served. Are you surprised that you can do this division of fractions problem in your head once you learn this connection?

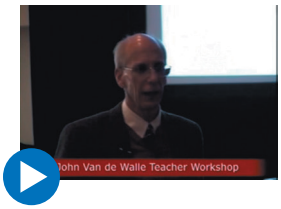
To respond to students' challenges, uncertainties, and frustrations, you may need to unlearn and relearn mathematical concepts, developing comprehensive conceptual understanding and a variety of representations along the way. Unearthing these ideas through looking at student work samples, even including errors, and discussing them is an important approach (McLaren, van Gog, Ganoe, Karabinos, & Yaron, 2016). Supporting your mathematics content knowledge on solid, well-supported terrain is important in making a lasting difference in your students' learning of mathematics—so be ready for change. What you already understand will provide you with many Aha moments as you read this book and connect new information to your current mathematics knowledge.

A Team Player. Your school should work as a unit where all teachers are supporting children not just for the one grade they teach but in a coherent manner across the grades. When this idea of a team is implemented, teachers agree to use the same mathematical language, symbols, models, and notation to give students a familiar thread that ties the concepts and procedures together year after year. This established understanding is called a Mathematics Whole School Agreement (Karp, Dougherty, & Bush, 2021), and your eager collaboration is essential in making this approach work well. These collaborative professional development efforts promote your school's capacity in terms of students' mathematics achievement and the ability to support your ongoing professional learning (Donaldson, 2018).

Self-Aware and Reflective. Maya Angelou said, "Do the best you can until you know better. Then when you know better, do better." No matter whether you are a pre-service teacher or an experienced teacher, there is always more to learn about the content and methodology of teaching mathematics. The ability to examine oneself for strengths or areas that need improvement or to reflect on successes and challenges is critical for growth and development. The best teachers are always trying to improve their practice through reading the latest article, attending the webinar or podcast with the most up-to-date information, or signing up for the next series of professional development opportunities. These teachers don't say, "Oh, that's what I am already doing"; instead, they identify and celebrate each new insight and idea they gain. Highly effective teachers never finish learning or exhaust the number of new mental connections they make, and, as a result, they never burn out. Sherman (1982) suggests, "You can't go back and make a new start, but you can start right now and make a brand new ending" (p. 45). Explore the following self-reflection Professional Growth Chart to list your strengths and indicate areas you wish to target for continued growth.

Think back to the quotations from the teachers at the beginning of this section. Again, what memories will you create for your students?

As you begin this adventure, let's be reminded of what John Van de Walle said with every new edition of the book: "Enjoy the journey!"



Pearson eText
Video Example 1.2
In this video late author John Van de Walle discusses the challenges of teaching in ways focused on a problem-solving approach. He emphasizes the importance of letting children do the thinking.

Pearson eText Teacher Resource: Professional Growth Chart

RESOURCES FOR CHAPTER 1

WRITING TO LEARN

At the end of each chapter of this book, you will find a series of questions under this same heading. The questions are designed to help you reflect on important ideas of the chapter. Writing (or talking with a peer) is an excellent way to explore new ideas and incorporate them into your own knowledge base.

1. What are the eight Standards for Mathematical Practice? How do they relate to the Mathematics Content Standards?
2. What are three purposes for learning mathematics as suggested by the NCTM *Catalyzing Change* document? How do these purposes align with your own mathematics instruction when you were a student?

RECOMMENDED READINGS

Articles

Buckheister, K., Jackson, C., & Taylor, C. (2015). An inside track: Fostering mathematical practices. *Teaching Children Mathematics*, *22*(1), 28–35.

The authors share a game used with early elementary students, and through teacher–student dialogue, they describe how the several mathematical practices can be developed.

Karp, K., Bush, S., & Dougherty, B. (2014). 13 rules that expire. *Teaching Children Mathematics*, *21*(1) 18–25.

This article helps students move away from overgeneralizations and rules that cause confusion while considering developing terminology and notation that enhances student understanding.

Books

Kobett, B., & Karp, K. (2020). *Strengths-based teaching and learning in mathematics: 5 teaching turnarounds for grades K–6*. Thousand Oaks, CA: Corwin: A Sage Publication; and Reston, VA: NCTM.

If you want to focus on students' mathematics strengths rather than identify their deficits, this book is the perfect read.

National Council of Teachers of Mathematics. (2020). *Catalyzing change in early childhood and elementary mathematics: Initiating critical conversations*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2020). *Catalyzing change in middle mathematics: Initiating critical conversations*. Reston, VA: NCTM.

The Catalyzing Change series explores important conversations about the policies, practices, and issues that are factors in influencing mathematics education.

CHAPTER

2

Exploring What It Means to Know and Do Mathematics

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

2.1 Describe essential components of mathematical proficiency.

2.2 Illustrate what it means to do mathematics.

2.3 Connect learning theories to mathematical proficiency and to mathematics teaching.

To help students learn mathematics, you must know *what* is important to learn, as well as *how* students learn. That is the focus of this chapter! Let's look at a relatively poorly understood topic, fraction division. If a student learned this topic, what will they know and what should they be able to do? The answer is more than being able to implement a procedure successfully (e.g., the invert-and-multiply algorithm). To know and understand division of fractions, a student can answer these questions:

- What does $3 \div \frac{1}{4}$ mean conceptually?
- What situation might this expression represent?
- Will the result be greater than or less than 3 and why?
- What strategies can we employ to solve this problem?
- What illustration or manipulative might illustrate this problem?
- How does this expression relate to subtraction? To multiplication?

Wondering the answers to some of these questions? See Chapter 15! The point here is that there is more *to* learn than procedures. This chapter focuses first on what it means to do mathematics, then engages you in doing mathematics, and finally explains how students learn mathematics.

What Is Mathematical Proficiency?

Learning Outcome 2.1 **Describe essential components of mathematical proficiency.**

Doing mathematics means generating strategies for solving a problem, applying an efficient strategy, and checking to see whether the answer makes sense. As you read in Chapter 1, standards documents describe these actions as Mathematical Practices (see Table 1.1 and Appendix A).

The Mathematical Practices describe specific actions of *mathematical proficiency*, as defined by the National Research Council in *Adding It Up* (2001) (see Figure 2.1). A person who is mathematically proficient doesn't just know content; they know how to navigate the process of solving problems. It is our responsibility as teachers of mathematics to help students develop each of these five strands of mathematical proficiency!

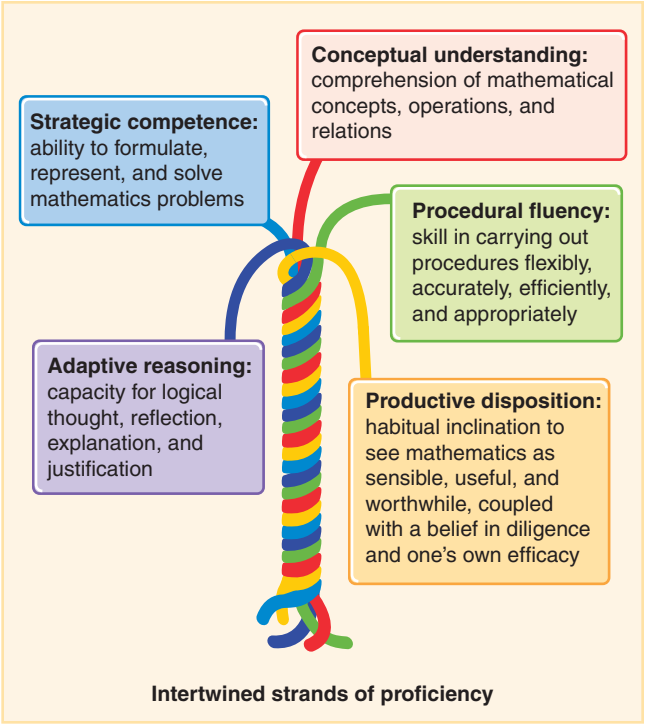

Conceptual understanding: comprehension of mathematical concepts, operations, and relations
Strategic competence: ability to formulate, represent, and solve mathematics problems
Procedural fluency: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
Adaptive reasoning: capacity for logical thought, reflection, explanation, and justification
Productive disposition: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy
Intertwined strands of proficiency

FIGURE 2.1 Five strands of mathematical proficiency

Source: National Research Council. (2001). *Adding It Up: Helping Children Learn Mathematics*, p. 5. Reprinted with permission from the National Academy of Sciences, courtesy of the National Academies Press, Washington, DC.

Conceptual Understanding

Conceptual understanding means having *connected* knowledge: "mental connections among mathematical facts, procedures, and ideas" (Hiebert & Grouws, 2007, p. 380). Consider what a person might know about a fraction such as $\frac{6}{8}$. Here is a partial list of what they might be able to do:

- Read the fraction.
- Identify the 6 and 8 as the numerator and denominator, respectively.
- Recognize that $\frac{6}{8}$ is equivalent to $\frac{3}{4}$.
- Notice that $\frac{6}{8}$ is more than $\frac{1}{2}$ (recognize relative size).
- Draw a region that is shaded in a way to show $\frac{6}{8}$.
- Locate $\frac{6}{8}$ on a number line.
- Illustrate $\frac{6}{8}$ of a set of 48 pennies or counters.
- State that there are an infinitely large number of equivalencies to $\frac{6}{8}$.
- Recognize $\frac{6}{8}$ as a rationale number.
- Describe $\frac{6}{8}$ as a ratio (girls to boys, for example).
- Represent $\frac{6}{8}$ as a decimal fraction.

At what point does a person know enough to claim they understand fractions? A number of items on this list refer to procedural knowledge (e.g., being able to find an equivalent fraction), and others refer to conceptual knowledge (e.g., recognizing that $\frac{6}{8}$ is greater than $\frac{1}{2}$ by analyzing the relationship between the numerator and denominator). A student may know that $\frac{6}{8}$ can be simplified to $\frac{3}{4}$ but not recognize that $\frac{3}{4}$ and $\frac{6}{8}$ are equivalent (having procedural knowledge without conceptual knowledge). A student may be able to find one fraction between $\frac{1}{2}$ and $\frac{6}{8}$ but not be able to find others, meaning they have some procedural knowledge but insufficient connections to recognize that they could also change denominators. The point is that *understanding* a mathematics topic is hard to define. It is best explained as a measure of the quality and quantity of connections a person can make between a new idea and their existing ideas. In other words, conceptual understanding is a flexible web of connections and relationships within and between ideas, interpretations, and images of mathematical concepts—a relational understanding.

Relational Understanding. Understanding exists along a continuum from an *instrumental understanding*—doing something without understanding (see Figure 2.2) to a *relational understanding*—knowing what to do and why (Skemp, 1978). This continuum continues to be a useful way to think about the depth of a student's conceptual understanding. In the $\frac{6}{8}$ example, a student who only knows a procedure for simplifying $\frac{6}{8}$ to $\frac{3}{4}$ has an understanding near the instrumental end of the continuum, whereas a student who can draw diagrams, give examples, and find numerous equivalencies has an understanding moving toward the relational end of the continuum.

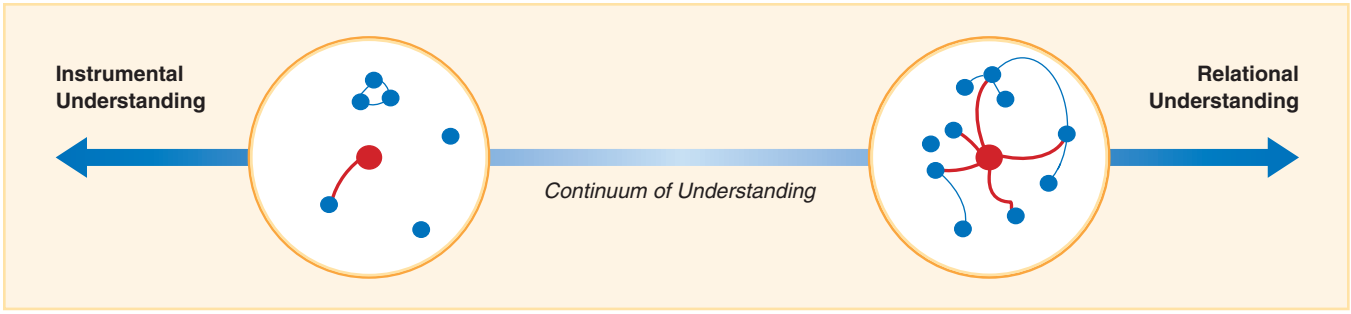


FIGURE 2.2 Continuum of understanding from instrumental to relational

Relational understanding is the goal for students across all mathematics content in the K–8 curriculum. Figure 2.3 provides a visual of skills and concepts related to understanding ratios. Note how much is involved in having a relational understanding.

Procedural Fluency

Procedural fluency is built on the foundation of conceptual understanding (NCTM, 2014b). Procedural fluency is sometimes confused with being able to do standard algorithms correctly and quickly, but it requires a much more comprehensive knowledge than that. *Procedural* fluency includes three components: (1) efficiency, (2) flexibility, and (3) accuracy (see Figure 2.4). Both efficiency and flexibility require appropriate strategy selection. Additionally, connected with procedural fluency is knowing ways to check for reasonableness, including estimating at the beginning of the problem and looking back to see whether the answer is close to that estimate (addressed in Part II of this book).

To illustrate fluency, we will explore two topics: addition with whole numbers and subtraction with decimals.

Adding Two-Digit Numbers. Solve the problem in a way that makes sense to you.

$$37 + 28 = ?$$

Figure 2.5 illustrates four strategies that lead to a correct answer (accurate). Which of these strategies is efficient? A fluent student does not automatically stack the numbers and apply the standard algorithm (though they know how to do this); the fluent student looks at the problem and considers which strategy will be *efficient*, given the numbers in the problem. An efficient strategy is one that may take fewer steps than the alternatives. In this case, a student might move 2 from the 37 to the 28 to adapt the equation to $35 + 30$. This is called a

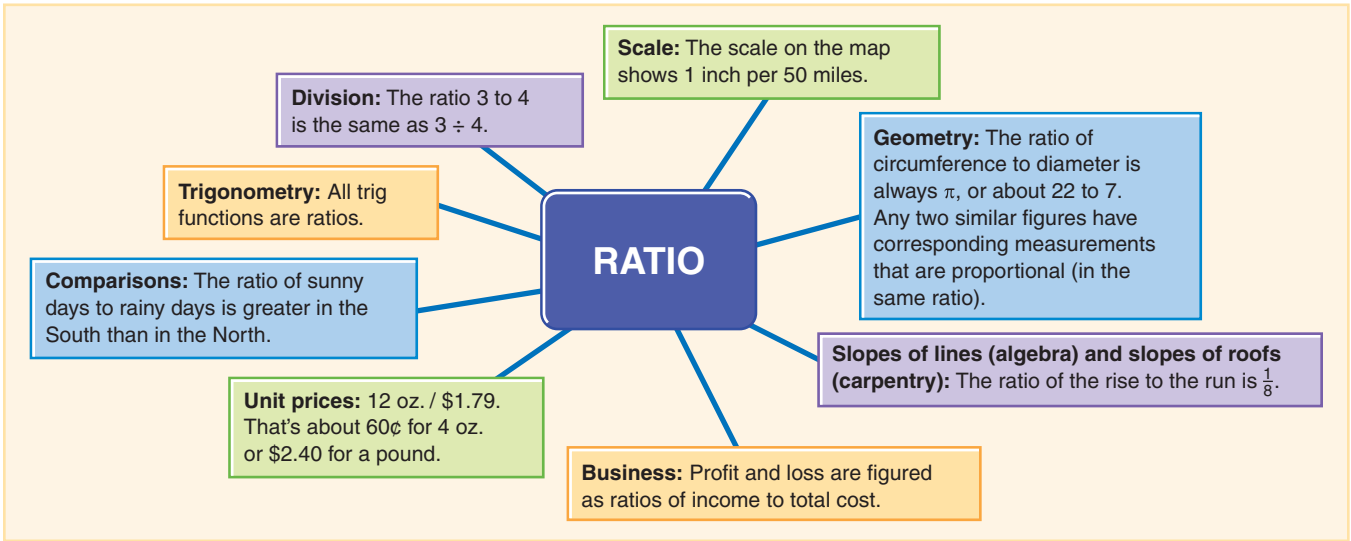

RATIO
Scale: The scale on the map shows 1 inch per 50 miles.
Division: The ratio 3 to 4 is the same as 3 ÷ 4.
Trigonometry: All trig functions are ratios.
Comparisons: The ratio of sunny days to rainy days is greater in the South than in the North.
Unit prices: 12 oz. / $1.79. That's about 60¢ for 4 oz. or $2.40 for a pound.
Business: Profit and loss are figured as ratios of income to total cost.
Geometry: The ratio of circumference to diameter is always π, or about 22 to 7. Any two similar figures have corresponding measurements that are proportional (in the same ratio).
Slopes of lines (algebra) and slopes of roofs (carpentry): The ratio of the rise to the run is $\frac{1}{8}$.

FIGURE 2.3 Connected ideas that contribute to a relational understanding of ratios

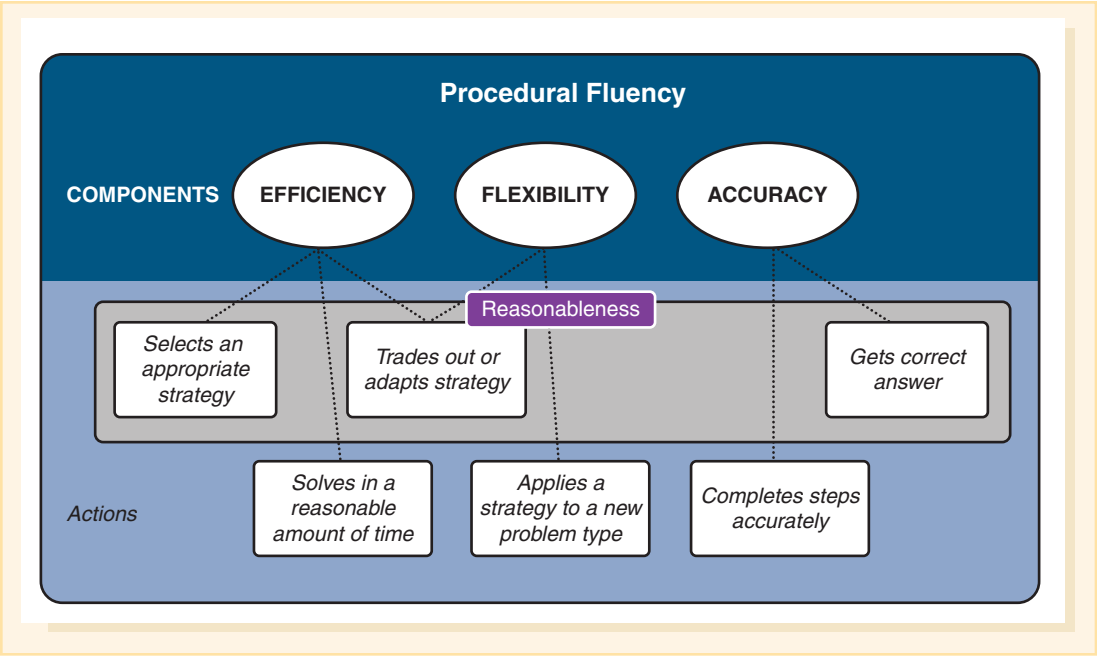


FIGURE 2.4 Procedural fluency components and actions

Source: Bay-Williams, J. M., & San Giovanni, J. J. (2021). *Figuring out fluency in mathematics teaching and learning, K-8.* Corwin. Used with permission.

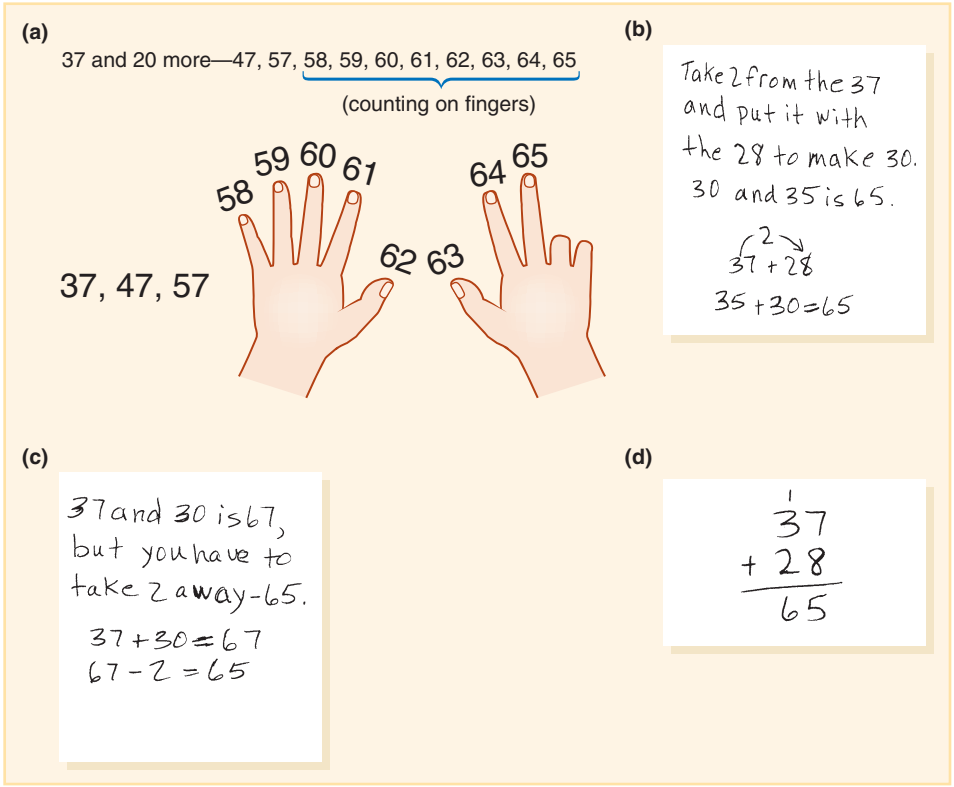


FIGURE 2.5 Different strategies for solving 37 + 28

Make Tens strategy (see Figure 2.5[b]). Or a student might add 2 to 28 to round to an easier number, add $37 + 30$ and then subtract 2 from the answer to counteract the addition. This is called a *Compensation strategy* (see Figure 2.5[c]). Either of these strategies may be done mentally. Which strategy or strategies are efficient? (Which strategy did you use?) Using Make Tens or Compensation uses fewer steps than the standard algorithm and can be done mentally, making them more efficient for this example.

Given different problems, a fluent student opts for different strategies, showing *flexibility*. Practice your fluency by selecting one of these strategies (Make Tens, Compensation, standard algorithm, or another strategy) to solve these four problems.

1. $57 + 98 =$ **2.** $49 + 48 =$ **3.** $62 + 24 =$ **4.** $37 + 46 =$

Make Tens or Compensation strategies are a good fit for the first two problems. The third problem involves no regrouping, so simply adding tens and adding ones is efficient. The final problem is not as good of a fit for Make Tens or Compensation, so a student might use a Partials Sums strategy, adding tens (70) and ones (13) and then combining (83). Or, they may use the standard algorithm (adding the digits in the ones first and then in the tens). How efficient and flexible were you in solving these four two-digit addition problems?

Subtracting Decimals. Solve it using an efficient method:

$$11.24 - 10.9$$

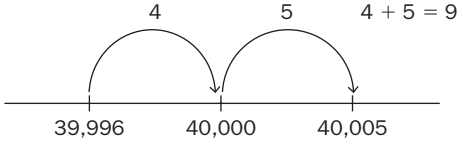
Too many people only know one way to solve this problem, which is to stack the numbers and regroup starting with the hundredths place. But there are more efficient options that require understanding what subtraction *means*. One meaning is "take away." For this equation, a take-away situation might be having \$11.24 and spending \$10.90. Subtraction also means to compare or find the difference. A compare situation might be asking the difference between times for running the 100-meter race for two runners.

Either interpretation (take-away or compare) can be used to find the difference. Here are three fourth-grade explanations:

Elisa: I started at 10.9, counted up 0.1 to 11, plus 0.24 more, so the answer is 0.34.
Monique: I added one tenth to both numbers. That makes 11.34 – 11. Then it was easy: 0.34.
Aaron: I subtracted 11 to get 0.24, then added the 0.1 back on to get 0.34.

Do any of these strategies match how you thought about it? Can you see which ones used take-away thinking and which ones used compare thinking? How do these strategies compare to the standard algorithm in terms of efficiency?

Think about the following problem and what you just read about subtraction: $40{,}005 - 39{,}996 =$ _____. Hopefully, you see how efficient a counting-up strategy is here.



In this example, the standard algorithm for subtraction involves regrouping across zeros, a sometimes tedious, inefficient, and prone-to-error method.

Procedural fluency supports conceptual understanding (and vice versa), and both contribute to student development of procedural flexibility (Schneider, Rittle-Johnson, & Star, 2011). Sadly, procedural fluency is often mistaken for learning standard algorithms and being able to do them quickly and accurately. An overemphasis on teaching standard algorithms—especially too early—can actually interfere with the development of fluency.

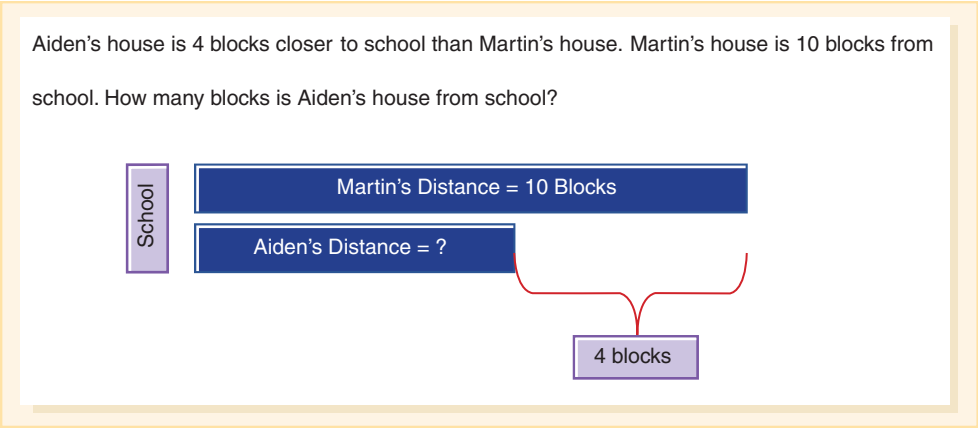

Aiden's house is 4 blocks closer to school than Martin's house. Martin's house is 10 blocks from school. How many blocks is Aiden's house from school?
School
Martin's Distance = 10 Blocks
Aiden's Distance = ?
4 blocks

FIGURE 2.6 A story problem translated into a bar diagram

Strategic Competence

This strand of proficiency is what is often called problem solving. *Strategic competence* includes reading a story problem or encountering a mathematical situation and figuring out what the problem is as well as determining a method for solving it. As part of strategic competence, students are able to create useful representations to help them reason about the problem. These depictions might include a number line, as illustrated earlier, or a bar diagram to make sense of a story problem, as illustrated in Figure 2.6.

Strategic competence also means a student can solve both routine and nonroutine problems. A routine problem is one that the student knows how to solve based on previous experiences, for example, the two-digit addition problems described earlier. A student with strategic competence can choose the best strategies for each problem. Nonroutine problems are problems the learner does not immediately know how to solve. Examples are provided in the next section (e.g., Two Machines, One Job). Students with strategic competence consider the situation, possible representations, and different approaches as they work to solve nonroutine problems. Like With procedural fluency, flexibility is the key to strategic competence. Some strategies fit one situation but not another. Strategic competence is knowing what features in a problem make it a good fit for a particular strategy. You can support students' strategic competence and procedural fluency by asking these questions:

- Why does this strategy work?
- Is this an efficient strategy?
- When does this strategy work?

Adaptive Reasoning

As described in Figure 2.1, *adaptive reasoning* is the capacity to think logically about the relationships among concepts and situations. It includes careful consideration at the beginning of solving a problem of a variety of solution approaches, the readiness to switch out for a new strategy in the middle of working a problem if one isn't fruitful, and being able to justify the solution at the end of solving a problem. When a teacher confirms that a student's answer is correct or not, they can undermine the student's development of adaptive reasoning. As described in *Adding It Up* (NRC, 2001), "students who disagree about a mathematical answer need not rely on checking with the teacher… they need only check that their reasoning is valid" (p. 129). Teachers support adaptive reasoning by asking students to justify their strategies and/or their answers. Doing so supports each student's understanding of mathematics and develops productive dispositions and self-regulation.

Productive Disposition

A productive disposition is a favorable attitude about being able to do mathematics. A student with a productive disposition tries to make sense of mathematics and believes that putting effort into learning mathematics is what is needed to be successful. This outlook is called a growth mindset (Boaler, 2015; Dweck, 2016). A person with a productive disposition sees themselves as an effective learner and doer of mathematics (NRC, 2001). They have the confidence to solve complex problems. In essence, they have a positive mathematics identity. A student's identity is a deeply held belief about themselves (Aguirre, Mayfield-Ingram, & Martin, 2013).

Although we addressed this myth in Chapter 1, it bears repeating: Far too many people think that mathematical ability is inherited and thereby limited to only a few people (Grady, 2016). This is simply not true, and such a fixed mindset leads to math anxiety (Sparks, 2020). Sadly, numerous studies show that anxiety specific to math (i.e., not to other areas such as reading) begins as early as first grade, and this anxiety negatively affects performance on achievement tests, on student learning of math skills and concepts, and on problem solving (Ramirez, Shaw, & Maloney, 2018). The truth is that there are not math people and non-math people. There are people who persevere and solve challenging tasks and those who quit. Telling yourself, your colleagues, your parents, and your students that effort matters more so than ability can help to turn their math anxiety into a productive disposition. Because students (and other stakeholders) may have a fixed mindset or experience math anxiety, this fifth strand of mathematical proficiency must receive more attention!

A productive disposition emerges when the other strands of mathematical proficiency are developed, and conversely, it supports the development of these other practices.

For which of the following questions might a student with a productive disposition typically say "yes"?

- When you read a problem you don't know how to solve, do you think, "Great, something challenging. I can solve this"?
- Do you consider several possible approaches before diving in to solve a problem?
- As you work, do you draw a picture or use a manipulative?
- Do you recognize a wrong path and try something else?
- When you finish a problem, do you wonder whether it is right? If there are other answers?

A "yes" response is evidence of mathematical proficiency and connects to at least one of the mathematical practices. When teachers are intentional and explicit about developing the mathematical practices, students' participation and confidence improve—they develop a productive disposition (Wilburne, Wildmann, Morret, & Stipanovic, 2014).

An Invitation to Do Mathematics

Learning Outcome 2.2 **Illustrate what it means to do mathematics.**

The purpose of this section is to provide *you* with opportunities to engage in doing mathematics so you know better how to engage your students. Three problems (tasks) are posed. For each, stop and solve. Then read the "Explore" section. Continue to engage with the task. As you do, reflect on your mathematical proficiency and your use of Mathematical Practices (or processes).

Task 1. One Up, One Down

This task has two parts, addition and multiplication. Either task can be used with students, or you can do one and then the other.

Task 1a One Up, One Down: Addition

When you add $7 + 7$, you get 14. When you make the first number 1 more and the second number 1 less, you get the same answer:

$$\begin{array}{ccccc} & \uparrow & & \downarrow & \\ 7 & + & 7 & = & 14 \\ 8 & + & 6 & = & 14 \end{array}$$

For what other numbers, does this one-up, one-down pattern work?

What else do you notice about this pattern?

Why is this pattern true? Explain and illustrate.

Pearson eText Activity Page: One Up, One Down: Addition

Task 1b One Up, One Down: Multiplication

When you multiply 7×7, you get 49. When you make the first number 1 more and the second number 1 less, you get an answer that is one less:

$$\begin{array}{ccccc} & \uparrow & & \downarrow & \\ 7 & \times & 7 & = & 49 \\ 8 & \times & 6 & = & 48 \end{array}$$

For what other numbers does this one-up, one-down pattern work?

What else do you notice about this pattern?

Why is this pattern true? Explain and illustrate.

Pearson eText Activity Page: One Up, One Down: Multiplication

Explore. If you explored both of these operations, you may have noticed that there are many more patterns or generalizations in the addition situation than in the multiplication situation. Consider:

- What manipulative or picture might illustrate the patterns?
- How is the pattern altered if the sums/products begin as two consecutive numbers (e.g., 8×7)? If they differed by 2 or by 3?
- What if you instead go two up, two down (e.g., $7 + 7$ to $9 + 5$ OR 7×7 to 9×5)?
- In what ways is the addition situation similar to and different from the multiplication situation?

Strategies. Let's look at the multiplication pattern, using illustrations. Draw a rectangle (or array) with dimensions of the original problem, 7×7 (see Figure 2.7[a]), and then draw the new rectangle (an 8-by-6) and compare the two rectangles. A second option is to represent the situation as equal groups, using stacks of chips (see Figure 2.7[b]). How might you move the counters to represent the one up, one down problem of 8×6? See how the stacks for each expression compare.

Have you made some mathematical connections and conjectures in exploring this problem? In doing so, you have hopefully felt a sense of accomplishment and excitement. Such a feeling strengthens a person's mathematics identity and productive disposition, which is why we must engage students in doing mathematical explorations like this one!

Task 2. Solving the Mystery

This task engages students in looking for a pattern. For middle school students, they can also explore this task by using variables.

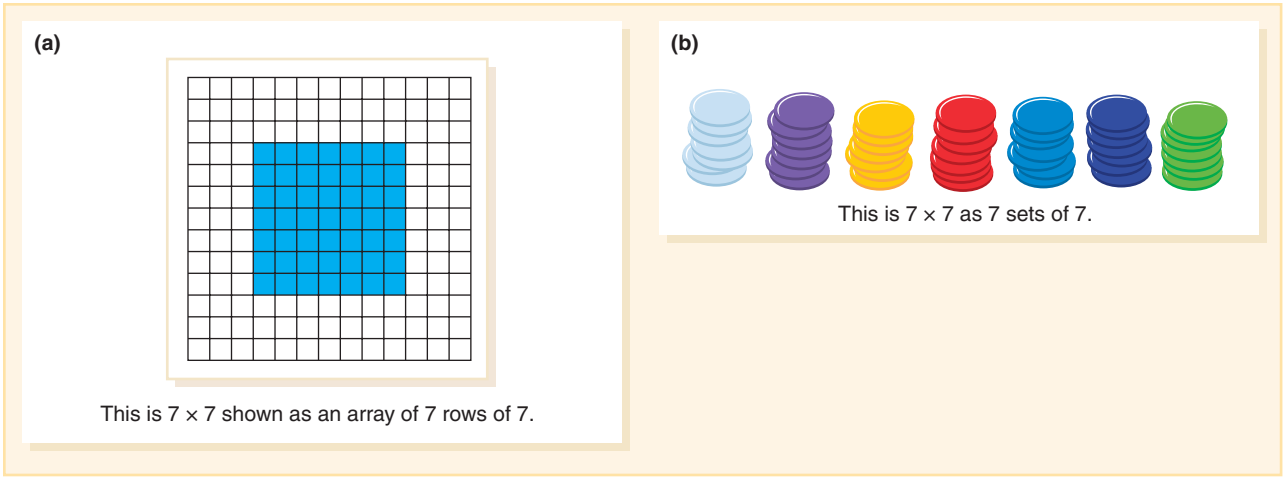


FIGURE 2.7 Two ways to illustrate One Up, One Down: Multiplication

Task 2 Solve the Mystery

Follow these five steps:

1. Write down any whole number.
2. Add to it the number that comes after it in counting.
3. Add 9.
4. Divide by 2.
5. Subtract your original number.

Did you get 5? Wow! How did we know? Can you solve the mystery of why this works?

Pearson eText Activity Page: Solving the Mystery

Explore. Here are ways to explore this mathematical mystery:

- Try different starting numbers and look for a pattern.
- Try to explore with a variable for your number (like n).
- Work backward to see what insights you might gain.
- Explain why this series of steps always leads to the answer of 5.

Extend. This four-step mystery could be simplified or made more complex (or students can write their own). The result can be the start number, or a set number (like 5). Try one of these:

- Create a three-step mystery that ends with the starting number. A simple example: Add 10, subtract 7, subtract 3. Be clever!
- Create a mystery puzzle that results in the answer of 0.
- Create a mystery puzzle that uses all four operations.

Task 3. Two Machines, One Job

This task may seem like it is a complicated ratios or proportion problem, but it is really about reasoning.

Task 3 Two Machines, One Job

Sophia's Paper Shredding Shop started with one paper-shredding machine. Business was good, so Sophia bought a new shredding machine. The old machine could shred a truckload of paper in 4 hours. The new machine could shred the same truckload in only 2 hours. How long will it take to shred one truckload of paper if Sophia runs both shredders at the same time?

Pearson eText Activity Page: Two Machines, One Job

Explore. This task may take a while to explore. As you practice perseverance, here are some things to try:

- Estimate how much time you think it should take the two machines. Estimating can sometimes lead to a new insight.
- Use a manipulative (chips, plastic cubes, pennies) to represent the size of the truckload.
- Draw a picture. For example, could you draw a rectangle or a line segment to represent the truckload? The time that was used or needed?

Strategies. Hopefully, you have solved this problem in some way that makes sense to you. Understanding other people's strategies can deepen your own understanding. And, teachers are always in a position where they must try to figure out how their students are thinking about a problem.

Figure 2.8 illustrates one method for solving the problem (based on Schifter & Fosnot, 1993). The whole paper strip represents the whole truckload. The new machine completes $\frac{1}{2}$ of the truckload in one hour (see purple shading). The old machine completes $\frac{1}{4}$ of the paper in one hour (green). Together, the machines shred $\frac{3}{4}$ of the truckload in one hour. So, the remaining one-fourth will take 20 minutes.

Another solution is to reason abstractly rather than to use visuals.

4 hours: 3 Truckloads (2 truckloads from new machine + 1 truckload from old machine)
If three truckloads take 4 hours, divide by 3 to figure out how much time for 1 truckload:
$\frac{4}{3}$ hours = $1\frac{1}{3}$ hours = 1 hour and 20 minutes

Reflecting on the Three Tasks

Now that you have solved these tasks, to what extent do you feel you demonstrated mathematical proficiency? In other words, did you notice your knowledge of concepts and procedures working together to solve the problems? Did you try a strategy and trade it out for another strategy along the way, demonstrating strategic competence? Did you adapt strategies and justify why you selected them and/or why they work? And, importantly, did you enter the problem solving with a productive disposition, thinking that with enough effort you could figure it out? More specifically, to what extent did you engage in the mathematical practices?

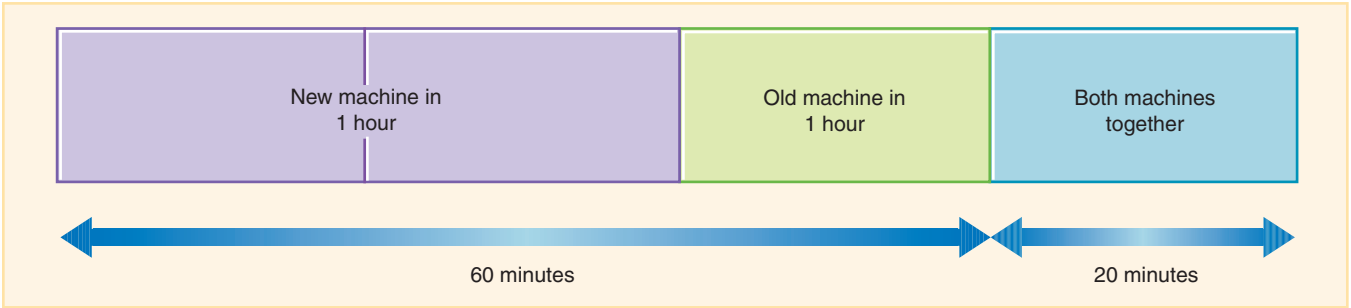


FIGURE 2.8 One illustration of Two Machines, One Job task

For example, to what extent did you make sense of the problem and persevere (MP 1)? Use a manipulative strategically (MP 5) or draw a mathematical representation (MP 4)? Look for structure or a pattern (MP 7 or 8)? Reason about the quantities in the problems (MP 2)? These practices (actions) are what it means to *do mathematics*!

If your answers are yes, you demonstrated mathematical proficiency. If your answers are no, reflect on why this might be the case. You may have learned mathematics by replicating procedures shown to you by your teachers, with the expectation that you should show all your steps. As a result, you may have felt uncomfortable and maybe even incapable of solving these problems. When we learn mathematics in a way that we are just asked to practice skills, rather than to figure out problems, it is more difficult to develop a positive, productive mathematics disposition. As a teacher, developing a productive disposition for yourself and for your students must be a priority! Showing students how to do an algorithm does not prepare them to be mathematically proficient. Imagine how much more engaging and meaningful math class can be when the focus is on thinking! And, in developing students' mathematical proficiency, you are also developing positive mathematics identity!

How Do Students Learn Mathematics?

Learning Outcome 2.3 **Connect learning theories to mathematical proficiency and to mathematics teaching.**

Given that the goal for every student is mathematical proficiency, you may be wondering how to ensure that they learn such things as strategic competence and reasoning. The answers to how you will carry this out are grounded in learning theory.

In this section, we briefly describe two compatible major learning theories and provide mathematics-specific teaching ideas that stem from them (Norton & D'Ambrosio, 2008).

Constructivism

Constructivism is rooted in Jean Piaget's work, developed in the 1930s and translated to English in the 1950s. At the heart of constructivism is the notion that learners are not blank slates but rather creators or constructors of their own learning (Fosnot, 2005). Integrated *networks*, or *cognitive schemas*, are formed by constructing knowledge, and they are used to build new knowledge. Through *reflective thought*—the effort to connect existing ideas to new information—people modify their existing schemas to incorporate new ideas (Fosnot, 2005). All people construct or give meaning to things they perceive or think about. As you read these words, you are giving meaning to them. Whether listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain is applying prior knowledge from your existing schemas to make sense of the new information.

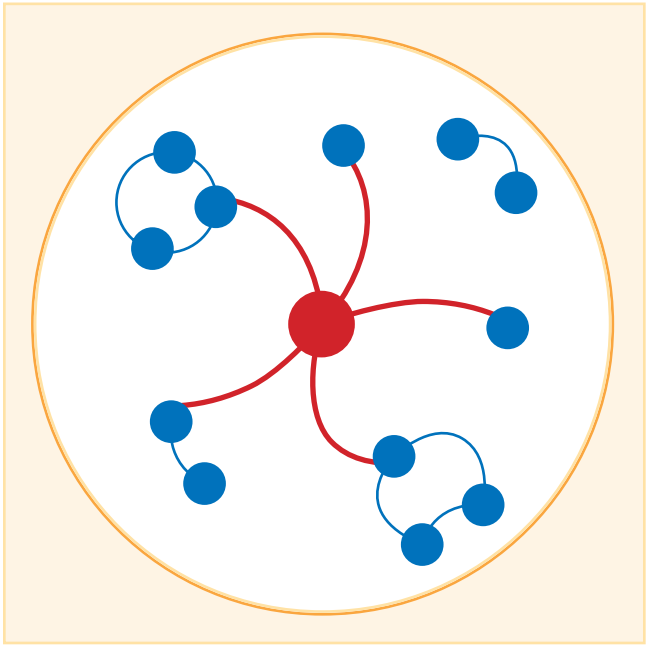


FIGURE 2.9 Learning connects prior knowledge (blue dots) to new knowledge (red dot)

To connect to the metaphor of building construction, the *tools* we use to build understanding are our existing ideas and knowledge. The *materials* we use are things we see, hear, or touch, as well as our own thoughts and ideas. In Figure 2.9, blue and red dots are used as symbols for ideas. The blue dots represent existing ideas and prior knowledge. The lines joining the ideas represent our logical connections or relationships that have developed between and among ideas. The red dot is an emerging idea, one that is being constructed. When existing ideas are used to build new knowledge, those connections remain; when new knowledge is learned in isolation, connections are not formed.

Sociocultural Theory

About 100 years ago, Lev Vygotsky, a Russian psychologist, began developing what is now called sociocultural theory. Fifty years later, it began to gain attention as a way to explain learning. Like constructivism, this theory assumes active meaning-seeking on the part of the learner. Key to this theory is the zone of proximal development (ZPD) (Vygotsky, 1978). Simply put, the ZPD refers to a range of knowledge that is out of reach for a person to learn on his or her own, but is accessible if the learner has support from more knowledgeable others. If a task is too easy, an individual can do it on their own, but they won't learn from it. If it is out of reach, they won't learn from it. An experience must be in the zone for learning to occur. That means for learning to occur, three things must be present: (1) guidance from someone with advanced knowledge, (2) social interaction, and (3) tasks that position the learner in their ZPD (Cherry, 2020). Social interaction is essential for learning! Additionally, the learner is influenced by both the classroom community and the broader social and historical culture of the members of the classroom (Forman, 2003).

Implications for Teaching Mathematics

Learning theories are not teaching strategies—theory *informs* teaching. This section outlines teaching strategies that are informed by constructivist and sociocultural perspectives. You will see these strategies revisited in more detail in Chapters 3 and 4, where a problem-based model for instruction is discussed, and in Part II of this book, where you learn how to apply these ideas to specific mathematics topics.

Use Manipulatives and Other Tools. *Manipulatives* are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics (e.g., connecting cubes) or for other purposes (e.g., buttons). A *tool* is any object, picture, or drawing that can be used to explore a concept. Choices for manipulatives (including virtual manipulatives) abound—from common objects such as lima beans to commercially produced materials such as pattern blocks. Manipulatives help students explore mathematical relationships concretely.

A tool, however, does not illustrate a concept. The tool is used to visualize a mathematical concept; only your mind can impose the mathematical relationship on the object (Suh, 2007a). Consider each of the concepts and the corresponding manipulative model in Figure 2.10. Try to separate the tool from the mathematical concept.

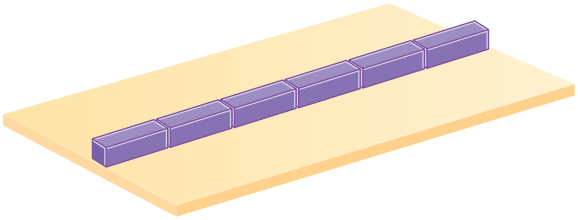
a. Beyond being part of a counting sequence, 6 represents a quantity. Quantities are relational. Adding 1 counter to a set of 6 counters means there is now *one more*: 7 is *one more than* 6.
b. The concept of "measure of length" is a comparison. The length measure of an object is a comparison relationship of the length of the object to the length of the unit.
c. The concept of "rectangle" includes both spatial and length relationships. The opposite sides are of equal length and parallel, and the adjacent sides meet at right angles.
d. The concept of "hundred" is not in the larger square but in the relationship of that square to the strip ("ten") and to the little square ("one").
e. "Chance" is a relationship between the frequency of an event and all possible outcomes. Spinners can represent relative frequencies with shading (e.g., 25% chance of yellow).
f. Location is based on direction and magnitude. Arrows on the number line can show direction (pointing) and magnitude (length).

Although tools can be used to support learning, they can be ineffective. One such misstep is when a teacher shows the students exactly what to do with the manipulative, and has students imitate that process. It is just as possible to move blocks mindlessly as it is to go through the steps of an algorithm mindlessly. On the other extreme, giving out manipulatives with no purpose for their use can lead to nonproductive and unsystematic investigation (Stein & Bovalino, 2001). The goal is to set up tasks, using the tools as ways to explore the concept so that students notice important mathematical relationships that can be discussed, connecting the concrete representations to abstract concepts.

(a)

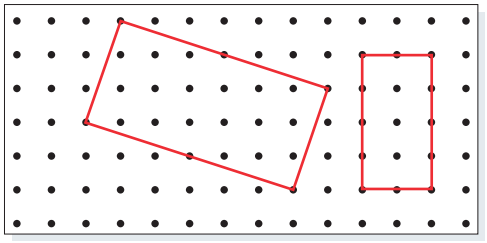
Countable objects can be used to model "number" and related ideas such as "one more than."

(b)



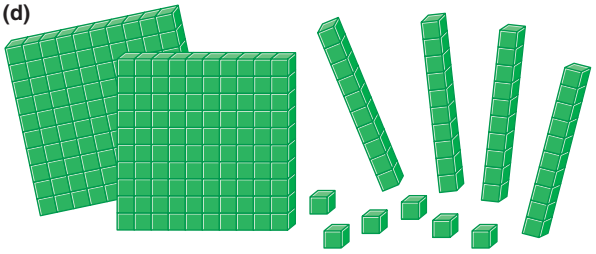
"Length" involves a comparison of the length attribute of different objects. Rods can be used to measure length.

(c)



"Rectangles" can be modeled on a dot grid. They involve length and spatial relrelationships.

(d)

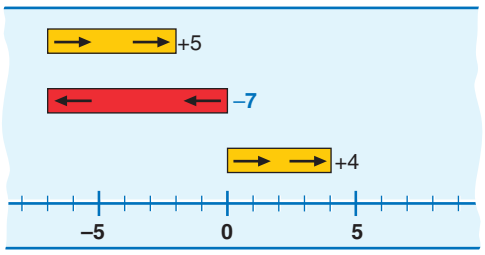


Base-ten concepts (ones, tens, hundreds) are frequently modeled with base 10 blocks. Sticks and bundles of sticks are also commonly used.

(e)

"Chance" can be modeled by comparing outcomes of a spinner.

(f)



"Positive" and "negative" integers can be modeled with arrows with different lengths and directions.

FIGURE 2.10 Examples of tools to illustrate mathematics concepts

Make Connections Explicit. Purposefully having students connect ideas is not only grounded in learning theory, it is well-established through research to improve students' conceptual understanding (Hiebert & Grouws, 2007). The teacher's role in making mathematical relationships explicit is to be sure that students are making the connections that are implied in a task. For example, asking students to relate today's topic to one they investigated last week, or by asking "How is Laila's strategy like Marco's strategy?" are ways to be help students be explicit about mathematical relationships.

Use Multiple Representations. Students need to connect ideas within a concept. That includes making connections across different representations. Figure 2.11 illustrates a web of representations showing ways to demonstrate understanding. Students who have difficulty translating a concept from one representation to another also have difficulty solving

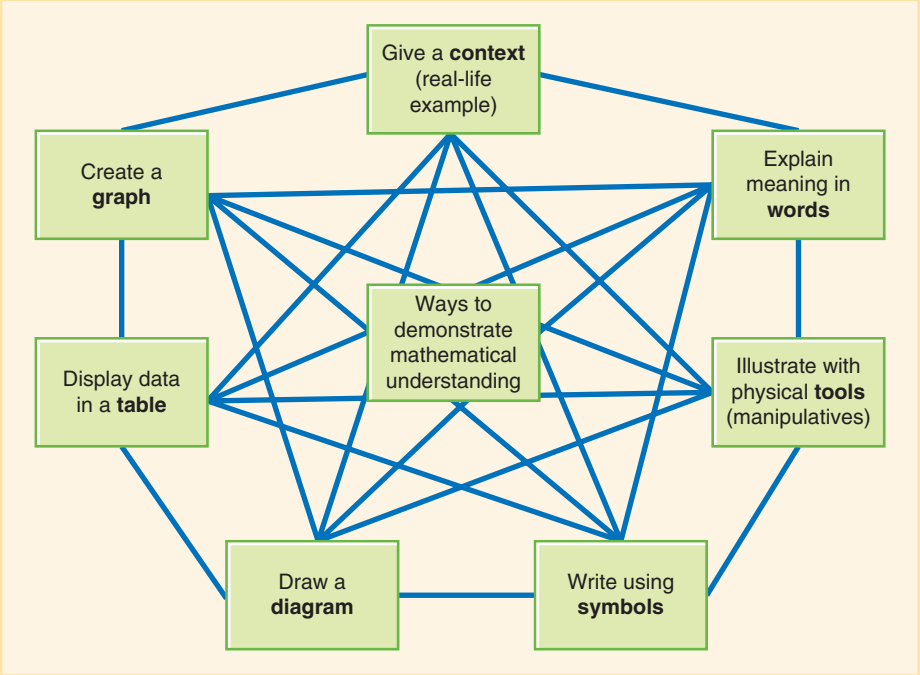

Give a **context** (real-life example)
Create a **graph**
Explain meaning in **words**
Ways to demonstrate mathematical understanding
Display data in a **table**
Illustrate with physical **tools** (manipulatives)
Draw a **diagram**
Write using **symbols**

FIGURE 2.11 Web of representations to demonstrate understanding

problems and understanding computations (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Therefore, strengthening students' ability to move between and among these multiple representations improves their understanding.

Earlier, the topic of ratios was explored to highlight conceptual (relational) knowledge (Figure 2.3). In Figure 2.12, the same topic is used to illustrate a web of representations. Using these visuals and contexts helps learners develop relational understanding.

Encourage Multiple Strategies. Encourage students to use strategies that make sense to them, building on their prior knowledge. Even learning a basic fact like 7×8 can have better results if a teacher promotes multiple strategies, for example asking, "Besides skip counting, how can you figure out 7×8 if you don't know the answer?" Student explanations include:

> **Jair: I know 5×8 equals 40, and 2×8 equals 16, so 7 groups of 8 is 56.**
> **Lina: I used the square of 7×7, which equals 49, and added one more 7 more to get 56.**
> **Cody: I know 4×7 equals 28, so I doubled 28 to get eight groups of 7, which was 56.**

Encouraging different strategies develops strategic competence, fosters a productive disposition, and develops procedural fluency.

Provide Opportunities to Communicate and Reflect. For a new idea to be interconnected in a rich web of interrelated ideas, children must reflect on the relevant ideas they possess (blue dots) and bring them to bear on the development of the new idea (red dot). In discussions with peers, students adapt and expand on their existing networks of concepts as they look at the representations and listen to the thinking of others. Pressing for students to explain further and justify their thinking increases student mathematical reasoning (i.e., use of the Mathematical Practices) (Sussman, Hammerman, Higgins, & Hochberg, 2019).

Engage Students in Productive Struggle. Have you ever just wanted to think through something yourself without being interrupted or told how to do it? Too often, teachers step in to help too soon or give too much guidance. In reality, these moves do not help the student learn. Engaging in productive struggle helps students learn mathematics (Hiebert & Grouws, 2007).

Pearson eText
Video Example 2.1
In this video consider how this class explores a problem-solving task and their discussion over two possible solutions to the problem.

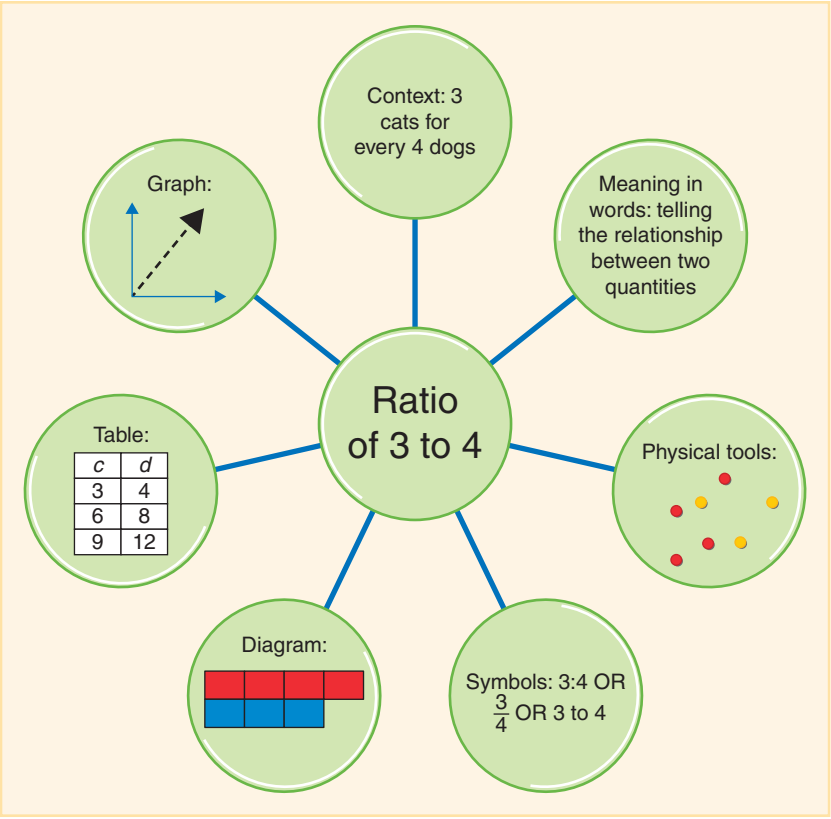


FIGURE 2.12 Multiple representations for ratio of 3 to 4

Pearson eText
Video Example 2.2
In this video consider how the class engages in a patterning problem where they attempt to guess the rule given by the teacher.

Productive struggle includes such actions as drawing a representation or seeking out a manipulative, asking a question of a peer or teacher, and persevering through a task by trying a second or third approach if needed (NCTM, 2014b; SanGiovanni, Katt, & Dykema, 2020). Productive struggle is *not* waiting to be shown how to do something, feeling despair because a problem doesn't initially make sense, or becoming frustrated with an inaccessible task (Baker, Jessup, Jacobs, Empson, & Case, 2020). When students are in productive struggle, they are in their ZPD. They are building conceptual connections. That is why they learn better than if they were just shown how to do something.

Treat Errors as Opportunities for Learning. When students make errors, it can mean a misapplication of their prior knowledge in a new situation. Students rarely give random responses, so their errors are an insight into limited understandings or misconceptions they might have. For example, students comparing decimals may incorrectly apply rules of whole numbers, thinking that longer means larger (Martinie, 2014). Often, one student's misconception is shared by others in the class, and discussing an error publicly can help other students understand the correct approach (Hoffman, Breyfogle, & Dressler, 2009; Rathouz, 2011a). This level of transparency about using errors to learn also communicates that doing mathematics involves making mistakes; it is just part of the discipline.

Scaffold Content. The practice of *scaffolding* is based on the idea that a task otherwise outside of a student's ZPD can become accessible if it is carefully structured. When students do not have skills they need to be successful with a grade-level topic, you can plan several days of instruction that start with what students know and sequence tasks each day to explore a culminating task that does meet the grade-level standard (Coleman, 2020). For example, if you are teaching multiplication of a whole number times a fraction (e.g., $5 \times 3\frac{1}{2}$), scaffolding might begin with prior knowledge: (1) exploring and explaining the meaning of whole-number multiplication (5×3), perhaps connecting to story situations, and (2) constructing

representations to show 5 groups of 3. Then scaffolding can build meaning, starting with concrete and semi-concrete representations and moving toward abstract understanding (CSA). That would mean exploring stories and using manipulatives and drawings to represent $5 \times 3\frac{1}{2}$. For concepts completely new to students, the learning may require more structure or assistance, including the use of tools (e.g., manipulatives, sketches) or more support from peers or knowledgeable others. As students become more comfortable with the content, the scaffolds are reduced and the student becomes more independent. Scaffolding can also be used to help students develop strategic competence by initially offering students simpler problems or removing the question prompt so that students focus on making sense of the situation first (Barlow, Duncan, Lischka, Hartland, & Willingham, 2017).

Honor Diversity. Finally, and importantly, these theories emphasize that each learner is unique, with a different collection of prior knowledge and cultural experiences. Because new knowledge is built on existing knowledge and experience, effective teaching incorporates and uses what the students bring to the classroom, honoring those earlier experiences. Thus, lesson contexts should be selected based on students' interests, knowledge, background, strengths, and experiences. (See also the discussion of culturally responsive mathematics instruction in Chapter 6.)

Connecting the Dots

It seems appropriate to close this chapter by once again connecting some dots, especially because the ideas represented here are the foundation for the approach to each topic in the content chapters found in Part II. This chapter began with discussing what is important to know about mathematics—the five strands of mathematical proficiency. Second, you explored what *doing* mathematics feels and looks like as you solved and reflected on three tasks. Each of these tasks offered opportunities to make connections between mathematics concepts—connecting the blue dots to new ideas represented as red dots. Finally, you read how learning theories—the importance of having opportunities to connect the dots—connects to mathematics learning. The best learning opportunities, according to constructivism and sociocultural theories, are those that engage learners in productive struggle, making connections, using representations, communicating ideas, and reflecting on their use of strategies, to name a few. Your teaching must focus on purposefully creating these types of opportunities for students to develop their own networks of blue dots and logically connect new red dots to this existing web of ideas. As you carefully plan and design instruction that challenges students to think critically and creatively, you nurture positive mathematics identities and support the development of mathematical proficiency!

RESOURCES FOR CHAPTER 2

WRITING TO LEARN

1. An elevator talk is an explanation that can be offered between an elevator door closing and reopening, usually to a complete stranger. In other words, simple, meaningful, and to the point. What might be your elevator talk for answering, "What are (a) mathematical proficiency and (b) Mathematical Practices or processes and/or (c) doing mathematics?"
2. How can you support students' relational understanding (and how can you avoid their development of only an instrumental understanding)?
3. Describe how to implement the One Up, One Down task with students in a way that reflects constructivist and/or sociocultural learning theory. Link specifically to aspects of the theory.
4. Select a major concept students learn in K–8. For this concept, select at least three of the Implications for Teaching Mathematics and describe how those implications support students in constructing meaning of that concept.

RECOMMENDED READINGS

Articles

Carter, S. (2008). Disequilibrium & questioning in the primary classroom: Establishing routines that help students learn. *Teaching Children Mathematics*, *15*(3), 134–137.

This is a wonderful teacher's story of how she infused the constructivist notion of disequilibrium and the related idea of productive struggle to support learning in her first-grade class.

Whitacre, I., Schoen, R. C., Champagne, Z., & Goddard, A. (2016). Relational thinking: What's the difference? *Teaching Children Mathematics*, *23*(5), 303–309.

These authors illustrate the importance of relational thinking by sharing the results of students' work on subtraction problems. They offer helpful strategies for developing students' procedural flexibility.

Baker, K., Jessup, N. A., Jacobs, V. R., Empson, S. B., & Case, J. (2020). Productive struggle in action. *Mathematics Teacher: Learning & Teaching PK-12*, *113*(5), 361–367.

Through sharing Ryan's experience in solving a problem, these authors show what engaging students in productive struggle looks like. They also offer an excellent list of takeaways.

Books

Kelemanik, G., Lucenta, A., & Creighton, S. J. (2016). *Routines for reasoning: Fostering the mathematical practices in all student*s. Portsmouth, NH: Heinemann.

These authors propose an organizational structure and excellent classroom teaching routines to ensure that your students become mathematically proficient.

Mohr, D., Walcott, C., & Kloosterman, P. (2019). *Mathematical thinking: From assessment items to challenging tasks*. Reston, VA: NCTM.

Using the National Assessment of Educational Progress (NAEP) items for inspiration, the authors share 36 excellent tasks along with full lesson plans. Suggestions for facilitating discussion and differentiating are also included. An excellent resource to get students doing mathematics!

CHAPTER 3

Teaching Problem-Based Mathematics

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **3.1** Describe specific practices for effective mathematics teaching.
- **3.2** Explain the roles of problem solving and learning progressions in setting goals for student learning.
- **3.3** Describe features of worthwhile tasks and methods for adapting tasks to make them better.
- **3.4** Illustrate the relationship between procedural fluency and conceptual understanding.
- **3.5** Describe and relate the Mathematics Teaching Practices related to facilitating mathematical discourse.

Imagine yourself in a mathematics classroom. What are students working on? What are they talking about? If that classroom embodies the ideas in Chapter 2 of doing mathematics, then you will see students working on a task carefully selected by the teacher that allows them to add to their prior knowledge, connect mathematical ideas, and learn important conceptual and procedural knowledge related to the topic. Effective teaching of mathematics is described in *Principles to Actions* (NCTM, 2014b) through eight Effective Mathematics Teaching Practices. These Teaching Practices support the development of the eight Mathematical Practices for students. These are different sets of practices (possibly confusing because there are eight on both lists and both lists use the word *practice*): The Mathematical Practices describe student actions; the Teaching Practices describe teacher actions. This chapter is about the Teaching Practices, providing lots of tasks, examples, and activities to illuminate what effective mathematics teaching looks like.

Effective Mathematics Teaching

Learning Outcome 3.1 **Describe specific practices for effective mathematics teaching.**

Classrooms where students are learning through doing mathematics do not happen by accident—they happen because the teacher implements specific Teaching Practices and establishes expectations that encourage risk taking, reasoning, the generation and sharing of ideas, and so forth. Teaching this way is a paradigm shift from traditional mathematics teaching

where students repeat what the teacher demonstrates. At first glance, it may seem that the teacher's role is less demanding because the students are doing the mathematics, but the teacher's role is actually more demanding in such classrooms. Why work harder? So that each and every student develops competence and confidence in their ability to do mathematics—and that outcome is worth your effort.

Success for Every Student

The NCTM Effective Mathematics Teaching Practices were designed to ensure that every child has access to high-quality mathematics learning experiences: "[E]ffective teaching is the nonnegotiable core that ensures that all students learn mathematics at high levels" (NCTM, 2014b, p. 4). Inclusive teaching, teaching that challenges and supports every child, engages students in *doing* mathematics (not watching the teacher do the mathematics). Teaching through engaging students in problem solving provides opportunities for every student to become mathematically proficient, because this approach:

- *Develops Mathematical Practices and processes*. By definition, teaching through problem solving positions students to be the doers, and as they are doing the mathematics, they are developing the eight Mathematical Practices (see Table 1.1 and Appendix A).
- *Focuses students' attention on ideas and sense making*. When solving problems, students are reflecting on the concepts inherent in the problems and making connections (e.g., between concepts and procedures).
- *Builds on students' strengths*. Students can choose and apply strategies that they understand. For example, students may solve 42 – 26 by applying various mental strategies, using a manipulative or a hundred chart, or by applying an algorithm.
- *Invites creativity*. Students enjoy the creative process of problem solving, searching for patterns, and showing how they figured something out.
- *Allows for extensions and elaborations*. Extensions and what-if questions can motivate and challenge all students as well as provide enrichment for advanced learners or quick finishers. Students can even generate their own mathematical questions to explore.
- *Develops student confidence and identities*. As students learn through problem solving, they begin to identify themselves as doers of mathematics.
- *Engages students so that there are fewer discipline problems*. Many discipline issues are the result of boredom, not understanding an explanation, or finding little relevance. Students like to be challenged, have choice, and work with peers, thus giving them less reason to disengage.
- *Provides formative assessment data*. As students use manipulatives, draw pictures, and talk, they provide the teacher with a steady stream of valuable information about their mathematical thinking that can be used to support continued learning.

When students have confidence, show perseverance, and enjoy mathematics, it makes sense that they will achieve at a higher level and want to continue learning about mathematics, opening many doors to them in the future.

The Mathematics Teaching Practices

Before exploring effective teaching, let's reflect on a traditional mathematics lesson and how it falls short. Stereotypical mathematics lessons include the teacher showing students how to follow the steps of a procedure and then students practicing that procedure on a worksheet with 20 or so straight-forward exercises. With this image in mind, how does that teaching relate to the *Mathematics Teaching Framework* in Figure 3.1? This framework organizes the eight Mathematics Teaching Practices described in *Principles to Actions* (Huinker, 2018; NCTM, 2014b). Let's look at the three boxes at the top of the framework and see how that traditional lesson matches up. The lesson goals focus learning on memorizing a procedure. Not a very robust learning goal for students. The tasks do not promote reasoning or problem solving, and the procedures were not connected to concepts. So far, the lesson is falling short of effective mathematics teaching.

When the top three boxes (Teaching Practices) are not well developed, the rest of the Mathematics Teaching Practices (in the "facilitate meaningful mathematical discourse" box) are virtually impossible to enact. There is little opportunity to pose questions beyond, "What answer did you get?" and therefore, it is difficult to see evidence of student thinking. Meanwhile, the students were shown *how* to do the procedures, so no productive struggle will happen, except when students forget what the teacher said, which is not the meaning of productive struggle (and requires the teacher to repeat constantly). Furthermore, there were likely no representations to connect. This whole scenario leads us to a critical fact about effective mathematics teaching: These first three Teaching Practices must be effectively implemented to open up the chance for effective discourse and meaningful and useful learning.

Too often, mathematics teaching still follows this I-do, We-do, You-do pattern of the teacher showing one way to perform a skill and students practicing that skill first together and then independently, using the same procedure. Unfortunately, this approach to mathematics teaching has not been successful. Here are a few shortcomings of a teach-by-telling approach:

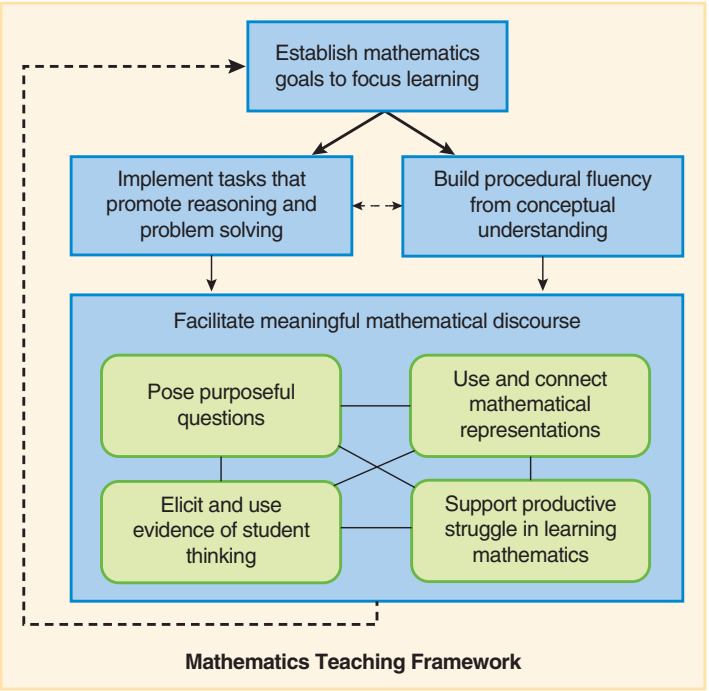

Establish mathematics goals to focus learning
Implement tasks that promote reasoning and problem solving
Build procedural fluency from conceptual understanding
Facilitate meaningful mathematical discourse
Pose purposeful questions
Use and connect mathematical representations
Elicit and use evidence of student thinking
Support productive struggle in learning mathematics
Mathematics Teaching Framework

FIGURE 3.1 Mathematics Teaching Framework (Huinker & Bill, 2017).

From D. Huinker & V. Bill. (2017). *Taking Action: Implementing Effective Mathematics Teaching Practices in Kindergarten–Grade 5*. Taking Action Series. NCTM.

- Students are not thinking (the teacher is doing the thinking for them) (Liljedahl, 2021).
- It communicates that there is only one way to solve the problem, misrepresenting the field of mathematics and disempowering students who naturally may want to make sense of mathematics.
- It positions the student as a passive learner, dependent on the teacher to present ideas, rather than as an independent thinker with the capability and responsibility for solving the problem.
- It assumes that all students have the necessary prior knowledge to understand the teacher's explanations—which is rarely the case.
- It decreases the likelihood of a student attempting to solve a nonroutine problem without instructions on how to solve it. Yet, doing mathematics means you can solve nonroutine problems.

We can do better for our students through effective mathematics teaching. That begins with attending to the eight Teaching Practices in Figure 3.1. This chapter begins with the top three and then focuses on the components related to facilitating meaningful discourse.

Establishing Goals to Focus Learning

Learning Outcome 3.2 **Explain the roles of problem solving and learning progressions in setting goals for student learning.**

A focus on goals for learning a mathematics lesson begins with thinking of goals that are useful in today's society. Skills needed in the 21st-century workplace are less about being able to compute and more about being able to design solutions to authentic problems, from distribution of food to conservation to the spread of disease. Important life skills include critical

thinking, communication, collaboration, creativity, and the ability to use technology effectively (Partnership for 21st Century Skills, n.d.). These life skills must be goals within mathematics teaching. We call this *problem-based* teaching.

Problem-Based Mathematics Teaching

Solving problems has always been a part of mathematics learning, but the way in which they are incorporated has varied based on our goals for students. Here we briefly review three approaches to problem solving (Schroeder & Lester, 1989). The distinctions among these ways are important because attending to only one of these approaches will not lead to your students becoming mathematically proficient.

Teaching for Problem Solving. Teaching *for* problem solving starts with learning the abstract concept and then moving to solving problems as a way to apply the learned skills (explain-practice-apply). For example, students learn the algorithm for adding fractions first and then solve story problems that involve adding fractions. This has been the traditional approach (practice skills first, and then solve story problems). The major shortcoming of this approach is that students learn very early in school that the stories they encounter will be solved using the skill they just learned. Therefore, there is no need to read and make sense of the story; they can just lift the numbers out and use the skill of that day's lesson. This habit results in students having difficulties with story problems, multistep problems, and high-level tasks. In other words, the pattern of explain-practice-apply works against preparing students to *do mathematics*. Solving application problems after a skill is learned *is* important. The key is to be sure that the application problem is complex enough that understanding the situation and using strategies are necessary to solving it.

Teaching Problem Solving. Students need guidance on how to problem solve. This includes (1) the process of problem solving and (2) general ways to solve problems (e.g., draw a picture). The mathematician George Pólya wrote a classic book, *How to Solve It* (1945), which outlined four steps for problem solving. These steps continue to be widely used today. For example, they are reflected in the first mathematical practice, "Making sense of problems and persevere in solving them." The four steps are summarized here:

1. **Understand the problem.** First, you must figure out what the problem is about and identify what question or problem is being posed.
2. **Devise a plan.** How might you solve the problem? Consider your options and make a choice: Will you want to write an equation? Will you want to model the problem with a manipulative?
3. **Carry out the plan.** Implement the plan you selected from step 2.
4. **Look back.** Does your answer make sense? If not, loop back to step 2 and select a different strategy to solve the problem, or loop back to step 3 to fix something within your strategy.

The beauty of Pólya's framework is its generalizability; it can and should be applied to many types of problems, from simple computational exercises to authentic and worthwhile multistep problems. Explicitly teaching these four steps to students can improve their ability to think mathematically.

Problem-solving strategies are identifiable methods for approaching a task. Students select or design a strategy as they devise a plan (Pólya's step 2). Table 3.1 provides a list of general problem-solving strategies in grades K–8, though not all of them are used at every grade level. These strategies become the tools by which students can enter into the process of solving unfamiliar and novel tasks. These strategies are not distinct, but interrelated. For example, creating a list is a way of looking for patterns. When students employ one of these strategies, it should be identified, highlighted, and discussed. Over time, these

TABLE 3.1 PROBLEM-SOLVING STRATEGIES IN K–8

Strategy	Brief Explanation	What It Might Look Like
Model the problem	The problem is interpreted in a concrete, visual way	• Uses manipulatives • Acts out the problem • Draws a picture or sketch (on paper or by using dynamic software)
Make an organized list	The problem is explored by recording possible solutions or examples in a logical way.	• Makes an organized list • Creates a table or chart
Predict and check for reasonableness	Possible realistic answers are tried out to see if they work (this is sometimes called the Guess & Check strategy).	• Tries out an answer (e.g., using a calculator or testing it in the equation) • Creates a list of guesses and adjustments (tries a value; if it's too low, tries a higher value, etc.)
Make a simpler problem	Student simplifies the quantities in a problem to make a situation easier to understand and analyze.	• Rereads the problem with no numbers in it or uses easy numbers in it • Writes and solves a similar problem that fits the situation, but with smaller values • Uses visuals to explore the simpler problem
Look for a pattern	Student looks across a series of values or problems to notice similarities to solve the problem.	• Solves simpler problems to see what happens • Looks across problems to see what is different and what is the same
Prove a conjecture	Student formulates a possible solution based on mathematical properties, for example, and justifies (proves) that it works.	• Makes a claim (e.g., the answer will be one more) • Uses manipulatives, visuals, or numbers to prove they are correct
Write equations	Student interprets the story numerically and/or symbolically	• Writes an expression or series of expressions • Writes an equation

problem-solving strategies will become habits of mind and help students become mathematically proficient.

Mathematical problem solving is grounded in curiosity. It is important not to proceduralize it. In other words, do not take the problem solving out of a problem-solving activity by telling students to "make a table to solve this problem." Instead, pose a problem that lends itself to different strategies and ask students to approach the problem in a way that makes the most sense to them. You can post the list of strategies from Table 3.1 (or one like it) to help students select a method.

The **Elbow Hello** problem (replacing the classic handshake problem) is an example of a nonroutine task that lends itself to many strategies:

Pearson eText
Video Example 3.1
In this video consider how the third-grade class engages in a problem-solving approach.

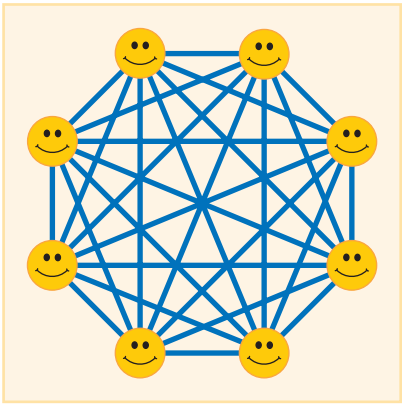
Task 1

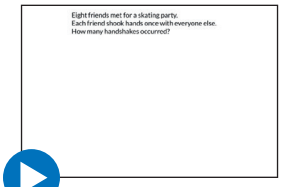
Eight friends met for a skating party. Each friend gave one elbow hello to everyone else at the party. How many elbow hellos occurred?

Pearson eText Activity Page: Elbow Hello Problem

Here are some problem-solving strategies for counting elbow bumps:

Model the Problem (could be a drawn picture or students actually acting it out):





Eight friends met for a skating party.
Each friend shook hands once with everyone else.
How many handshakes occurred?
7 hands
6 hands
5 hands

Pearson eText
Video Example 3.2 and 3.3
Each of these two videos illustrates one student's 'think aloud' for the Handshake (Elbow Hello) Problem. What do you notice about each student's thinking?

Make an Organized List (also incorporates Make a Simpler Problem):

Number of friends	2	3	4	5	6	7	8
Number of elbow bumps	1	3	6	10	15	21	28

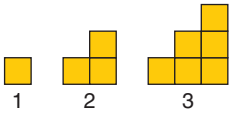
Write an Equation: $7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = 28$. Student explains: The first friend elbow-bumps 7 friends, the next friend has only 6 friends to greet (she already elbow-bumped the first friend), and so on down to 0 elbow-bumps for the last friend.

During the sharing of strategies, you can help students see other options and make connections among the strategies. Additionally, you can highlight a particular strategy so that more students can use that strategy on a future task. Discussing the many ways to approach this task supports the development of the Mathematical Practices (Albrecht, 2016).

Similar problems can be posed to support the development and reinforcement of these problem-solving strategies; for example:

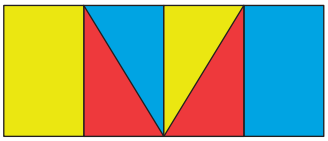
Task 2

1. If six softball teams play each other once in a round-robin tournament, how many games will be needed?
2. How many blocks are needed for the 10th staircase?

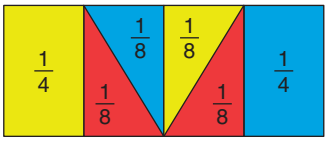


Teaching through Problem Solving. This approach means that students learn mathematics content through doing problems. They explore real contexts, problems, situations, and models, and from those explorations they learn concepts and skills. Teaching *through* problem solving is essentially turning the teaching *for* problem solving approach upside down—with the problem or task presented at the *beginning* of a lesson or unit and related knowledge or skills emerging from exploring the problem(s). As such, the students are involved in *thinking* in order to make sense of the mathematics (Liljedahl, 2021). Consider the following task, given to students who have not learned the algorithm for adding fractions with different denominators:

What fraction of this flag is blue?



By partitioning, students might label each section with their fraction of the whole, labeling the blue parts as illustrated here:



As students work, they recognize that they need equal-sized pieces to combine them. They change $\frac{1}{4}$ to $\frac{2}{8}$ (either by partitioning or knowing the fraction equivalency) and then add the blue pieces to get $\frac{3}{8}$. *After* students have solved the task, the teacher convenes the class to highlight important mathematical ideas—in this case, that you can most efficiently combine same-sized pieces. Notice, with this approach, that mathematical ideas are the outcomes of the problem-solving experience rather than of problem solving explained beforehand. Children are *learning* mathematics by *doing* mathematics; and by doing mathematics, they are learning mathematics (Cai, 2010).

Teaching through problem solving has a positive impact on students' performance. A recent national survey of fourth graders (National Center for Education Statistics, 2019) reports that students with higher mathematics scores (1) have teachers who place a strong emphasis on *using alternate strategies* to solve problems, (2) enjoy solving complex problems, (3) are more confident. Students in problem-based classrooms learn to ask: Why? What would happen if? What is another way? How does this way compare to that way? Will this always work? Generating this habit of consistent inquiry develops a disposition of openness, curiosity, and wonder (Clifford & Marinucci, 2008). This curiosity comes naturally to young students. Our goal as teachers is to nurture this disposition.

Learning Progressions

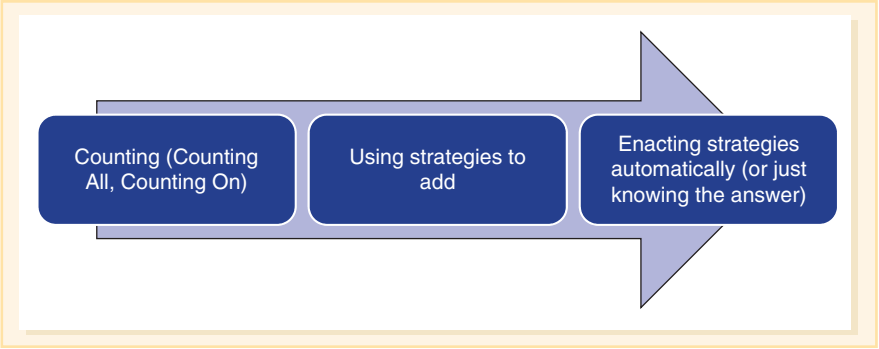
Learning goals (and related objectives) must fit within learning progressions. Mathematics is developmental. Teaching developmentally means that each lesson is part of a learning progression or trajectory (both terms are used interchangeably). *Learning progressions* describe the development of student thinking and strategies over time in terms of sophistication of both conceptual understanding and procedural fluency (Clements & Sarama 2021; Daro, Mosher, & Corcoran, 2011; Siemon et al., 2017; Sztajn, Confrey, Wilson, & Edgington, 2012). Recall that the Mathematics Teaching Framework (Figure 3.1) begins with the Teaching Practice related to establishing goals. That Teaching Practice states:

> Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

Lesson goals and objectives are the focus of Chapter 4 (planning). Part II of this book includes learning progressions as they relate to each chapter's content. Regardless of what you are teaching, as you establish goals, ask yourself:

- How does this topic relate to students' prior mathematics knowledge and experiences?
- How does this idea connect to future mathematics learning?

After selecting a topic within a learning progression, you establish learning goals and then day-by-day objectives for students. Each day, the lesson goals must make sense to students. As an example, let's look at addition facts. These are the single-digit sums. The learning progression to reach automaticity with these facts looks like this (Baroody, 2006):


Counting (Counting All, Counting On)
Using strategies to add
Enacting strategies automatically (or just knowing the answer)

Through formative assessments, you notice that students are able to count on and ready to learn strategies. A goal might be to have students use a Make Tens Strategy. Lesson objectives are specific and measureable. Examples for this goal are:

- Students will be able to rethink a single-digit addition problem (e.g., $9 + 6$) as a $10 + \_\_\_\_$ sum, using a Make Tens strategy.
- Students will be able to explain why $9 + 6 = 10 + 5$.

Once students learn this strategy, they keep practicing it until they automatically use the strategy and determine an answer to $9 + 6$. Effective teaching means knowing where your daily goals and objectives fit within the broader landscape.

Just restating the standards is not likely to help your students know what they are doing or why they are doing it and, on many days, you are working on an aspect of a standard, not the whole standard. Instead, think of student-friendly ways to help them understand the focus of that day's lesson. Additionally, on some days, the lesson's purpose reveals itself *after* students explore a task and discover an important mathematical idea. On such days, telling students what they will learn can spoil the surprise. Instead, consider helping students connect the outcome of the task to a big idea in mathematics and conclude by sharing that this was the purpose of the day's lesson. You can pose questions such as: What did we learn from solving this task? What big ideas can we take away from our work today? Ask yourself:

- What mathematics is being learned?
- Why is this mathematics important to learn?
- How can I introduce the big ideas (without taking away an aha in the lesson itself)?

With a focus on problem solving and learning progressions, the next important step is selecting worthwhile tasks.

Tasks That Promote Problem Solving

Learning Outcome 3.3 **Describe features of worthwhile tasks and methods for adapting tasks to make them better.**

To create classroom experiences in which students engage in problem solving, you must select tasks that promote problem solving. Such a worthwhile task may take on many forms. It might be clearly defined or open-ended, it may involve problem solving or problem posing, it may

include words or be purely symbols, it may take only a few minutes to solve or weeks to investigate, it may be real-life or abstract. Also, a task can promote problem solving initially but then become routine as a student's knowledge and experience grows.

Worthwhile Tasks

For a task to lend itself to problem solving—in other words, to be problematic—it must pose a question for which (1) there are no prescribed rules or methods to solve and (2) there is not a perception that there is one correct solution method (Hiebert et al., 1997).

Here is a task for you to try:

Task 3

$$10 + \blacksquare = 4 + (3 + \blacktriangle)$$

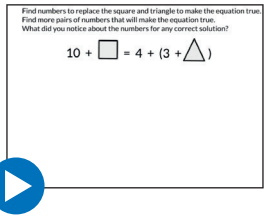
Find a number to replace the square and a number to replace the triangle so that the equation is true.

Find more pairs of numbers that will make the equation true.

What pattern do you notice about the numbers for any correct solution?

Pearson eText Activity Page: Make the Equation True

Does this task promote problem solving? It does not have a prescribed approach, and there are numerous ways to approach the problem, so it meets the first criterion of being problematic. Worthwhile tasks have other features, namely that they are cognitively demanding and open, and/or include relevant contexts. Each is discussed here.


Find numbers to replace the square and triangle to make the equation true.
Find more pairs of numbers that will make the equation true.
What did you notice about the numbers for any correct solution?
10 + □ = 4 + (3 + △)

Pearson eText
Video Example 3.4
In this video consider how the student solves the Make the Equation True problem.

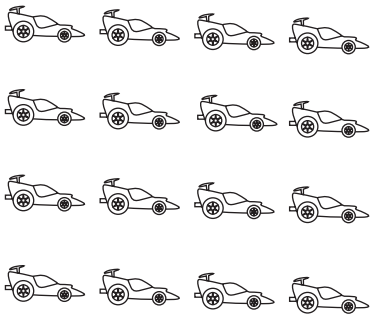
High Cognitive Demand. Tasks that promote problem solving are cognitively demanding, meaning they involve high-level thinking. Low cognitive demand–tasks (also called routine problems or lower-level tasks) involve stating facts or following known procedures or computations. High-level cognitive demand tasks, on the other hand, involve (1) procedures with connections or (2) doing mathematics (Smith & Stein, 1998). Such tasks engage students in understanding, analyzing, applying, and evaluating. When trying to determine whether a task is high-level cognitive demand, look for these characteristics (adapted from NCTM, 2014b):

- Has a solution pathway that is not immediately obvious
- Makes connections between procedures and mathematical concepts and ideas
- Focuses attention on generalizations and underlying structures
- Includes the potential use of multiple representations
- Involves non-algorithmic (nonroutine) thinking
- Engages students in exploration and reflection (e.g., Is this approach getting me somewhere?)

Open Tasks. As implied by the list you just read, worthwhile tasks can be solved in a variety of ways, even if it is a computation problem like $597 + 332$. Open-ended tasks are tasks that have more than one answer, like Make the Equation True. Open-ended may also mean that there are different ways to represent your answer (e.g., various pictures, demonstrations with manipulatives, acting out a problem, using a table or a graph, etc.). However, many worthwhile problems have one answer, one end. Yet, how to reach that result is open. In other words, the middle of the task is open, hence the term *open middle tasks* (https://www.openmiddle.com) (Kaplinsky, n.d.). Whether open-middle or open-ended, worthwhile tasks provide students with opportunities to reason and make decisions. Open tasks support diverse learners because students are encouraged to employ their prior learning and experiences, and having a choice of strategies can lower the anxiety of students, particularly that of multilingual learners (Murrey, 2008). Compare the K-1 tasks in Figure 3.2.

TASK 1: [The teacher places a bowl of objects (e.g., toy cars) on the table.] He asks, "Do we have enough [toy cars] for everyone in the class?"

TASK 2: [The teacher gives each student a page with pictures of cars copied in rows]. He asks, "Do we have enough cars for everyone in the class?"



(see **Do We Have Enough? Activity Page**).

FIGURE 3.2 Open task examples

Pearson eText Activity Page: Do We Have Enough?

At first glance, the first task might seem more engaging because it includes actual manipulatives, but in this case, having toy cars (or a cube representing a car) available might lead to a low-level strategy: passing the cars out to see whether there are enough for each child. The second task is more challenging. Students might count the cars. As they count, you can observe their thinking: Do they start at the top and count across the rows? Or do they haphazardly count and miss a car or double-count? Do they count by ones? By twos? By fours? Instead of counting cars, students might count their friends in the class first. Or, students might assign a car to each friend, writing a name on each car. Counting the cars is just one aspect of the task; students must also decide how that number compares to the number of students in the class. Does a student just know that the number of cars is greater or fewer than the number of students? Do they represent each child in the class with a counter and match a counter with a pictured car? Do they look for the two numbers on a hundred chart or number line to compare? Because the second task is more challenging, it offers many more options for how students engage in the task and how they can show a solution, making it an open and worthwhile task for the class.

Figure 3.3(a) provides an open task on probability (grades 7–8).

Pause & Reflect

Before studying the solutions in Figure 3.3(b), read the problem, select a strategy, and solve it. ●

Figure 3.3(b) illustrates a range of students' solutions. Student (b) used percentages as a way to compare; student (d) found simplified fractions to compare quantities; student (g) used part–part ratios to reason about the quantities. Solutions (a) and (d) reveal student misconceptions. During a classroom discussion, strategies are shared so that students can make connections among the strategies and to important mathematical ideas (i.e., ratio, fractions, percentages, and probability) and clear up misconceptions (Smith, Bill, & Hughes, 2008).

(a)

Ms. Rhee's mathematics class was studying statistics. She brought in three bags containing red and blue marbles. The three bags were labeled as shown below:

75 red
25 blue

Bag X
Total =
100 marbles

40 red
20 blue

Bag Y
Total =
60 marbles

100 red
25 blue

Bag Z
Total =
125 marbles

Ms. Rhee shook each bag. She asked the class, "If you close your eyes, reach into a bag, and remove 1 marble, which bag would give you the best chance of picking a blue marble?"

Which bag would you choose?

Explain why this bag gives you the best chance of picking a blue marble. You may use the diagram above in your explanation.

(b)

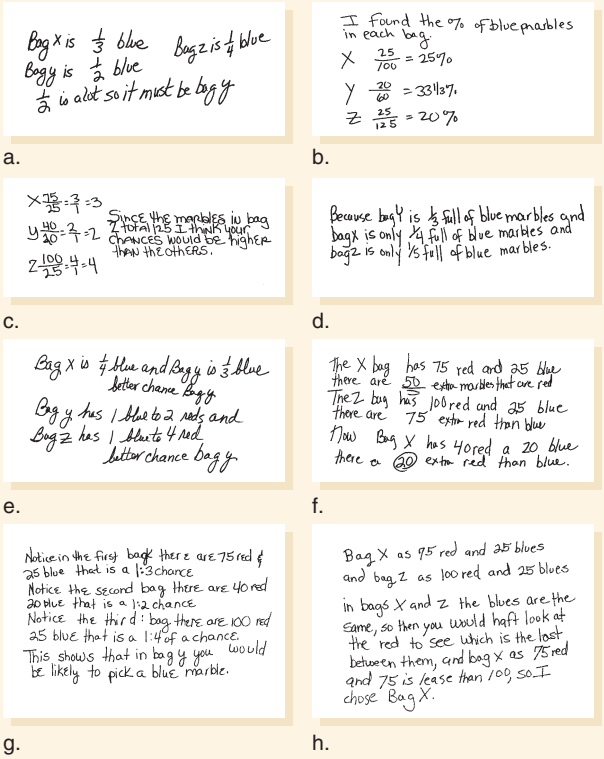

Bag X is 1/3 blue Bag Z is 1/4 blue
Bag Y is 1/2 blue
1/2 is alot so it must be bag y
a.
I found the % of blue marbles in each bag.
X 25/100 = 25%
Y 20/60 = 33 1/3%
Z 25/125 = 20%
b.
X 75/25 = 3/1 = 3
Y 40/20 = 2/1 = 2
Z 100/25 = 4/1 = 4
Since the marbles in bag Z total 125 I think your chances would be higher than the others.
c.
Because bag Y is 1/3 full of blue marbles and bag X is only 1/4 full of blue marbles and bag Z is only 1/5 full of blue marbles.
d.
Bag X is 1/4 blue and Bag y is 1/3 blue better chance Bag y.
Bag y has 1 blue to 2 reds and Bag Z has 1 blue to 4 red better chance bag y.
e.
The X bag has 75 red and 25 blue there are 50 extra marbles that are red
The Z bag has 100 red and 25 blue there are 75 extra red than blue
Now Bag X has 40 red a 20 blue there a 20 extra red than blue.
f.
Notice in the first bag there are 75 red & 25 blue that is a 1:3 chance
Notice the second bag there are 40 red 20 blue that is a 1:2 chance
Notice the third bag there are 100 red 25 blue that is a 1:4 of a chance.
This shows that in bag y you would be likely to pick a blue marble.
g.
Bag X as 75 red and 25 blues and bag Z as 100 red and 25 blues
in bags X and Z the blues are the same, so then you would haft look at the red to see which is the best between them, and bag X as 75 red and 75 is lease than 100, so I chose Bag X.
h.

FIGURE 3.3 The best chance of blue: An open task for middle school

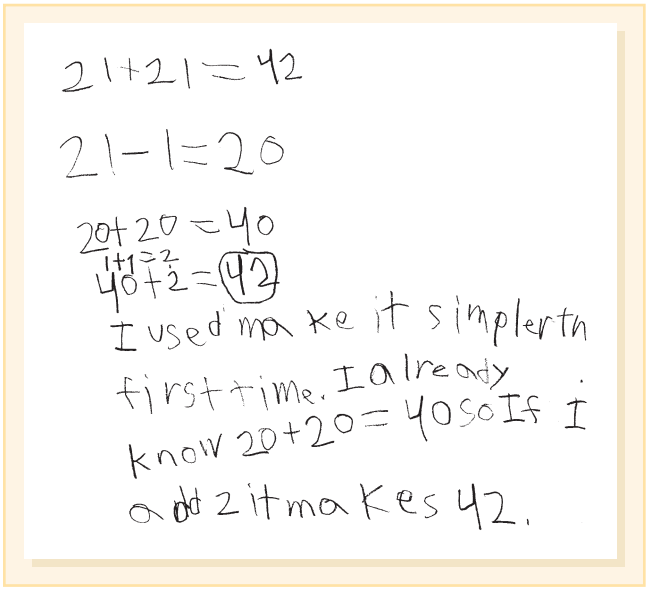
Source: Smith, M.S., Bill, V., & Hughes, E. K. (2008). Thinking through a lesson: Successfully implementing high-level tasks. *Mathematics Teaching in the Middle School, 14*(3), 132–138. Reprinted with permission.

Use Relevant Contexts. A positive feature of a task is when the problem gets students excited about doing mathematics. Compare the following tasks on multiplying 2 two-digit numbers. Which one do you think would be more relevant to fourth-grade students?

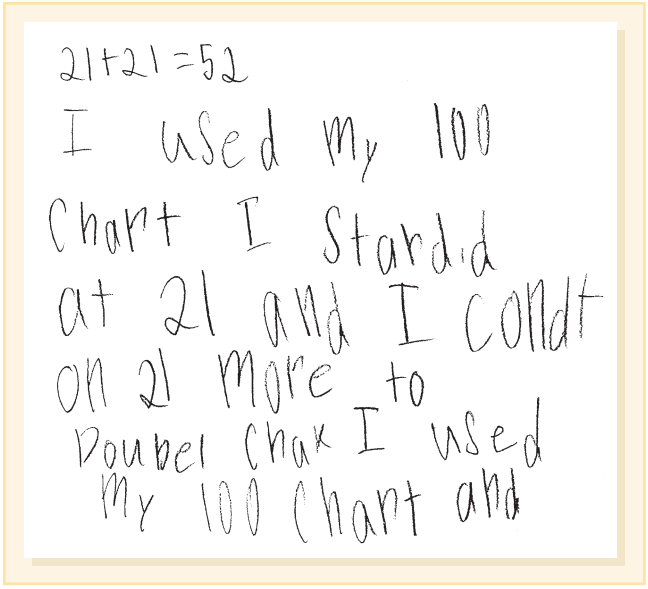
> **Classroom A:** "Today we are going to use grid paper to show the sub-products when we multiply 2 two-digit numbers."
>
> **Classroom B:** "The school is planning a fall festival and the class is selling water. The principal said we have 14 cases of bottled water in the storage closet. I looked in the storage closet and could see that a case had 7 rows, with 5 bottles in each row. How can we use the information about one case to figure out how many bottles of water we have? If we sell each bottle for $2.00, how much money might we make?"

Contexts must reflect the cultures and interests of the students in your classroom (the focus of Chapter 6). Here we briefly share two ways to incorporate relevant contexts—using literature and connecting to other disciplines.

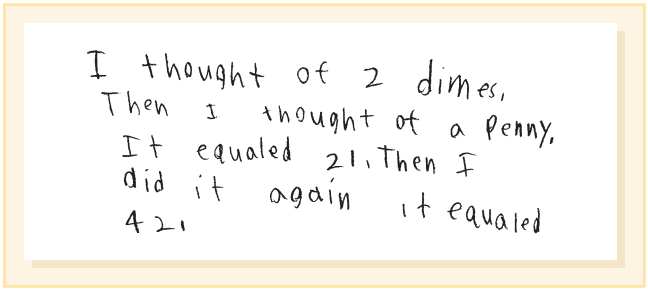
1. Use Literature. Children's literature and young adolescent literature are rich sources of problems. Picture books, poems, media, and chapter books can be used to create cognitively demanding, open tasks. As you read this book, look for "Literature Connections" at the end of every chapter in Part II. Here is one example: Two of Everything (Hong, 1993), a Chinese folktale, involves finding a pot that doubles whatever is put into it. (Imagine where the story goes when Mrs. Haktak falls in the pot!) Like many stories, read-alouds can be found on YouTube (e.g., https://youtu.be/JML_7tsqlmU). Students can explore the following open task: How many students would be in our class if our whole class fell in the Magic Pot? Figure 3.4 illustrates different second-grade solutions.



Robbie adds tens and ones to solve.



Kylee uses a hundred chart and counts on.



Benjamin uses the context of money to combine.

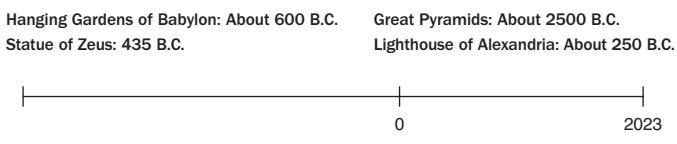
FIGURE 3.4 Three solutions to doubling a class of 21 children

Many books are written to focus on numeracy, but it is also important to see mathematical reasoning opportunities in favorite children's books commonly used in reading aloud (Linder & Bennett, 2020). Exploring the mathematics present in stories that are not explicitly mathematics-oriented helps students see their world mathematically. Another great way to use literature is to explore nonfiction (picture books, chapter books, newspapers, magazines, and the web), which truly is real-world problem solving. There are books of lists (e.g., *The Curious Book of Lists* by Turner & Selmes, 2019) and world-record books, for example, which provide many great contexts for exploring the globe and comparing world data to your class (see Bay-Williams & Martinie, 2009 and Petersen, 2004 for lessons). Epic! (https://www.getepic.com) is a free service for teachers that contains many collections of online books, videos, audiobooks, and so on. It can be searched or filtered by grade, subject, topic, Lexile, Accelerated Reading level, Developmental Reading level, Grade Level reading level, and so on. (Families can also subscribe.)

2. Connect to Other Disciplines. Interdisciplinary lessons help students see connections among the courses and topics they are studying, which may feel completely separate to them. The Mathematical Practices are closely connected to the Science and Engineering Practices, illuminating how closely related these two fields are (de Araujo, Hanuscin, & Otten, 2020). In kindergarten, students can link their study of natural systems in science to mathematics by sorting leaves based on color, smooth or jagged edges, feel of the leaf, and shape. Students learn about rules for sorting and can use Venn diagrams to keep track of their sorts. They can observe and analyze what is common and different in leaves from different trees. Older students can find the perimeter and area of various types of leaves and learn about why these perimeters and areas differ.

The social studies curriculum is rich with opportunities to do mathematics. Timelines of historic events are excellent opportunities for students to work on the relative sizes of numbers and to make better sense of history. Dates lend to creating number lines. For example, the dates for the seven wonders of the ancient world can be used to explore negative numbers.

Where on the number line do these wonders of the ancient world go? Can you find the other wonders and place them on the number line?


Hanging Gardens of Babylon: About 600 B.C.
Great Pyramids: About 2500 B.C.
Statue of Zeus: 435 B.C.
Lighthouse of Alexandria: About 250 B.C.
0
2023

Evaluating and Adapting Tasks

Throughout this book, in student textbooks, on the Internet, at workshops and webinars you attend, and in articles you read, you will find suggestions for tasks that someone believes

Worthwhile Task Evaluation Guide	
Task Potential	Try it and ask… ❑ What mathematical goals does the task address (and are they aligned with what you are seeking)? ❑ What key concepts and/or misconceptions might this task elicit? ❑ What is problematic about the task?
Problem-Solving Strategies	Will the task elicit more than one problem-solving strategy? Which strategies are possible? ❑ Model the Problem ❑ Make an Organized List ❑ Predict and Check for Reasonableness ❑ Make a Simpler Problem ❑ Look for a Pattern ❑ Prove a Conjecture ❑ Write an Equation
Features	To what extent does the task have these key features? ❑ High Cognitive Demand ❑ Open ❑ Relevant Context(s)

FIGURE 3.5 A tool to evaluate a task

are effective for teaching a particular mathematics concept. Yet a large quantity of what is readily available falls short when measured against the standards of being cognitively demanding, open, and relevant to students. Beware of the low-level cognitive-demand tasks cloaked in attractive artwork—they may look fun, but if the mathematics is not problematic, the tasks are not worthwhile. Make sure it is the mathematics itself that is clever and engaging.

An intentional selection process can help you pinpoint high-quality tasks, such as applying the Task Evaluation Guide in Figure 3.5, which includes the ideas discussed in this chapter. All boxes do not need to be checked. A task may include a number of problem-solving strategies but fall short in terms of relevance for students. In this case, you may decide to change the context. Or the task is complete with the features, but it does not match your mathematical goals for the lesson; hence, you don't use it.

Let's explore how this guide can help with selecting a worthwhile task. Imagine you are teaching fourth grade and are seeking a worthwhile task for this goal:

> **4.G.A.2: Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles. (NGA Center & CCSSO, 2010, p. 32)**

You type "classify triangles" into a web search. Hundreds of links and worksheet images appear, like the one in Figure 3.6.

Pause & Reflect

How does this task measure up on the Task Evaluation Guide? How might you adapt the task so that it rates higher on some of these measures of a worthwhile task? ●

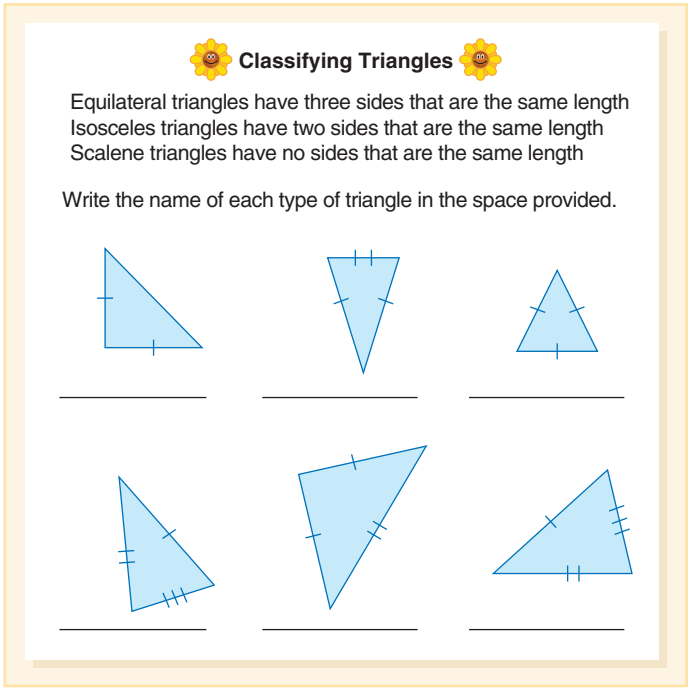

Classifying Triangles
Equilateral triangles have three sides that are the same length
Isosceles triangles have two sides that are the same length
Scalene triangles have no sides that are the same length
Write the name of each type of triangle in the space provided.

FIGURE 3.6 Example of a categorizing triangles worksheet

The worksheet in Figure 3.6 appears to match the learning goals, but it is not cognitively demanding, not open, or has no context. Here are some ways you might adapt it:

1. Remove the tick marks telling which sides are the same length; distribute just the set of triangles, letter them, and ask students to write similarities and differences between pairs of triangles. This opens the task and requires a higher level of cognitive demand.
2. Only use the list of terms with definitions at the top and ask students to identify examples of each in the room or in a picture book.
3. Cut out the triangles; ask students to create piles of what they consider the same triangles and give names to their groups. In a later discussion, different possible ways to sort triangles can be discussed, and appropriate terminology can be reinforced.

Extensions to make this task stronger could be added, such as asking students, "Can you build two triangles of different sizes that are both isosceles?" "Can you create a triangle with three obtuse angles? Why or why not?" "If a triangle is classified as a right triangle, then which classifications for sides are possible or impossible?" Each of these adaptations takes very little time and greatly increases the quality of the task.

Certainly, computation tasks (i.e., worksheets) are often devoid of the features of worthwhile tasks, but they must have these features for students to develop procedural fluency. Here are some suggestions for adapting low-quality computation tasks (adapted from Boaler & Sengupta-Irving, 2016):

1. Allow multiple *methods*: Explicitly ask students to use multiple methods, strategies, and representations to solve.
2. Make it an exploration: Change the task so there is more to it than a single computation. For example, rather than ask kindergartners to add $3 + 5$, ask them to find numbers they can add to equal the number 8.
3. Add a visual requirement: Visualization enhances understanding. Students can use two different manipulatives to justify their solution, or show how an equation fits a drawing of a story situation.
4. Reason and convince: Require students to create convincing mathematical arguments and expect the same from their peers. Ask them to be skeptics and ask clarifying questions of each other. (Modeling the use of logical arguments and follow-up questioning is important to be sure students focus on supporting each other's thinking.)
5. Postpone teaching a standard method: Begin with students' intuition about how to solve a problem type before learning about conventional methods. Delay the teaching of algorithms until other strategies are learned.

Just as students become adept at problem-solving strategies, with time and commitment you will become adept at evaluating and adapting tasks to support student learning better.

Procedural Fluency from Conceptual Understanding

Learning Outcome 3.4 **Illustrate the relationship between procedural fluency and conceptual understanding.**

In Chapter 2, you read that mathematical proficiency includes conceptual understanding *and* procedural proficiency. It is not surprising that effective teaching involves making connections between concepts and procedures. We illustrate this by exploring five tasks.

Tasks That Build Procedural Fluency from Conceptual Understanding

Try out each of the tasks in this section. Ask yourself, "What procedures am I exploring? What concepts am I exploring? How are connections made between concepts and procedures?"

Topic 1: Partitioning **Grades: K–1**

Task: Bowls on Two Tables

Six bowls of cereal are placed on two tables. Draw a picture to show a way that six bowls might be placed on two tables. Write the equation that matches your picture. Find more ways and write the equations. Finally, tell how many ways you think there are to place six bowls on two tables.

Pearson eText Activity Page: Bowls on Two Tables

In kindergarten or grade 1, contexts are necessary, because they are concrete ways for students to think about the mathematical relationships. Mathematically, students may determine one or two ways to decompose 6 or may find all the ways (open). Students can share how they thought about the problem and what patterns they noticed as they found new combinations (high cognitive demand). For example, a student might note that as one table gets a bowl, the other table loses a bowl, so one bowl moves over. They may notice that there are then seven possible ways. You can extend student thinking, asking, "How do you know you have them all?" The task can also be adapted to other totals and other contexts, such as, "How many ways can you put 10 toys into 2 baskets?" or "If there are 10 bowls, would there be 11 possible ways?" Students begin to notice that numbers can be taken apart and put back together in different ways (conceptual understanding). They also begin to learn about addition by recording number sentences such as $6 = 5 + 1$ (procedural knowledge), discussing the meaning of the symbols, and showing how the numbers represent the situation (connecting concepts and procedures).

Topic 2: Adding Two-Digit Whole Numbers **Grades: 1–2**

Select a strategy to solve $48 + 25$. Illustrate or explain your strategy.

Even though there is no story or situation to resolve, this task can be worthwhile. (Check the list of five suggestions in the previous section.) In this case, students were asked to choose a method. The teacher recorded selected student thinking on the board:

Sam	Erin	Jeremiah	Leah
$40 + 20 = 60$ $8 + 5 = 13$ $60 + 13 = 73$	$50 + 25 = 75$ $75 + 2 - 73$	$50 + 23 = 73$	$48 + 20 = 68$ $68 + 5 = 73$

As students share their solution pathways, they are deepening their conceptual understanding of place value and properties, they are learning procedures, and they are developing a foundation for learning the standard algorithm. All operations for whole numbers, decimals, fractions, and integers can be explored in a similar manner.

Topic 3: Area of a Rectangle **Grades: 3–4**

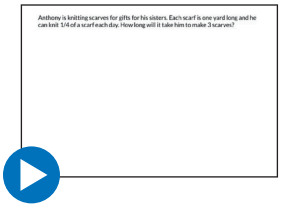
Task: Tiling Book Covers

Find the area of the cover of your math book by covering it with color tiles. Repeat for the areas of books of various sizes. What patterns do you notice in covering the book? Is this pattern or rule for covering any rectangle?

Pearson eText Activity Page: Tiling Book Covers

As students begin to cover surfaces, they may run out of tiles, or they may just get tired of placing tiles. They notice that each row has the same number of tiles, so they just need to know how many rows and columns will cover the book and they can skip count or multiply to find the total number of tiles. This investigation develops the concept of area, strengthens understanding of multiplication as equal rows, and leads to the procedure for area of a rectangle.

Topic 4: Division of Fractions **Grades: 5–7**
Task: Making Scarves
Anthony is knitting scarves for gifts for his three sisters. Each scarf is one yard long, and he can knit $\frac{1}{4}$ of a scarf each day. How long will it take him to make the scarves?


Anthony is knitting scarves for gifts for his sisters. Each scarf is one yard long and he can knit 1/4 of a scarf each day. How long will it take him to make 3 scarves?

Pearson eText
Video Example 3.5
In this video consider how the student solves the Making Scarves problem.

When students explore this task without the "division of fractions" label, they can approach the problem in multiple ways. Leah, Kelly, Jaden, and MacKenna solved the task by applying what they knew, including skip counting by fourths (Leah), measuring equivalencies (Kelly), and figuring ratios of yards to days (Jaden) and rates of days per yards (MacKenna). To extend their thinking about scarf-making, they were asked, "What would happen if Anthony decided to make $\frac{3}{4}$ of a scarf in one day?" A review of their second task shows that they used both their strategies and their answers from the first task in interesting ways. After more experiences with scarf-making at other rates, students can begin to connect to the concept of division involving fractions. Series of related tasks can help students connect concepts and procedures. The scarf length and the amount completed per day can vary to help students look for patterns across the problem set.

Pearson eText Leah's Solution
Pearson eText Kelly's Solution
Pearson eText Jaden's Solution
Pearson eText MacKenna's Solution

Topic 5: Ratios and Proportions **Grades: 6–8**
Task: First to the Water
Jack and Jill were at the same location at the bottom of a hill, hoping to fetch a pail of water. They both begin walking up the hill, Jack walking 5 yards every 25 seconds and Jill walking 3 yards every 10 seconds. Assuming a constant walking rate, who will get to the water first?

Pearson eText Activity Page: First to the Water

Students can engage with this task in a variety of ways. They can represent the problem visually with jumps on a number line or symbolically by using a rate approach (determining the number of yards per second for each person). By specializing and considering several examples, students will begin to generalize a procedure for comparing ratios, which is the essence of proportional reasoning. For a full lesson with this context and others, see Expanded Lesson: It's a Matter of Rates.

Pearson eText Expanded Lesson: It's a Matter of Rates

What about Drill and Practice?

The phrase "drill and practice" slips off the tongue so rapidly that the two words, *drill* and *practice*, appear to be synonyms—and, for the most part, they have been. In the interest of developing a new perspective on drill and practice, consider definitions that differentiate between these terms as different types of activities.

> *Drill* refers to repetitive exercises designed to replicate a procedure or algorithm.
>
> *Practice* refers to varied tasks or experiences focused on a particular concept or procedure.

Drill. Finding pages for drill is easy; there is an endless supply of such worksheets on the Internet. But where has this drill gotten us? Drill has created generations of people who don't remember what they learned, do not like mathematics, and purposefully do not pursue professions that involve mathematics. Drill may improve procedural knowledge but not conceptual understanding, but when the number of problems is reduced and time is then spent discussing problems, both procedural and conceptual knowledge are supported (Franke, Kazemi, & Battey, 2007).

Practice. High-quality practice is not as easy to find. Such practice gives students opportunities to solidify concepts, explore alternative and flexible strategies for solving problems, and make connections between concepts and procedures. Using worthwhile tasks as practice sends a clear message that mathematics is about figuring things out. Routines, games, and activities, like those found throughout this book, can engage students in meaningful, enjoyable practice.

Facilitating Mathematical Discourse

Learning Outcome 3.5 **Describe and relate the Mathematics Teaching Practices related to facilitating mathematical discourse.**

With the first three Mathematics Teaching Practices implemented effectively, the focus of teaching turns to classroom discourse. To help students become productive mathematical thinkers, teachers must ask key questions, be able to respond with targeted feedback to students, probe student thinking, prompt students to reflect on their thinking, be comfortable with uncertainty, and know the difference between productive and unproductive struggle (Kazemi & Hintz, 2014; NCTM, 2014b). Teachers themselves must display productive dispositions, showing students they are willing to explore, experiment, and make conjectures; recognize multiple solution pathways; make connections among strategies; and monitor and reflect on their work. The goal of productive discourse is to keep the cognitive demand high while students are learning and formalizing mathematical concepts. Note that the purpose is not for students to tell their answers and get validation from the teacher. Orchestrating classroom discourse is so important and complex that it incorporates five of the eight Teaching Practices (see Figure 3.1).

A lot of research connects mathematical discourse to student learning (e.g., Kazemi & Hintz, 2014; Staples & King, 2017; Wood, Williams, & McNeal, 2006). Suggestions from these studies include:

- Encourage student–student dialogue rather than student–teacher conversations that exclude the rest of the class. When students have differing solutions, have them work these ideas out as a class. "George, I noticed that you got a different answer than Tomeka did. What do you think about her explanation?"
- Encourage students to ask questions. "Pete, did you understand how they did that? Do you want to ask Antonio a question?"

- Ask follow-up questions whether the answer is right or wrong. Your role is to understand student thinking, not to lead students to the correct answer. So follow up with probes to learn more about their answers. Sometimes you will find that what you assumed they were thinking is not accurate. And if you only follow up on wrong answers, students quickly figure this out and get nervous when you ask them to explain their thinking.
- Call on students in such a way that, over time, all students are able to participate. Use times when students are working in small groups to identify interesting solutions that you will highlight during the sharing portion of the lesson. Be intentional about the order in which the solutions are shared; for example, select two that you would like to compare presented back to back.
- Demonstrate to students that it is okay to be confused and that asking clarifying questions is expected. Tell them that this confusion, or disequilibrium, just means they are engaged in doing real mathematics and is an indication they are learning.
- Move students to more conceptually based explanations when appropriate. For example, if a student says that he knows 4.17 is more than 4.1638, you can ask him (or another student) to explain why this is so. Say, "I see what you did, but I think some of us are interested in why you did it that way."
- Be sure all students are involved in the discussion. Strategies to ensure participation include using sentence starters or examples to help students know what kind of responses you are hoping to hear; asking students to practice their explanations with a peer; and inviting students to use illustrations and actual objects to support their explanations.

At first, facilitating discussions may seem uncomfortable, especially if you are used to a teacher-directed approach to learning mathematics. Just keep in mind that the purpose is to unveil student thinking and help students understand each other's thinking. Warning signs that you are taking over children's thinking include interrupting a child's strategy or explanation, manipulating the tools for the child instead of allowing the child to do so, and asking a string of closed questions (Jacobs, Martin, Ambrose, & Philipp, 2014). Taking over children's thinking sends the message that you do not believe they are able to do it on their own, and that can inhibit the very discourse you are trying to encourage.

Elicit and Use Evidence of Student Thinking

The value of student talk in mathematics lessons cannot be overemphasized. As students describe and evaluate solutions to tasks, share approaches, and make conjectures, learning will occur in ways that are otherwise unlikely to take place. Students—in particular multilingual learners, other students with more limited language skills, and students with learning special needs in math—need to use precise mathematical vocabulary and articulate mathematics ideas to learn both the language and the concepts of mathematics. Students begin to take ownership of ideas (strategic competence) and develop a sense of power in making sense of mathematics (productive disposition). As they listen to other students' ideas, they come to see the varied approaches to how mathematics can be solved and see mathematics as something that they can do.

Smith and Stein (2018) describe five teacher actions for orchestrating productive mathematics discussions: anticipating, monitoring, selecting, sequencing, and connecting. These discussions occur after having selected a worthwhile task. The first action, anticipating, takes place before the lesson begins. During the lesson, the teacher monitors students as they work, observing strategies they are using on the task and asking questions, such as:

- How did you decide what to do? Did you use more than one strategy?
- What did you do that helped you make sense of the problem?
- Did you find any numbers or information you didn't need? How did you know that the information was not important?
- Did you try something that didn't work? How did you figure out that it would not work?

These questions are designed to help students reflect on their own strategy use and help the teacher determine which strategies to select for a group discussion after the lesson. Having selected a range of strategies to be shared, the teacher strategically sequences the presentations so that particular mathematical ideas can be emphasized. Perhaps most important, the

teacher designs questions that connect strategies and mathematical concepts. These tend to be questions that are specific to the task, but some general questions include:

- How did Tracie represent her solution? What mathematical terms, symbols, or tools did she use? How is this like/different from [Jada's] strategy?
- Was there something in the task that reminded you of another problem we've done before?
- What might you do the same or differently the next time you encounter a similar problem?

Notice that these questions focus on the problem-solving process as well as on the answer, and what worked as well as what didn't work.

Because of the benefits of talking about mathematics, it is critical to make sure that everyone participates in the classroom discussion. You may need to discuss explicitly with students why discussions are important and what it means to listen actively and respond to others' ideas. For example, children can demonstrate they are listening by making eye contact with the speaker and through nonverbal cues (e.g., nodding), letting the speaker finish before sharing questions or ideas, and responding appropriately and respectfully by asking questions or summarizing the speaker's ideas (Wagganer, 2015).

Next, you use *talk moves.* A talk move is an action you take (or a student might take) in order to support discussion (Chapin, O'Conner, Anderson, 2013). To begin, a talk move is *Wait time*, which is critical to ensuring participation and thinking (Roake, 2013). When you have asked an important question, providing 10–30 seconds of quiet time provides necessary time for every student to think through a response. A second move is *Partner talk* is an effective way to have lots of students talking (and listening), which can be in preparation for a whole class discussion, or used within a whole class discussion when a big idea needs more think time. A very useful strategy is called think-pair-share. The think step is the wait time, the pair step is the partner time, and the share step is the whole class discussion.

Supporting a whole class discussion so that everyone in the class is participating and learning requires purposefully selecting and using talk moves (Chapin, O'Conner, Anderson, 2013). The talk moves help students understand the ideas being shared:

1. *Revoice a student:* You restate what a student has said, as a way to ensure you captured what a student has shared and that it has been heard. This is also a way to insert appropriate language.

   "So, you found the product of 7×19 by thinking of 19 as 20, multiplying, and then compensating by subtracting a 7."

2. *Repeat Request:* You ask another student to say again what they heard. Note: this is *not* the way to punish someone who wasn't listening. This is a way to have everyone in the class hear the idea again, with slightly different words.

   "Can you repeat or rephrase how Aniyah thought about 7×19?"

3. *Say More:* You ask the speaker to elaborate on what they have shared or you ask other students to add to the first speaker's idea.

   "Can you say more about why you used compensation?"

4. *Critique:* You ask students what they think of an idea or suggestion that was shared.

   "Does this idea work?" "Is this a reasonable way to solve this problem? Why or why not?"

5. *Revise Thinking:* You ask if students have changed their thinking or their strategy.

   "How has your thinking/understanding changed?"

6. *Reason:* You ask students to explain why or when something is true.
7. *Connect:* You ask students to find similarities and/or differences between strategies, ideas, or representations.

   "What do you notice about these two solutions?"

These talk moves do not need to occur in order—you decide which one is most fitting in the moment. Imagine students have just solved Task 3 *Make the Equation True* (p. 39). They have shared answers that work, they have worked with partners, and now you are starting a conversation about the final question, "What pattern do you notice about the numbers for any correct solution?" Table 3.2 shows how that discussion might go, as students offer various thoughts, and you decide what talk move will support understanding and participation.

TABLE 3.2 VIGNETTE OF CLASSROOM DISCUSSION OF MAKE THE EQUATION TRUE TASK

Student says...	You think...	So, your talk move is...
"I notice that if one number is odd, the other is even."	This is unexpected, but interesting for us to think about.	Repeat: "Can someone rephrase what ____ said?" Reason: "Why do you think this is true?"
"I kept going up with both numbers and it always worked."	I am not sure what this student is saying.	"Say more: "Can you tell us what you mean by going up with both?"
"The number in the box is always smaller than the number in the triangle."	"This is a good start."	"Revoice: "The number here" (pointing at square) "is always less than the number that goes here" (pointing at triangle)."
"The triangle number is actually three more than the other number."	I wonder if others have noticed this yet. I want others to consider this pattern. I want to extend student thinking.	Repeat Request: "Who can repeat what ____ just explained, using numbers as examples?" Critique: "What do others think about this?" Reason: "Why do you think this is the case?" [wait time] Reason: "How does this pattern (that one number is three more than the other) connect to the odd/even pattern ____ shared earlier?" [partner talk]

The following exchange illustrates an example of discourse with a small group of students discussing how to solve $27 - 19 =$ _____. The teacher is asking two students (Tyler and Aleah) to think about their different answers.

Tyler: Well, I added one to nineteen to get twenty. So then I did twenty-seven take away twenty and got seven. But I added one, so I needed to take one away from the seven, and I got six.
Teacher: What do you think of that, Aleah?
Aleah: That is not what I got.
Teacher: Yes, I know that, but what do you think of Tyler's explanation?
Aleah: Well, it can't be right, because I just counted up. I added one to nineteen to get twenty and then added seven more to get twenty-seven. So, I counted eight altogether. Six can't be right.
Teacher: Tyler, what do you think of Aleah's explanation?
Tyler: That makes sense, too. I should have counted.
Teacher: So, do you think both answers are right?
Tyler: No.
Aleah: No. If it was twenty-seven minus twenty, the answer would be seven, because you count up seven. So, if it is nineteen, it has to be eight.
Tyler: Oh, wait. I see something. I did get the seven . . . ? See, I got the twenty-seven take away twenty is seven. But then . . . ? I see . . . ? It's twenty-seven take away nineteen. I took away twenty! I took away too many so I have to add one to the seven. I get eight, just like Aleah! (Kline, 2008, p. 148)

Pause & Reflect

What talk moves do you notice in the previous vignette? See if you can identify two. ●

Not only is the teacher able to help her students better articulate their mathematical thinking (connecting to Mathematical Practices 3 and 6), she is able informally to assess what they know. This is what the Teaching Practice means by "use evidence." The teacher has evidence from this exchange related to whether these students are ready to move on or whether they need more experiences. Chapter 5, on assessment, provides much more guidance on this notion of eliciting and using evidence of student thinking.

Posing Purposeful Questions

The preceding section included many purposeful questions, so you might be wondering why this header now? Those questions are designed to facilitate a classroom discussion; the focus here is on the mathematical goals of the lesson. Questioning is complex! It is something to work on throughout your career. Here are some major considerations in the art of questioning.

The Purpose of the Question. We ask questions throughout a lesson with different purposes in mind. Hallman-Thrasher & Spangler (2020) describe four purposes:

1. Elicit thinking (e.g., How did you solve? Say more…)
2. Generate ideas (e.g., What are we trying to figure out? What might we try?)
3. Clarify/revise explanation (e.g., What do you mean by… .? What if…)
4. Justify claims (e.g., How do you know that is true? Why did this method work?)

These purposeful questions help students understand the mathematics more deeply as they revise and refine their explanations and illustrations, and this supports their learning (Jansen, 2020).

Level of the Question. Questions are leveled in various models. For example, Bloom's Taxonomy (revised) includes six levels, with remembering considered to be the lowest level, and each of the others one more cognitively demanding than the previous ones (Anderson & Krathwohl, 2001). Smith & Stein's (1998) high-cognitive demand levels are another example.

Both procedural and conceptual knowledge are important, and questions must target both and connect the two. If questions are limited to questions such as "How did you solve this?" or "What are the steps?" students will be thinking about procedures but not about related concepts. Questions more focused on conceptual knowledge and making connections include:

- Why does that strategy (or procedure) work?
- Why did you use common denominators to add fractions?
- How does the equation you wrote connect to the picture?
- How do these two procedures compare?
- When would you use (this method)?
- When would you avoid (this method)?

Pattern of Questioning. Some patterns of questioning are not as effective as others in encouraging student reasoning and problem solving (Herbel-Eisenmann & Breyfogle, 2005). One common pattern goes like this: The teacher asks a question, the student answers the question, and the teacher confirms or challenges the answer. This initiation–response–feedback, or IRF pattern, does not lead to classroom discussions that encourage all students to think. Another pattern is funneling, when a teacher continues to probe students to lead them to a particular answer. This is different from a focusing pattern, which uses probing questions to help students understand the mathematics. The talk moves described previously are intended for a focusing pattern of questioning.

Who Is Doing the Thinking? You must be sure your questions engage each and every student! When you ask a great question, and only expect one student to respond, students realize they don't need to think of an answer (and your great question is wasted). Instead, use strategies to be sure everyone is accountable to think about the question you posed. Ask students first to write their ideas on a notecard or individual whiteboard, to talk to a partner about the question, and finally to ask for contributions as a whole class. This think-pair-share strategy maximizes all students' participation.

How You Respond to an Answer. When you confirm a correct solution (e.g., "Yes, correct!") rather than use one of the talk moves, you lose an opportunity to engage students in meaningful discussions about mathematics and thereby limit the potential learning. Use student answers to find out whether other students think the conclusions made are correct, whether they can justify why, and whether there are other strategies or solutions to the problem.

Using and Connecting Mathematical Representations

There is strong evidence that teaching students how to use representations improves their mathematics achievement (Woodward et al., 2018). Using representation was also mentioned in Chapter 2 because it is so central to developing Mathematical Practices and consistent with learning theories (see Figure 2.11). Representations provide concrete and/or visual ways to make sense of abstract ideas. Different representations make student thinking visible, so they are critical to orchestrating classroom discourse and providing evidence of individual thinking for assessment and future planning.

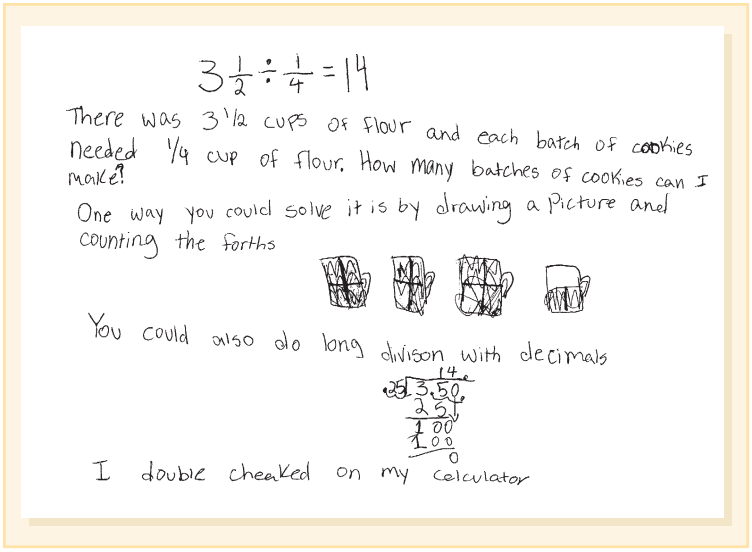

$3\frac{1}{2} \div \frac{1}{4} = 14$
There was 3 1/2 cups of flour and each batch of cookies needed 1/4 cup of flour. How many batches of cookies can I make?
One way you could solve it is by drawing a picture and counting the forths
You could also do long divison with decimals
14
25 3.50
25
100
100
0
I doubie cheaked on my calculator

FIGURE 3.7 A student gives an example and explains how to solve $3\frac{1}{2} \div \frac{1}{4}$

Writing. Writing plays a critical role in classrooms. Like questioning, writing can serve various purposes (Firmender, Casa, & Colonnese, 2017). Those purposes can be:

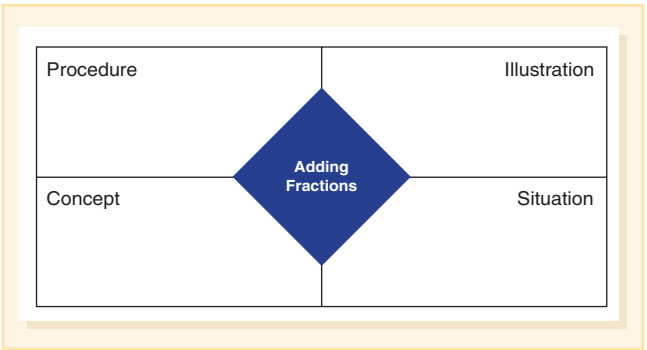
1. Exploratory (to tell about what a problem means or how it relates to you).
2. Explanatory (to describe your thinking).
3. Argumentative (to justify *why* something works or critique a solution).
4. Creative (convey original ideas, flexibility in thinking, elaborate on ideas).

Writing can serve as a rehearsal for a classroom discussion. It can be difficult for students to remember how they solved a problem. A written record or a pre-drawn picture can be used for reference during discussions. In addition, students can revise their rough-draft thinking to illustrate their new insights about the mathematics they learned after a classroom discussion (Jansen, 2020). Writing helps students focus on the need for precise language in mathematics and understand that illustrations can support a good explanation. Posting different written solutions, just like hearing about solutions, engages students in reflecting on which strategy makes sense, and a written product provides evidence for you to analyze student understandings further after the lesson. For example, the student work in Figure 3.7 illustrates how a fifth-grade student is thinking about division by the fraction $\frac{1}{4}$.

Graphic Organizers. Graphic organizers can be used to help students connect representations. The classic Frayer Model is a two-by-two grid with a concept, word, or equation in the middle, allowing students to record their thinking for each category and opening the opportunity to make connections:

Classic Frayer: Definition, Characteristics, Examples, and Nonexamples
Algebraic Thinking: Graph, Equation, Table, and Situation (GETS)
Geometry: Definition, Properties, Examples, and Nonexamples
Operations: Procedure, Illustration, Concept, and Situation (PICS) (McGatha, Bay-Williams, Kobett, & Wray, 2018).

You do not need a copy of such a graphic organizer; you can have students fold a piece of paper into fourths, fold the inside corner down, and open it up, as illustrated here:



These graphic organizers lend themselves to many engaging uses. Students can work in groups of four, each student assigned one box. Students can work with a partner on just one box and then pass their page to another pair who has to add on a new completed box before passing it on. Students can do gallery walks, comparing what their peers used for illustrations. And, so much more!

For younger children, four sections are too many. They can fold a paper in half to *show* (manipulatives or pictures) *and tell* (words and symbols). Venn diagrams with two or

three can be used to compare types of numbers, shapes, operations, and so on. *Teaching Mathematics with Foldables* (Zike, n.d.) is also an engaging way to have students create graphic organizers.

Photographs. Taking photographs of student work can help other students see their presentations and serve as an excellent way to assess student thinking formatively (Roller, Cunningham, & Marin, 2019). Using photographs is particularly valuable with young children, because their mathematics learning is mostly focused on using manipulatives and sharing how they are thinking verbally. You can help students learn from each other by posting the pictures and inviting students to go on a gallery walk to see their friends' strategies (Farkas, 2019). Document cameras can also be used to help students see their friends' strategies.

Supporting Productive Struggle

Are you thinking that showing students is the helpful approach, preventing students from struggling while also saving time? If so, you are not alone. But, there is strong research showing that engaging students in productive struggle leads to increased conceptual understanding (Hiebert & Grouws, 2007; NCTM, 2014b). Productive struggle is grappling with a task, trying out different mathematical strategies to reach a solution. Effective teaching involves finding ways to engage students in productive struggle (while not frustrating them). Here are some pragmatic suggestions. (Many connect back to what you just read!) (NCTM, 2014b; Warshauer, 2015):

1. Establish classroom expectations from day one that focus on perseverance. Yeh & Chao (2019) share this list of classroom Math Rules, posted in the classroom and co-constructed with students: (1) Give it a try. (2) No problem is unsolvable. (3) Pay attention to precision. (4) Being fast is *not* all that. (5) It's *not* too hard. (6) It's all right to make mistakes. (7) Try your best!
2. Use worthwhile tasks.
3. Ask assessing questions (to assess what a student knows and is able to do) and/or advancing questions (to help move a student's thinking forward, but without doing the mathematics for the student) (Smith & Stein, 2018).
4. Encourage effort, not just answers. Explicitly tell students that mathematics is about perseverance, not speed. Mathematicians are successful because they stay with a problem and keep trying different things, not because they are fast.
5. Give adequate time to solve problems. Be ready with ways to extend the task for those who finish early (rather than send them off to do something unrelated to math).
6. During sharing time, call on students with right and wrong answers, and facilitate discourse about why a strategy or idea does or doesn't work in that situation.
7. Focus feedback on perseverance (e.g., "I noticed that you traded out your first strategy to try something different—good effort" and "How do you know your answer is correct?").
8. Avoid feedback that works against perseverance (e.g., "Wow, you solved that so fast!" and "Good job, that's right.").

One of the most perplexing dilemmas for teachers is how much information and direction to provide to students during mathematical inquiry. On one hand, telling can diminish what is learned and lower the level of cognitive demand in a lesson by eliminating the productive struggle that is the key to conceptual understanding. On the other hand, to tell too little can sometimes leave students floundering. Three things that teachers need to tell students are (Hiebert et al., 1996):

- *Mathematical conventions*. Symbols such as $+$ and $=$, terminology, and labels are conventions. As a rule of thumb, symbolism and terminology should be introduced *after* concepts have been developed as a means of expressing or labeling ideas.
- *Alternative methods*. When an important strategy does not emerge naturally from students, then the teacher needs to introduce the strategy, being careful to introduce it as *another* way, not the only or the best way.
- *Clarification or formalization of students' methods*. It is appropriate and necessary to help students clarify or interpret their ideas and point out related ideas. A student may add 38 and 5 by noting that 38 and 2 more is 40 with 3 more equaling 43. This strategy can

be related to the Make Tens strategy used to add $8 + 5$. Drawing everyone's attention to this connection can help other students see the strategy while also building the confidence of the student(s) who originally proposed the strategy.

Teaching Practices At-a-Glance

To summarize this chapter on effective mathematics teaching, Table 3.3 lists the eight Mathematics Teaching Practices along with the teacher actions that relate to that practice. If it seems like a lot, it is! No matter how long you are teaching, you continue to work on becoming more and more effective. This list can serve as a reflection tool as you grow professionally. As you probably have already established, these Teaching Practices require careful planning—far more so than lecturing! And that is the focus of the next chapter.

TABLE 3.3 MATHEMATICAL TEACHING PRACTICES AND RELATED TEACHER ACTIONS

Mathematics Teaching Practices and Related Teacher Actions At-a-Glance	
Teaching Practice	**To Enact the Mathematics Teaching Practice, a Teacher:**
1. Establish mathematics goals to focus learning.	• Uses learning progressions to identify learning goals. • Articulates clear learning goals that identify the mathematics students will learn in a lesson or lessons. • Helps students understand how the tasks they are doing relate to the learning goals. • Uses the articulated goals to inform instructional decisions.
2. Implement tasks that promote reasoning and problem solving.	• Selects tasks that: ❍ Have maximum potential to build and extend students' current mathematical understanding. ❍ Are open-middle or open-ended. ❍ Require a high level of cognitive demand. ❍ Are relevant to student. • Supports students to make sense of and solve tasks, using multiple strategies and representations, without doing the thinking for the students.
3. Use and connect mathematical representations.	• Expects students to use representations to support their reasoning and explanations. • Allows students to choose representations. • Introduces representations when appropriate. • Helps students attend to a mathematical idea represented in a variety of ways.
4. Facilitate meaningful mathematical discourse.	• Facilitates productive discussions among students by focusing on reasoning and justification. • Strategically selects and sequences students' strategies for whole class discussion. • Makes explicit connections between students' strategies and ideas.
5. Pose purposeful questions.	• Asks assessing and/or advancing questions that: ❍ Probe students' thinking and that require explanation and justification. ❍ Build on students' ideas and avoid funneling. ❍ Make students' ideas and the mathematics more visible so learners can examine the ideas more closely. ❍ Provides appropriate amounts of wait time.
6. Build procedural fluency from conceptual understanding.	• Encourages students to make sense of, use, and explain their own reasoning and strategies to solve tasks. • Makes explicit connections between strategies produced by students and conventional strategies and procedures.
7. Support productive struggle in learning mathematics.	• Anticipates potential difficulties and prepares ways to support students' thinking while not taking over their thinking. • Allows students time to grapple with problems. • Helps students see struggles, errors, or misconceptions as opportunities for learning. • Praises students for their efforts and perseverance in problem solving.
8. Elicit and use evidence of student thinking.	• Decides what will count as evidence of students' understanding. • Gathers evidence of students' understanding at key points during lesson. • Interprets students' thinking to gauge understanding toward learning goals. • Decides how to respond to students to support and extend their thinking. • Uses evidence of students' learning to guide subsequent instruction.

Source: Based on *Principles to Actions: Ensuring Mathematical Success for All* (NCTM), © 2014.

RESOURCES FOR CHAPTER 3

WRITING TO LEARN

1. Describe two important takeaways related to the first three mathematics Teaching Practices (goals, tasks, and procedural fluency).
2. What does it mean to teach through problem solving?
3. What is meant by an open task? Why are open tasks important?
4. How does orchestrating a classroom discussion compare to students going over their answers together as a class?
5. This chapter is full of question prompts. What might be your top five go-to questions? Why are they important? How will you make sure you won't forget to ask them?
6. How do this chapter's ideas on effective mathematics teaching change or confirm your ideas about how to help students learn mathematics?

RECOMMENDED READINGS

Articles

Battista, L. N., & Chapin, S. H. (2019). Creating a discourse community. *Teaching Children Mathematics*, *25*(5), 298–304.

Wondering how to build your math community? This article is full of fun ways to develop norms and get your math discussions and class community off to a great start!

Firmender, J. M., Casa, T. M., & Colonnese, M. M. (2017). Write on. *Teaching Children Mathematics*, *24*(2), 84–92.

This article offers excellent writing prompts that support student reasoning, creativity, and mathematical understanding. A great resource for teaching and assessing!

Hallman-Thrasher, A., & Spangler, D. A. (2020). *Purposeful questioning with high cognitive demand tasks. Mathematics Teacher: Learning & Teaching PK-12*, *113*(6), 446–459.

This is a fantastic article for illuminating what questions to ask across different purposes, and how to connect questioning to specific tasks, K–12.

Roake, J. (2013). Planning for processing time yields deeper learning. *ASCD Education Update*, *55*(8), 1, 6–7.

This article provides a brief description of 10 tips for building think time into discussions, with connections to brain research. A quick read with important ideas!

Books

Chapin, S., O'Conner, C., & Anderson, N. C. (2013). *Talk moves: A teacher's guide for using classroom discussions in math* (3rd ed.). Sausalito, CA: Math Solutions.

As described in this chapter, this book provides specific, strategic talk moves to engage learners. The book includes a plethora of examples, videos, and other resources to support your professional learning.

Jansen, A. (2020). *Rough draft math: Revising to learn*. Portsmouth, NH: Stenhouse.

This book shares multiple techniques to support students in sharing their beginning ideas on a variety of concepts through revision orally and in writing.

Smith, M. S., Bill, V. L., & Sherin, M. G. (2020). *The five practices in practice: Successfully orchestrating mathematics discussions in your [elementary or middle school] classroom*. Corwin and NCTM.

These two pragmatic books focus on how to implement the 5 practices effectively—anticipating, monitoring, selecting, sequencing, and connecting—to make your classroom discussions meaningful, engaging, and effective.

Thiessen, D. (Ed.) (2017). *Exploring math through literature*. Reston, VA: NCTM.

This e-book has dozens and dozens of great literature-based lesson ideas, including student activity pages. You can search by content and by grade. Related articles provide descriptions of how to use the activities and photos and illustrations of how they were used in classrooms.

CHAPTER

4

Planning in the Problem-Based Classroom

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded tasks and reflections, you will be able to:

- **4.1** Explain the features of a three-phase lesson plan format for problem-based lessons.
- **4.2** Design lessons, using a planning process focused on mathematical inquiry.
- **4.3** Identify and implement routines to support core instruction.
- **4.4** Evaluate ways to differentiate a lesson plan to support a range of learners.
- **4.5** Describe ways to engage families and use effective homework practices.

Designing instruction that engages students in problem solving looks quite different from a traditional "explain, then practice" style. The Mathematical Practices such as modeling mathematics, reasoning quantitatively, and looking for generalizations and structure are rarely developed in a tell-students-how lesson design (Munter, Stein, & Smith, 2015). A problem-solving lesson approaches learning as a complex process, builds on each student's prior knowledge, and prioritizes making connections among mathematical ideas. The three-phase lesson format and the process for preparing such a lesson discussed in this chapter are intended to support the creation of equitable mathematics teaching (Chapter 3) that will support the development of mathematical proficiency (Chapter 2) for all learners (Chapter 6).

A Three-Phase Lesson Format

Learning Outcome 4.1 **Explain the features of a three-phase lesson plan format for problem-based lessons.**

A three-phase lesson format provides a structure for lessons where students focus on a topic of inquiry, engage in action, and follow up with discussion, connections, and reflection. We refer to these three lesson phases as *before*, *during*, and *after*—what happens to set up that inquiry, what happens as they explore, and what happens after the task is solved (see Figure 4.1). Of course, having a high-quality task is a necessary element of such a lesson. In this section, we describe each phase's goals, along with appropriate teacher actions and specific examples across the mathematics curriculum.

The *Before* Lesson Phase

The *before* phase is the first 5 to 10 minutes of a lesson and is intended to get students ready for engagement in a focus task or exploration that involves them in higher-level thinking and using the Mathematical Practices. In the *before* phase, you must find the right balance of giving enough structure and guidance so that students understand the intent of the task, without giving so much structure and guidance that you have told students how to solve the task (lowering the cognitive demand). There are three goals in the *before* phase of a lesson.

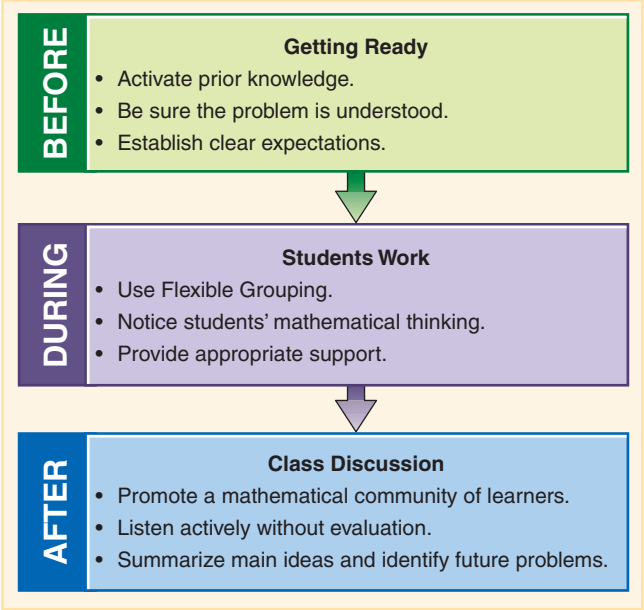


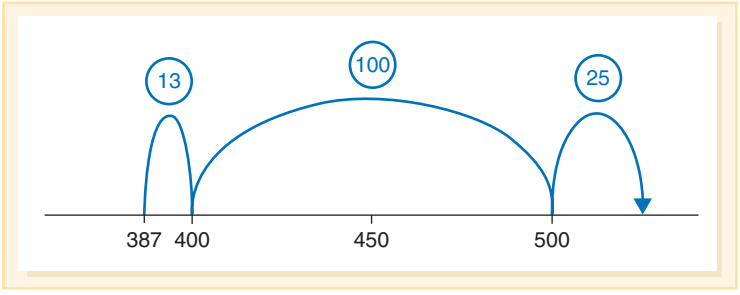
FIGURE 4.1 Teaching through problem solving lends itself to a three-phase structure for lessons

Activate Prior Knowledge. This goal means both to remind students of what they have previously learned and connect to their personal experience as it relates to the day's mathematical learning goal(s) and objective(s). What form this prompting of previous learning might take will vary with the topic. Let's look at an example for a lesson on understanding, Think Addition, as a method for subtracting. The following story connects to the idea of think-addition because it has a context that asks, "How much more?" It is a good choice to explore the idea of think-addition concretely.

Concept: Subtraction (Using Think-Addition) **Grades: 2–3**

Task: To pay for plants for a school garden, the third grade is trying to raise $525. After selling candles, they had raised $387. How much more money needs to be raised?

For this concept, using think-addition efficiently requires knowing combinations that equal 100. Notice that one method to solve the focus task, using a count-up strategy, is this:



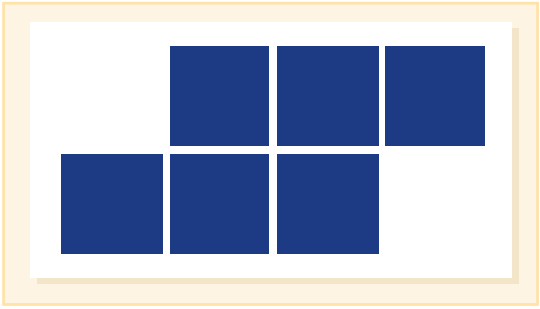
The first step is to know how far 387 is from 400 (a combination of 100). Given that students have already explored combinations that equal 100 (it is prior knowledge), the *before* phase is a good time to activate that knowledge. You might use a routine, "How much to 100?" and invite one partner to write a two-digit number and hold it up while the other partner figures out how much more is needed to equal 100.

Be Sure the Problem Is Understood. You must be sure that students understand the problem or task before they can begin working on it. This does not mean to explain *how* to solve it, rather that students know what the task entails. To do this well, you must first analyze the problem and anticipate student approaches, based on their prior knowledge (Wallace, 2007). Then you consider ways to engage students in thinking about the problem prior to thinking

about how they will solve it. (See Language-Focused Routines later in this chapter for ideas.) Even noncontextual situations must be understood. This problem's context is visual, not verbal:

Goal: Explaining Relationship between Perimeter and Area **Grades: 4–6**

Assume that the edge of each square is 1 unit. Add square tiles to this shape so that it has a perimeter of 18 units.



Understanding five specialized words is necessary to explore this task: perimeter, area, edge, square, and unit. You can activate prior knowledge in one of the following ways:

- Draw a 3-by-5 rectangle of squares on the board and ask students what they know about the shape. (It's a rectangle. It has squares. There are 15 squares. There are three rows of five.) If no one mentions the words *area* or *perimeter*, you could write them on the board and ask whether those words can be used in talking about this figure.
- Provide students with some square tiles and say, "Make a shape that has a perimeter of 12 units." After a short time, have several students share their shapes.

As students explore this initial task, you are helping them understand the context, the essential language, and the meaning of perimeter and area.

Establish Clear Expectations. With the intent of the problem understood, it is to transition to the *during* phase by establishing what's expected. Students need to know (1) how they will work and (2) what products they are to prepare for the after phase. One strategy for ensuring that students have individual accountability while sharing with their classmates is to implement a think-write-pair-share routine (Buschman, 2003). Students are given five minutes of think time and prompted to write down their thoughts. Then, they transition to working with a partner.

These *before* phase goals are not linear and may occur in any order. For example, for some lessons, you might do a short activity to activate students' prior knowledge for the problem and then present the problem and clarify expectations. Other times, you might first explain expectations and then pose an opening task that will revisit relevant, previously learned content as a refresher.

The *During* Lesson Phase

Because it takes time for students to explore (perhaps trying various approaches before finding a solution) and discuss ideas with their partners, this phase is the bulk and essence of a problem-based lesson. Students may engage in the task alone and then with partners or in small groups to explore. You may pause everyone at some point to share a common question or highlight some key ideas that have emerged, and then send the students back to exploring the focus task. In this *during* phase, students need to have access to tools, such as models, images, diagrams, and notation.

Use Flexible Grouping. Collaboration is an important life skill. Students feel that working in groups improves their confidence, engagement, and understanding (Nebesniak & Heaton, 2010). Collaborating on tasks provides students with the opportunity to communicate about mathematics, learn from their peers, and support their peers' learning. When students work with partners or small groups, they are able to use one another as resources, develop

respect among students, and support the development of mathematical agency (Bartell et al., 2017b). In other words, working in small groups supports equitable mathematics teaching. To accomplish such goals requires flexible grouping so that the size and makeup of small groups vary in a purposeful and strategic manner. The goal is to support the learning of every student. In other words, sometimes students are working in partners because the nature of the task best suits two people; other times, they are in groups of four because the task has enough tasks or roles to warrant a larger team.

To be clear, flexible grouping is *not* ability grouping. Ability grouping places those needing more support most frequently in a low group, which inherently has different (lesser) learning goals than students in higher groups. "Any ability grouping in mathematics education is an inequitable structure that perpetuates privilege for a few and marginality for others" (NCTM, 2020, p. 27). Ability grouping puts the students in the low group further behind, positions only some students as mathematically competent, and damages students' mathematics identities and sense of agency (Boaler, 2022; NCTM, 2020). Avoid ability grouping!

Group work should require (1) *individual accountability* and (2) *shared responsibility*. Teach students how to be accountable for their own learning and support the learning of their group members. For example, while setting clear expectations in the *before* phase, you might say, "What might you say to a partner if they don't get the right answer?" and model various responses as well as active listening (Ghousseini, Lord, & Cardon, 2017).

Pearson eText
Video Example 4.1
In this video, Jo Boaler connects ability grouping to students' mindsets.
https://www.youtube.com/watch?v=R4iAwShVIBE

Let Go! Once students understand what the problem is asking, and are in an individual or grouping format, it is time to *let go*. Encourage students to embrace the productive struggle. Doing mathematics takes time, and solutions to meaningful tasks are not always obvious. It is important to communicate to students that spending time on a task, trying different approaches, using multiple representations, and consulting each other are important to learning and understanding mathematics. As groups are working, you can support their interdependence in various ways. When someone asks you a question, for example, rather than answer, pose it to the rest of the group. When observing groups as you move from learner to learner, rather than ask Angela what she is doing, you can ask Bernard to explain what Angela is doing. The more you use these teaching moves (and others like them) that build effective collaboration, the more effectively groups will function and the more students will learn. When the group is stuck, ask questions like:

"What is this problem asking you to do?"

"How have you organized the information?"

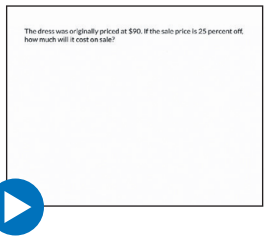
"What about this problem is difficult?" and

"Is there a different strategy (or manipulative) that you might try?"

These questions support student thinking, yet do not tell how to solve the problem.

Letting go includes allowing students to make mistakes and continue to grapple with the problem. Avoid being the source of right and wrong, and instead have students' mathematical reasoning be the source (Bertolone-Smith & Gillette-Koyen, 2019). To respond to questions like, "Is this right?" ask, "How might you decide?" or "What ways can you check your answer?" These responses from the teacher remind students that the first step in ensuring the correctness of an answer lies in the justification, not in the teacher's brain or answer key. In the *after* portion of the lesson, students will have an opportunity to explain, justify, defend, and challenge solutions and strategies. This cyclical process of making mistakes and trying out a strategy that does not work is a natural part of doing mathematics, and you must make it feel comfortable (not awkward) in your classroom.

Notice Students' Mathematical Thinking. Take the *during* time to listen, encourage, and attend to student thinking. "Professional noticing" means that you are trying to understand a student's approach to a problem and decide an appropriate response to extend that student's thinking in the moment (Jacobs, Lamb, & Philipp, 2010). Consequently, your questions should be based on the students' work and responses to your questions. This approach is very different from listening for or leading students toward an answer or approach



Pearson eText
Video Example 4.2
In this video, consider how the student solves the following problem: The dress was originally priced at $90. If the sale price is 25 percent off, how much will it cost on sale? What do you notice about their thinking and how might you support their emerging understanding?

that you have pre-selected or assumed. As you notice the range of strategies students use, consider how they are related and in what order you might sequence the sharing of solutions in the *after* phase (Smith & Stein, 2011). Any of the following probing questions can help you notice what students know and are thinking:

- Tell me what you are doing.
- I see you have started to [multiply/subtract/etc.] these numbers. Can you tell me why?
- Can you tell me more about…?
- Why did you…?
- How did you solve the problem?
- How does your picture connect to your equation?
- I am curious about what you have done here. Will you explain it so I can understand?

Don't be afraid to say you don't understand their strategy. When you are open to learning, you help students feel more comfortable with engaging in the mathematics.

Your actions may inadvertently shut down student thinking. Avoid these two phrases: "It's easy (or simple)" and "Let me help you." Think about the message each one sends. If a student is stuck and you say, "It's easy," then you inadvertently say, "You are not very clever or you wouldn't be stuck." "Let me help you" may communicate that you think the student cannot solve the problem without help. Probing questions like the ones listed here communicate that doing mathematics takes time, actions, and thinking. You are saying, "You can do it—let's see what you know and let's add on to that."

Provide Appropriate Support. Appropriate support keeps students productively engaged with the task, without giving away how to solve it. If a student is stuck, first be sure they understand the task and then ask students what they have tried: "What have you tried so far?" "Does this remind you of another task we've done lately?" "Where did you get stuck?" Third, you can suggest general strategies, for example, "Can you imagine this situation?" "Have you thought about drawing a picture?" "What if you used cubes to act out this problem?" These teacher moves support student thinking (Jacobs & Ambrose, 2008). Consider the following high-quality task:

Concept: Percent Increase and Decrease **Grades: 6–8**

In Fern's Furniture Store, Fern has priced all of her furniture at 20 percent over wholesale cost. In preparation for a sale, she tells her staff to cut all prices by 10 percent. Will Fern be making a 10 percent profit, less than a 10 percent profit, or more than a 10 percent profit? Explain your answer.

Imagine a pair of students is stuck and you stop by their group to help. They say they don't know what to do. How will you respond? Consider the following suggestions:

- "Try drawing a picture or a diagram that shows what 10 percent off and 20 percent more means."
- "Have you tried picking a price and seeing what happens when you increase the price by 20 percent and then reduce the price by 10 percent?"

Notice that these suggestions are not directive but, rather, serve as starters. After offering a tip or suggestion, walk away—this keeps you from helping too much and the students from relying on you too much.

Some students may solve the problem easily. You support these students by extending the task in an interesting way. For example, the furniture problem could be extended by asking for a student to figure out why a 10% increase followed by a 10% decrease doesn't result in the same starting amount. Or ask a student to propose a plan for putting furniture on sale at 10% off and still making a 20% profit from wholesale. For the perimeter task earlier in this chapter, you might ask an early finisher, "Do you think there are other solutions?" This may be enough

to send the students into a flurry of effort to find different ways. You can also extend their thinking by asking questions like:

- Which shape(s) with a perimeter of 18 has the largest area? The smallest area?
- Does the perimeter always change when you add another tile?

Questions that begin "What if…?" or "Would that same idea work for…?" are ways to extend student thinking in a motivating way.

The *After* Lesson Phase

In the *after* phase of the lesson, you facilitate students' engagement in a meaningful mathematical discourse (see Teaching Practices in Chapter 3). Your students will share, justify, challenge, and compare various solutions to the task they have solved. Ideas generated in the *during* phase must have a chance to bump against each other so that mathematical reasoning can emerge (Davis & Simmt, 2006, p. 312). The *after* phase is an important time to make drawings, notations, and writing visible to others; to make connections between the ideas that have emerged; and to create spaces for students to take up, try on, and expand on the ideas of others. It is in the *after* phase where much of the learning will occur, as students reflect individually and collectively on the solution strategies they have explored. The expectations for the *after* phase require careful consideration of possible student responses, recognition of the strategies and representations generated in the *during* phase, and a willingness to be open to unanticipated approaches.

It is challenging but critical to plan sufficient time for a discussion. Twenty minutes is not at all unreasonable for a class discussion. It is not necessary for every student to have finished, but all students need to be prepared to have something to share. This is not a time to check answers, but time for the class to communicate thinking and compare ideas.

Promote a Mathematical Community of Learners. A community of learners is one in which students value the contributions of their peers and recognize the importance of their own contributions to the small group and to the whole class. Creating such a community begins at the start of a school year and involves students in setting norms. For example, you might create a two-column chart on a poster that lists what it means to "Do Math" on one side and "Our Class Norms" on the other side. A community of learners is implicit in the effective Teaching Practices, in particular, the Practices in the Mathematics Teaching Framework box, "Facilitate Meaningful Mathematical Discourse" (see Figure 3.1, p. 33).

Promoting a community of learners in a lesson includes encouraging students to take risks, share their ideas, and respect each other's strategies even when they disagree. You must teach your students about your expectations for having a productive classroom discussion! Bertolone-Smith & Gillette-Koyen (2019) suggest a sharing routine to create a safe environment for students to share: The class listens quietly to a student's justification without interrupting. The class applauds and the student who shared *sits down*. Then the teacher asks, "What good math did you see?" "Do you see something here you want to discuss?" This keeps the focus on making sense of mathematics while positioning students as competent mathematicians.

Listen Actively without Evaluation. A goal of this phase is making students' thinking visible to other students. When you serve as a member of the classroom community and not as the sole evaluator, students will be more willing to share their ideas during discussions. When you say, "That's correct, Dewain," there is no longer a reason for other students to think about and evaluate the response. Had students disagreed with Dewain's response or had a question about it, they will not challenge or question it because you've said it was correct. You can support student thinking without evaluation: "What do others think about what Dewain just said?"

Likewise, use praise cautiously. Comments such as "Good job!" roll off the tongue easily, but do nothing to help the student or those listening know what was good about the work, statement, and idea. In place of praise, Schwartz (1996) suggests comments of interest and extension: "I wonder what would happen if you tried…" or "Please tell me how you figured that out." Notice that these phrases express interest and value the student's thinking. For example, if Ethan is sharing his work (see Figure 4.2), rather than compliment, express

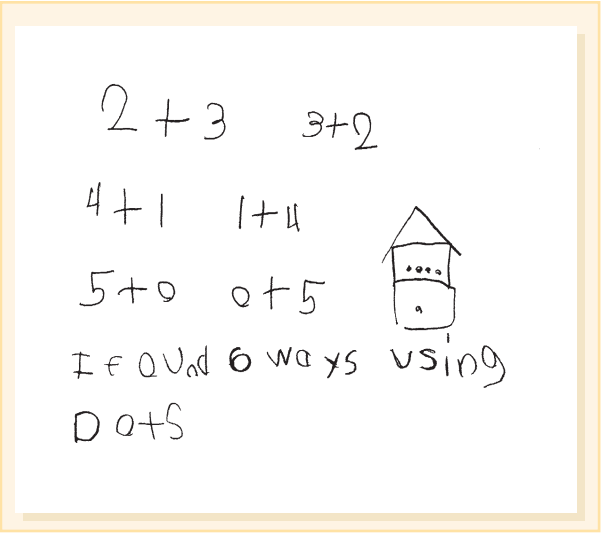


FIGURE 4.2 Ethan's work for "How many ways 5 people might be on two floors of a house."

curiosity: "Are all of these ways different?" and "I wonder if there are other ways?" "How do you know if you have all the ways possible?" These prompts engage all students in thinking about Ethan's solution thereby valuing his thinking and extending everyone's thinking. (See Talk Moves in Chapter 3 for more ideas.)

Summarize Main Ideas and Identify Future Tasks. The *after* phase is designed to formalize the main ideas of the lesson, making connections between strategies or different mathematical ideas. If a task involves multiple methods of computing, list the different strategies on the board. These can be labeled with the student's name and an example. Ask students questions that help them understand the strategies, comparing and making connections among them. It is also a time to reinforce appropriate terminology, definitions, or symbols.

There are numerous instructional strategies for engaging students in describing what they learned in a lesson. A few ideas include a partner exchange (e.g., one partner tells one key idea and the other partner gives an example of that idea), exit tasks (students record written responses to one or two prompts to explain the main takeaways of the lesson), or journal reflections on important information presented. And you can be creative—ask students to write a newspaper headline to highlight the day's mathematics activity and a brief column to describe and summarize it.

Finally, challenge students to think beyond the current task. Each task is just a specific example of a general category of tasks (Goldenberg & Mason, 2008). Ask students to make conjectures and look for generalizations to connect that day's learning to what has been learned before and what they will be doing going forward. For the problem in Figure 4.3, you might ask, "What might happen if there are six people? Ten people? What predictions do you have if there are three floors?"

The Planning Process

Learning Outcome 4.2 **Design lessons, using a planning process focused on mathematical inquiry.**

This section focuses on the intensive *decision-making* that goes into designing instruction. The first column (green) in Figure 4.3 illustrates the decisions to make prior to planning the actual three-phase lesson. None of these decisions is made in isolation. You might decide to adapt the task based on the mathematics in the task or because of your students' learning needs. The assessment decisions take into consideration the task, the students, and the content goals. Once the content and task decisions are made, the three-phase lesson is ready to be designed (see purple-shaded steps in Figure 4.3). Finally, it is important to review and finalize the plan, taking into consideration the flow of the lesson, anticipated student responses, and possible challenges. Here we describe each planning step, using an example lesson titled "Fixed Areas Expanded Lesson," to illustrate the decision-making process.

Pearson eText Expanded Lesson: Fixed Areas

Step 1: Determine the Learning Goals

To begin, it is important to remember the discussions in Chapters 1 through 3 about standards, learning progressions, and Mathematical Practices or Process Standards. Each state has grade-by-grade K–8 standards. Each grade also begins by describing three to five critical

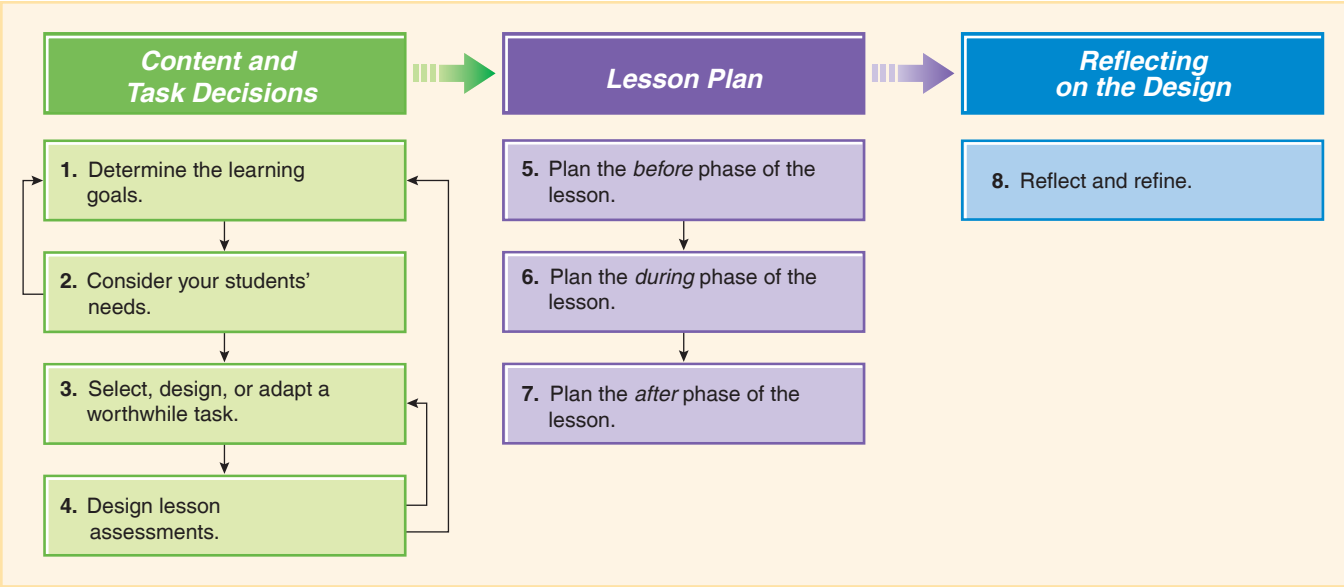

Content and Task Decisions
Lesson Plan
Reflecting on the Design
1. Determine the learning goals.
2. Consider your students' needs.
3. Select, design, or adapt a worthwhile task.
4. Design lesson assessments.
5. Plan the *before* phase of the lesson.
6. Plan the *during* phase of the lesson.
7. Plan the *after* phase of the lesson.
8. Reflect and refine.

FIGURE 4.3 Eight-step process for planning a lesson

areas or big ideas. Focus on these essential understandings that are important for you to know and for your students to learn (a great idea for a bulletin board!) (Kobett, Miles, & Williams, 2018). Content standards are not all of equal importance in the curriculum. Some may be addressed in a few days; others may take over a month. In addition, recall that standards have Mathematical Practices or processes in addition to content. These require specific attention in lesson planning. Finally, on any given day, you are working on a part of that unit goal, which in some way is helping to develop the big ideas of your grade. The key is that students see the lesson as adding to this interconnected whole of big ideas, and not as small, isolated skills (Earnest & Amador, 2018). In fact, mathematics lessons are situated within learning progressions, which are discussed in Chapter 3 and revisited throughout Part II as we share examples of learning progressions for various topics.

With this backdrop in mind, we zoom in on the thinking that goes into lesson planning. It is important to ask:

- What should my students be able to *do* when this lesson is over?
- How does the lesson goal connect to the unit goal? To the big ideas for my grade?
- What content is important?
- What Mathematical Practices/Processes will be developed?

Content Goals: Fixed Areas. A big idea in developing measurement concepts is the relationship between area and perimeter. In looking at the standards for fourth grade, you read, "Apply the area and perimeter formulas for rectangles in real world and mathematical problems" (NGA Center & CCSSO, 2010, p. 25). A possible goal for one lesson on this topic is for students to explore the relationship between area and perimeter—specifically that one can change while the other stays the same.

This goal leads to the development of observable and measurable *objectives*. There are numerous formats for lesson objectives, but the key is that they tell the things you want your students to *do* or *say* to demonstrate what they know. For Fixed Areas, the objectives are:

1. Students will be able to draw a variety of rectangles with a given area and accurately determine the perimeter of each.
2. Students will be able to explain relationships between area and perimeter.
3. Students will describe a process (their own algorithm) for finding the perimeter of a rectangle (MP 7).

As a counterexample, the objective "Students will understand that the perimeter can change and the area can stay the same" is not well-designed because "understanding" is not observable or measurable.

Step 2: Consider Your Students' Needs

What do your students already know or understand about the selected mathematics concepts? Perhaps they already have some prior knowledge of the content you have been working on, which this lesson is aimed at expanding or refining. Examine the relevant learning expectations from previous grades and for the next grade. Be sure that the mathematics you identified in step 1 includes something new for your students. At the same time, be certain that your objectives are not completely out of reach. Attend to the needs of different learners (see Chapter 6).

Questions to consider include:

- What context might be relevant to my students?
- What might students already know about this topic that can serve as a launching point?
- What visuals or models might support student understanding?
- What vocabulary support might be needed?

Student Considerations: Fixed Areas. By grade 4, many students will have had prior experiences with perimeter and area, but may confuse these terms. Some students may think that for a given area, there is only one perimeter (this lesson challenges the incorrect assumption that if you know the perimeter you can predict the area). Students will use manipulatives and test conjectures on how area relates to perimeter.

Step 3: Select, Design, or Adapt a Worthwhile Task

With content goals and students in mind, you are ready to consider what task, activity, or exercise you might use. Chapter 3 provided extensive discussion on what to consider for this step (see Tasks That Promote Problem Solving), so here we simply offer reflective questions:

- Does the task I am considering address the content goals (step 1) and the needs of my students (step 2)?
- Does the task have potential to engage my students in the Mathematical Practices?
- Will the task require students to apply problem-solving strategies?

If the answers to these questions are yes, the task can become the basis for your lesson. You may still decide on minor adaptations, like bringing in a children's book for the initial context or changing the context to connect better to your students. If you find the task does not fit your content and student needs, then you will need either to make substantial modifications or find a new task.

It is absolutely essential that you do the task yourself first. It is only in exploring the task that you can anticipate student approaches and possible challenges, and ensure that the task adequately addresses your learning objectives. Teachers who consider ways students might solve the task are better able to facilitate the lesson in ways that support student learning (Matthews, Hlas, & Finken, 2009; Stein, Remillard, & Smith, 2007).

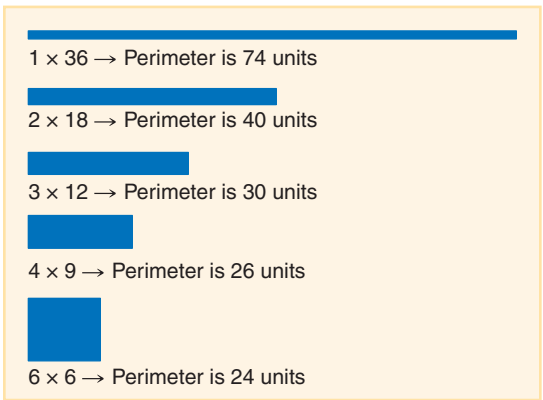

1 × 36 → Perimeter is 74 units
2 × 18 → Perimeter is 40 units
3 × 12 → Perimeter is 30 units
4 × 9 → Perimeter is 26 units
6 × 6 → Perimeter is 24 units

FIGURE 4.4 Possible rectangles with whole-number sides and total area of 36 square units.

Task Considerations: Fixed Areas. A valuable focus of inquiry is posing this task: **What perimeters are possible for rectangles with an area of 36 square units?** Within this broad question, you can ask, "How many are possible? What rectangle will have the smallest perimeter? the largest perimeter? After exploring, you might decide that 36 square units is a good choice because it results in five choices, including a square (see Figure 4.4). Students might wonder whether they should consider 1×36 the same as 36×1 and whether a square is considered a rectangle. You also might want to attend to the fact that as shape becomes more square, the perimeter decreases.

Step 4: Design Lesson Assessments

Thinking about what you want students to experience and how they are going to demonstrate their understanding of the content goals is an important consideration that occurs early in the planning process, not at the end. It is important to assess in a variety of ways (see Chapter 5). *Formative assessment* allows you to gather information that can be used to adjust the direction of the lesson midstream or make changes for the next day as well as inform the questions you pose in the discussion of the task for the *after* phase of the lesson. *Summative assessment* captures whether students have learned the objectives you have listed for the lesson.

Assessment Considerations: Fixed Areas. Three objectives are listed in step 1, each of which needs to be assessed. Here are some possibilities:

Objective	Assessments
1. Students will be able to draw a variety of rectangles with a given area and accurately determine the perimeter of each.	• In the *during* phase, use a look-for tool to observe whether each student can create at least three different rectangles with the same area and accurately find the perimeter. • In the *during* phase, ask each student, "How did you figure out the perimeter of that [pick one] rectangle?"
2. Students will be able to explain relationships between area and perimeter.	• In the *during* phase, ask, "What have you noticed about the relationship between area and perimeter of these rectangles?" and "If you know a rectangle's area, do you know its perimeter?" • At the end of lesson, ask students to complete an exit task (written summary) to explain the relationship between area and perimeter of a rectangle, using illustrations to support their explanation.
3. Students will describe a process (their own algorithm) for finding the perimeter of a rectangle.	• In the *during* phase, ask, "How are you finding perimeter? Are you seeing any patterns or shortcuts? Explain your thinking to me." • In the *after* phase, discuss different strategies students used to find the perimeter.

Step 5: Plan the *Before* Phase

As discussed earlier, the *before* phase should elicit students' prior knowledge, provide context, and establish expectations. Questions to guide your thinking include:

- How might you activate prior knowledge, introduce vocabulary, and/or clarify expectations?
- In what ways can you help students understand the problem? How might you connect the task with students' experiences?
- How will you clarify expectations for the learning?

Students need to know (1) the resources or tools they might use, (2) whether they will work independently or in groups, (3) if in groups, how groups will be organized, including assigned roles, and (4) how their work will be presented (e.g., completing a handout, writing in a journal, preparing a team poster) (Smith, Bill, & Hughes, 2008).

Planning the Before Phase: Fixed Areas. Plan to begin by distributing square tiles to students and asking them to build a rectangle, using 12 tiles (a simpler version of the task). This concrete experience helps to elicit prior knowledge. Clarify vocabulary (perimeter and area) and expectations (that a rectangle showing 3×4 is considered the same as 4×3).

Decide on a context (optional). For example, with a school play on the horizon, you might use the context of a stage, asking students to think about building a stage that has an area of 36 square meters. Consider questions to raise curiosity like, "Does it matter what the length and width are for the stage floor in terms of how much space we have for doing our play? Would one shape of rectangle be better or worse than another for a stage? Find different possibilities and then pick one that will serve the performers best."

Step 6: Plan the *During* Phase

The *during* phase is an opportunity for students to engage fully in the task and for you to notice student thinking and provide support as needed. Decide how you will group students and for how long. Decide what materials to have available. Prepare prompts that can help students get started without taking away the challenge of the task. Prepare extensions or challenges to pose to students.

***During* Phase: Fixed Areas.** Decisions that are made include giving every student about 20 square tiles, providing individual explore time for 10 minutes, and then pairing with a learning partner for five minutes to exchange ideas. Next, students will return to their individual explorations as they look for ways to attend to relationships between area and perimeter as well as consider a general rule for finding perimeter. You also plan to take one trip around the room to see that students are actually building a rectangle and recording its dimensions. On a second round, you will ask the questions you prepared in step 4 for each objective. If students finish early, an extension is to have students apply their conjecture of the best dimensions for a stage that is 48 square meters.

Step 7: Plan the *After* Phase

The *after* phase is when you connect the task to the learning goals. Even if the mathematics is obvious to you, students may complete the activity without making the intended connections. The following questions should be taken into consideration (Smith, Bill, & Hughes, 2008):

1. How will you organize the discussion to accomplish the mathematical goals (e.g., which student-generated solutions will be shared and in what order)?
2. What questions will you ask to help students make sense of the mathematics, make connections to other mathematics, see patterns, and make generalizations?
3. How will you involve all students (over time, not in every lesson)?
4. What evidence are you seeking that will tell you the students understand?

Importantly, the point of the *after* phase is not just to hear student solutions, but to compare and analyze those strategies and representations, making connections and generalizations about the mathematics. Plan time for such comparing and connecting. This is critical to student learning!

***After* Phase: Fixed Areas.** First, ask different groups to draw one of their rectangles, with its measurements, on the interactive whiteboard. Second, ask the following questions:

- What strategies did you use to find the different rectangles?
- What do you notice about these possible rectangles?
- How did you find the perimeters of these rectangles?
- If you were the stage architect, which of the rectangles would you pick and why?

Ask students to complete their exit task (see assessment table in Step 4).

Step 8: Reflect and Refine

A well-prepared lesson that maximizes the opportunity for students to learn must be focused and aligned. Steps 5, 6, and 7 result in a tentative lesson plan. The final step is to review this tentative plan in light of the lesson considerations determined in steps 1–4, making changes or additions as needed. There is often a temptation to do a series of fun activities that seem to relate to a topic but that have different learning goals. Look to see that objectives, assessments, and questions are aligned. If the questions all target only one objective, add questions to address each objective or remove the objective that is not addressed.

Questioning is critically important to the potential learning (see discussion in Chapter 3). Research on questioning indicates that teachers don't ask enough high-level questions—this is your chance to review and be sure that you have included some challenging questions that ask students to extend, analyze, compare, generalize, and synthesize. And be sure you have a plan to make sure *each* student is thinking about and responding to your high-level questions.

Reflect and Refine: Fixed Areas. This lesson is well-aligned, with a worthwhile task, a lesson plan that engages students in mathematical reasoning, and formative assessments aligned to each objective. Higher-level questions based on the objectives are posed to students in the *during* and *after* phases. Additional questions to extend learning are needed:

> Is there a number of tiles for which there is only one option for building a rectangle? What is its area and perimeter? Are there other examples?
>
> For what number of tiles (fewer than 100) are there the most possible rectangles?
>
> If you make a collection of rectangles with the same perimeter, can you predict the area for each?

For an At-a-Glance list of reflection questions across the eight steps for all students, with additional considerations for multilingual students, see the Reflective Planning Questions for Planning Teacher Resource.

Pearson eText Teacher Resource: Reflective Planning Questions for Mathematics Teaching

Routines and Other Worthwhile Learning Experiences

Learning Outcome 4.3 **Identify and implement routines to support core instruction.**

Mathematical routines are generally short experiences with a specific protocol or structure that is intended to strengthen student fluency, reasoning, justification, and sense-making (McCoy, Barnett, & Combs, 2013; Newell & Orton, 2018). Most routines are designed to be 10–15 minutes and are commonly used as a supplement to the mathematics lesson. In other words, a mathematics lesson might be: routine-before-during-after, but a routine can effectively serve as a launch to a full lesson, thereby becoming a component of the before phase of a lesson. Or, your mathematics routine may be tucked into a different time in your school day. Centers and special lessons (e.g., Three-Act Math tasks) are also important learning experiences.

Because routines involve student reasoning and listening to the reasoning of others, it is worthwhile to spend time establishing norms and building a safe space for students to share their ideas. Many team-building activities are available online—for example, *Team Building Games on a Shoestring* (a free downloadable PDF) (Heck, 2006) and "28 Awesome Team-Building Games and Activities for Kids" (Mulvahill, 2019), accessed at https://weareteachers.com.

This section highlights effective experiences that may be used to supplement, support, or extend your mathematics lessons.

Language-Focused Routines

Language-focused routines help students consume a problem without telling them how to solve it. As such, they appear in the *before* phase of a lesson and then may reappear in the other phases. One widely used and effective routine is Three Reads (Kelemanik, Lucenta, & Creighton, 2016). The routine can vary slightly in implementation. Two common options are shared in Figure 4.5. The first option is a good way to begin: The question is missing, so

students can't rush ahead to solving the problem. The second option reflects what mathematicians really do to make sense of story problems.

Try out one of the Three Reads processes on the following story. (If Option 1, don't read the last sentence.)

Trading Cards Problem

The local toy store purchased tradeable cards in large cartons because they got a discount. The owner, Ms. Sweets, read on the box that each carton included 12 boxes of cards. She opened up one of the boxes and found that each box had 25 packs of cards in it. She was very happy because she paid $450 for the carton. She is wondering how much she paid for each pack of cards, and then how much to charge her customers if she plans to charge them twice as much as she paid. What was Ms. Sweet's buying price and what should be her selling price?

	Option 1: Remove the Question	**Option 2: Read Like a Mathematician**
Read 1 (Teacher Reads)	Students listen in order to share, "What is happening in the story?" (ignoring any numbers).	
Read 2 (Student reads)	Listen to answer: What quantities are in this problem?	Listen to answer: What are we being asked to figure out?
Read 3 (Different student reads)	Students listen to share: What problems could we solve with the quantities we have?	Students listen to share: What quantities do we need from this story to answer the question?

FIGURE 4.5 Two options for the Three Reads process in mathematics

Understanding the problem requires knowing the mathematical meaning of the terms in the task. For the Trading Cards problem, the meaning of packs, boxes, and cartons is important to making sense of the nesting of containers. The first read of the three reads protocol can help surface these terms so that language support can be embedded in reasoning about the task. The second and third reads help students understand what the task entails (what they are being asked to find, and what information they need from the story).

Other routines help students engage in the language and content of mathematics. The https://www.fosteringmathpractices.com/ site offers ready-to-go presentation slides and other visuals to support *Three Reads* as well as several other routines. The Stanford Center for Assessment, Learning and Equity has prepared a collection of routines that support sense-making, cultivate conversations, and engage students in metacognition. For more information, go to http://achievethecore.org/ and look for Developing Math Language Routines or search for the Stanford document, which is a downloadable PDF. Using these routines supports effective mathematics teaching.

Number Talks

Number talks are a powerful routine that customarily focuses on one to four problems to be solved mentally using reasoning. It is a brief (5–15 minutes) routine intended to develop procedural fluency (efficiency, flexibility, and accuracy) (Parrish, 2014; Parrish & Dominick, 2016). Many teachers use number talks as a jumping-off point, inviting students up to the carpet and posing each equation one by one on the board or on chart paper. The routine is provided in Figure 4.6. Many useful protocols and slides can also be found through online searches.

Number Talk Routine

1. Teacher reveals the first expression.
2. Wait time. Students silently and mentally solve the problem, giving a thumbs up on their chest when they have figured out a method. Teacher waits for each student to be ready.
   - Students with one way to solve keep thinking of more ways, adding a new finger with each additional way they have found. Three fingers means three ways to find the answer.
3. Teacher asks, "What is the answer?" and records *all* answers.
4. Teacher invites a student to explain, recording their idea, using expressions or equations. Teacher asks for a student who solved the problem *a different way*, and again records their method. This may go on for three or four methods, depending on the problem.
5. After listening to methods, students look back at the answers and reflect on which one is correct.
6. Repeat for the next problem in the set.
7. After the set, teacher summarizes (or asks students to summarize) what they noticed across the problems.

FIGURE 4.6 Steps for a number talk routine

In the following table are two number talks that focus on reasoning strategies.

NUMBER TALKS: WHOLE-NUMBER EXAMPLES

Talk 1: Addition Strategies	Talk 2: Multiplication Fact Strategies
1. $39 + 21$	1. $4 \times 7 =$
2. $28 + 34$	2. $6 \times 4 =$
3. $48 + 53$	3. $6 \times 8 =$
	4. $9 \times 4 =$

In step 4, the teacher records student thinking. For example, for the first addition problem, students might solve it in any of these ways:

- $39 + 21 = 40 + 20$ [Student explains that they moved "1" over to "make 40."]
- $40 + 21 = 61$, $61 - 1 = 60$ [Student explains they pretended the 39 was a 40 to make the adding easier, then removed the extra 1 from their answer.]
- $30 + 20 = 50$ and $9 + 1 = 10$, $50 + 10 = 60$ [Student explains they broke apart their tens and ones to add and then put them back together.]
- Start at 39, 49, 59, 60. [Student explains they counted on by 10s and then 1s.]

In Step 7 (summarizing), the teacher might ask, "How are our equations alike or different?" or "Which strategies would you like to remember to use when you are doing addition problems this week?"

Students must also develop flexibility with fractions and decimals! Consider a (rational) number talk for this problem: $3\frac{1}{2} - 2\frac{5}{8} =$ _____. How might students solve it? One way might be to think of one-half as four-eighths and then subtract with regrouping, but mentally, there are more efficient strategies. For example, a student might notice that $2\frac{1}{2}$ and $3\frac{1}{2}$ are one apart (e.g., one inch), but the difference is $\frac{1}{8}$ less than one whole, so $\frac{7}{8}$. Another student might count up three-eighths from $2\frac{5}{8}$ to 3 and then four more eighths to different-sized fractions. Students may be using a number line to visualize or solve the task (concretely or mentally), or picturing circles or other manipulatives.

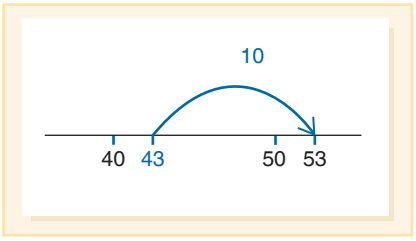
Number Strings

Number strings are much like number talks, with carefully selected problems revealed one at a time in a planned sequence. But whereas number talks invite a range of strategies, number strings are designed to highlight a particular strategy or big idea (Bofferding & Kemmerle, 2015; Lambert, Imm, & Williams, 2017). Mathematical models (e.g., number lines) are central to a number string routine. Let's say that instead of focusing on a range of strategies for

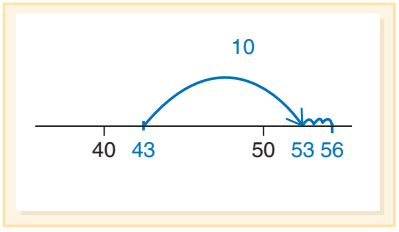
adding two-digit numbers, a teacher wants to focus on a counting-up strategy. She selects the following number string (that links counting by 10s only to counting by 10s and 1s).

(1) 43 + 10 (2) 43 + 13 (3) 43 + 19 (4) 38 + 40 (5) 38 + 41 (6) 38 + 39

She follows the Number Talk protocol and uses an open number line to record students' thinking:

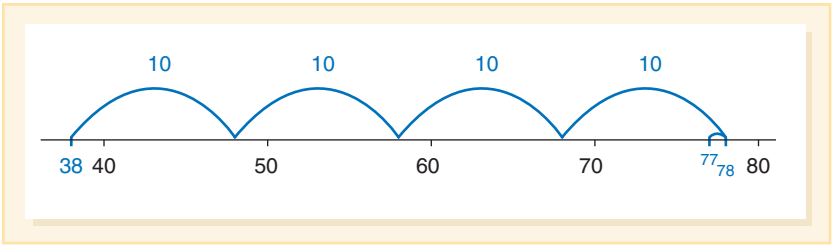


(1)

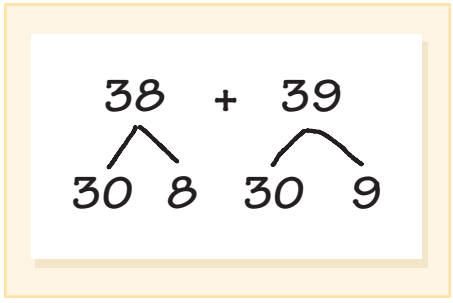


(2)

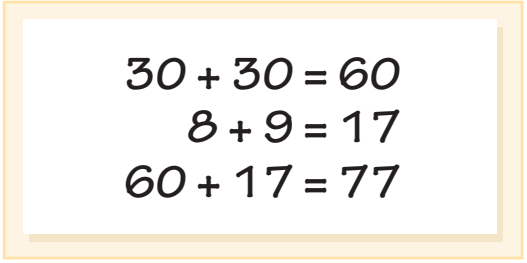
The last three problems help students make sense of the compensation strategy. Problem 6, for example, can be solved by adding 40 and subtracting 1:



During the number string talk, students are likely to share other methods, which will also be recorded using models. For example, if a student describes a break-apart method for (6), the teacher might write it like this:



and then record the related equations:



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Video Example 4.3

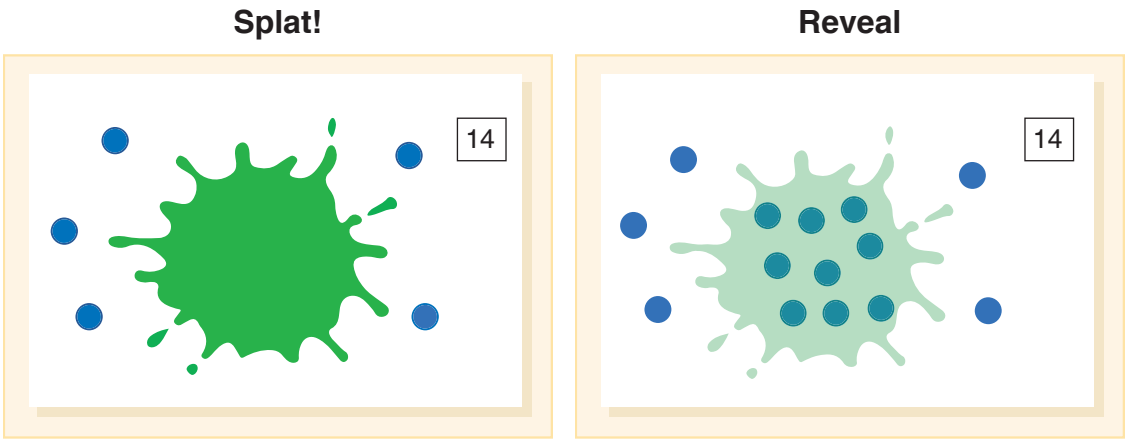
In this video, consider how these kindergarten students engage in a number talk using dot cards.

https://www.youtube.com/watch?v=DPChN-LdvDQ

The patterns and relationships built into number strings make strategies visible to students and are therefore a critical tool for developing students' strategic competence and procedural fluency.

Splat!

Splat! (Wyborney, 2017) is a routine in which visuals of a splat cover up part of the quantities, therefore requiring reasoning to solve the task. Here is an example of a Splat!



You begin with showing this visual and saying, "Splat," and then ask how many are under the splat? How do you know? Then a reveal shows what is under the splat. Finally, you ask, "What can we learn from this picture?" (Wyborney, 2017). These creative visuals are very engaging to students, and many are available for free on the related website at https://stevewyborney.com/.

Which One Doesn't Belong (WODB)

This routine involves four pictures, numbers, graphs, expressions, equations, or shapes placed in a two-by-two grid. Ask students, "Which one doesn't belong? Why?" Provide think time. Students should know that *each* of the four has one reason it doesn't belong. So, they continue to think of reasons to share why each element doesn't belong. An example for fractions is provided here:

$\frac{3}{4}+\frac{5}{8}$	$\frac{5}{6}+\frac{2}{12}$
$\frac{1}{4}+\frac{1}{8}$	$\frac{1}{5}+\frac{1}{10}+\frac{1}{2}$

Pause & Reflect

What justifications can you generate for each of the squares in this WODB? ●

There is no single answer for each of these, but a sampling of responses includes: Top left is the only answer greater than 1; top right is the only equation equal to a whole number or the only one based on twelfths; bottom left is the only sum less than $\frac{1}{2}$. Bottom right is the only one with three addends or the only one with tenths as the size of pieces to add or the only one wherein more than one fraction needs to be changed to an equivalent version. Many more of these are available at wodb.ca, and you can add your ideas there too!

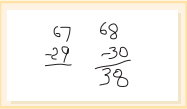
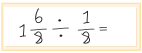
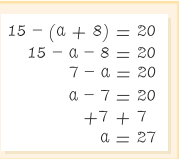
Correct Worked Example	Partial Worked Example	Incorrect Worked Example
Jaime correctly solved this subtraction problem: Problem: 67 – 29 = _____  Ask: What did Jaime do? Why does Jaime's strategy work? When is it a good idea to use this strategy?	Problem: $1\frac{3}{4} \div \frac{1}{8} =$ Annie started solving this problem:  Ask: Why did Annie do this as her first step? What does she need to do to finish it? When would you use Annie's strategy?	Shawn made a mistake in solving this problem:  Ask: What is correct? What might Shawn be thinking? What is incorrect and why? How can the error be corrected?

FIGURE 4.7 Three types of Worked Examples.

Worked Examples

Like it sounds, a worked example is an already-solved (or partially solved) task from an anonymous student (or one of your students). Worked examples should include correct, partially finished, and incorrect examples (See Figure 4.7).

Using worked examples has been found to improve student procedural and conceptual knowledge (Booth, Lange, Koedinger, & Newton, 2013; Star & Verschaffel, 2016). And using incorrect worked examples is particularly beneficial for students with special needs or students who struggle with mathematics (Booth, Cooper, Donovan, Huyghe, Kodeinger, & Paré-Blagoev, 2015). Worked examples can bring to light alternative solution strategies or common errors and misconceptions or introduce new content (Barlow, Gerstenschlager, & Harmon, 2016; McGinn, Lange, & Booth, 2015). A worked example could be used in a number talk, in a full lesson, or as homework. For example, a homework problem could be a worked example in which students are asked to identify the error and then solve the problem correctly (Walk & Lassak, 2017). To design an effective worked example:

- Analyze the topic you are teaching and anticipate important strategies and common errors.
- Write or find a problem that highlights the strategy or common error.
- Complete the problem (as an anonymous student) in a way that brings attention to the strategy or error.
- Construct questions that require students to explain the strategy (how it works, why it works, when it works) or mistake (what it is, why it doesn't work).

After students have had an opportunity to complete and/or discuss the worked example, give students related problems to solve.

Three-Act Math

Three-act math tasks are not short routines but a full lesson, playing out across three acts (like lesson phases!). Three-act tasks engage students in mathematical modeling, which includes identifying mathematical problems in our world, gathering information and determining which details will help us solve a problem, and developing and revising mathematical models of situations. Even very young children benefit from modeling, using manipulatives, diagrams, and equations to represent the situation (Suh & Seshaiyer, 2017; Lomax, Alfonzo, Dietz, Kleyman, & Kazemi, 2017). In fact, three-act math, with its use of visuals and connection to real life, are excellent lessons for young children. In Act 1, the teacher shares a highly engaging visual context, oftentimes a short video. *Students*

generate questions and conflicts from the video (or picture). Students make assumptions, predictions, and estimations and eventually settle on a problem to solve. In Act 2, students identify the variables (unknown quantities) needed to solve their problem and define a solution path. Act 3 concludes with a reveal through digital media, which is followed by sharing out and discussing the mathematics. Excellent three-act math tasks can be found at https://blog.mrmeyer.com/, http://www.gfletchy.com, and http://www.101Qs.com. Also check out Dane Ehlert's blog post, "Promoting Growth Mindset with 3-Act Math" on NCTM's website (www.nctm.org), where more links to three-act math tasks are provided (Elhert, 2015). Finally, once you get the idea of three-act math, create your own to connect to your own students' interests and local community.

Estimation 180

Estimation 180 (www.Estimation180.com), like three-act tasks, begins with a visual (a picture) and ends with a reveal. The focus, as you might guess, is on estimation. Highly engaging daily problems are carefully sequenced so that students can use their reasoning from previous days. Students learn strategies for estimation, for example, determining a reasonable range for their answer (it must be greater than _____ and less than _____). Lesson plans are provided, and new content is added regularly.

Learning Centers

Learning centers are a conceptual tool used to create stations where students focus on different ways of making sense of a concept. They are also concrete locations in the classroom. To make learning centers concrete, you may even arrange the classroom so each corner represents one of the stations. Setting up multiple centers is an excellent idea for the *during* phase of a lesson, when there are multiple ways to represent a concept. For example, knowing that a number can be decomposed into two other numbers is an important relationship for young children to understand. To develop this understanding, students need multiple opportunities to create part-part-whole relationships, using different objects. A lesson might focus on the number 6 and include four learning stations:

Station 1: Students sort 6 counters onto two plates, showing how 6 children might be in two classrooms.

Station 2: Students create towers that are six cubes tall and write how many of each color they used.

Station 3: Students find dominoes that equal 6.

Station 4: No tools are available, just paper and crayons. Children are asked to find ways two friends can share 6 plums.

The *before* phase is important when using learning centers, because it is still important to elicit prior knowledge, ensure that the task is understood, and establish clear expectations. It may include modeling what happens at each center. The *during* phase is still the time when students engage in the task, but they are stopping and rotating to new centers within this phase of the lesson. It is important to ask questions and keep track of strategies students are using to highlight later. In the *after* phase, you may decide to focus on one particular center, or you may begin with the center that was the least challenging for students and progress to the one that was most challenging.

A math learning center is also a single location in the classroom with an engaging and independent task, designed to provide an extra challenge or reinforcement of concepts. Learning centers for independent work need tasks that are motivating and can be used over and over. Games serve this purpose, in particular games that can be played with one or two players. Technology applications can also be the focus of an independent learning center. With games and apps, the key is that they engage students in reasoning and sense-making, not rote practice.

Differentiating Instruction

Learning Outcome 4.4 **Evaluate ways to differentiate a lesson plan to support a range of learners.**

Perhaps the most important work of a teacher is to be able to plan (and teach) lessons that support and challenge *all* students to learn important mathematics. The three-phase lesson plan is highly effective in meeting the needs of a wide range of learners. Students are asked to make sense of the task (*before*), bring their own skills and ideas to the problem to solve it in their own way (*during*), and then articulate their thinking as well as hear the thinking of others (*after*). In contrast, in a traditional, highly teacher-directed lesson, the assumption is made that all students will understand one approach. Students who may not be ready yet to understand the ideas presented must focus their attention on following rules or directions in an instrumental manner (i.e., without a conceptual understanding). In addition to using a problem-based approach, there are specific adaptations that can meet the needs of the diverse learners in your classroom; this is the focus of Chapter 6. This section focuses on ways to make a task accessible and challenging so that every child has access to the task *and* is challenged. This is sometimes referred to as a low floor (everyone can engage in the task) and high ceiling (extends to challenge anyone).

Opening Closed Questions

Many questions in textbooks are closed, meaning there is one answer and, often, only one way to get there. Such a task cannot meet the needs of a range of learners in the classroom. *Open questions* are broad-based questions that invite meaningful responses from students at many developmental levels (Small, 2017). For example, one number is about one-half of another. Their product is close to 100. What might the numbers be?

You can make a closed task open—for example, rather than ask students to add $8 + 12$, say, use a reversibility question and say, "The answer is 20, what is the equation?" You can give clues and ask students to write an equation that fits the clues. Note that once a problem is turned into an open task, it could be used as a number talk!

Open questions are particularly useful in turning low cognitive demand tasks into high cognitive demand tasks. Compare this pair of tasks:

Task A

Write 35,045,011 in scientific notation.

Task B

Write two numbers that are more easily compared when in scientific notation. Write two numbers that are difficult to compare when in scientific notation.

Differentiating Tasks

Differentiating tasks does not lower the cognitive demand but, in other ways, alters the task to meet students' learning needs. For example, in a class of first graders, some students may be working on single-digit addition while others are working on adding larger values. A teacher adapts Task A to Task B:

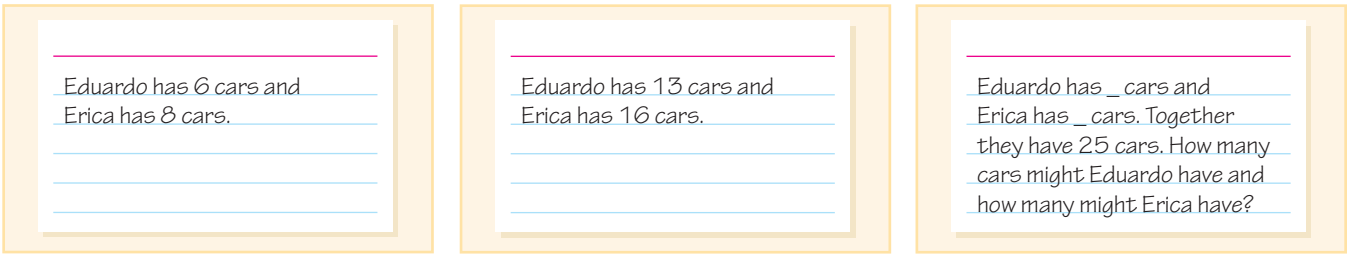
Task A

Eduardo had 9 toy cars. Erica came over to play and brought 8 toy cars. Can you figure out how many cars Eduardo and Erica have together? Explain how you know.

Task B

Eduardo had some toy cars. Erica came over to play and brought her toy cars. Can you figure out how many cars Eduardo and Erica have together? Explain how you know.

The teacher then asks students what is happening in the story and explains that they will be given more information. Then he distributes different task cards to different students that tell how many cars Eduardo and Erica have.


Eduardo has 6 cars and Erica has 8 cars.
Eduardo has 13 cars and Erica has 16 cars.
Eduardo has _ cars and Erica has _ cars. Together they have 25 cars. How many cars might Eduardo have and how many might Erica have?

The teacher can scaffold this task, having each student do all three cards; can give students a choice of card; or can assign cards.

The following example illustrates how to differentiate a lesson by adapting how much *structure* is provided. Notice that all tasks focus on the same learning goal of identifying properties of parallelograms.

Topic: Properties of Parallelograms **Grades: 5–6**

Students are given a collection of parallelograms, including squares and rectangles as well as nonrectangular parallelograms. The following task cards are distributed to different groups based on their learning needs and prior knowledge of quadrilaterals:

- **Option A, open-ended: Explore the set of parallelograms. Measure angles and sides, using your ruler and protractor. Make a list of the defining properties that you think are true for every parallelogram.**
- **Option B, slightly structured: Use your ruler and protractor to measure the parallelograms. Record any patterns or attributes that are true for all of the parallelograms related to sides, angles, and diagonals.**
- **Option C, highly structured: Explore the parallelograms to find patterns and rules that define the shapes as parallelograms. First, sort the parallelograms into rectangles and non-rectangles. Use a ruler to measure the sides and a protractor to measure the angles.**

1. **What pattern do you notice about the measures of the *sides* of all the parallelograms in the *non-rectangle* set?**

What pattern do you notice about the measures of the *sides* of all the parallelograms in the *rectangle* set?

2. **What pattern do you notice about the measures of the *angles* of all the parallelograms in the *non-rectangle* set?**

What pattern do you notice about the measures of the *angles* of all the parallelograms in the *rectangle* set?

Incorporating Choice

Choice tasks, or parallel tasks, involve a story problem or task where students choose the values they want to use (Small, 2017). In addition, the series of options can all be solved as they get progressively more challenging (like a number string). Giving a choice increases student motivation and helps students become more self-directed learners (Bray, 2009; Gilbert & Musu, 2008). Examples are provided in Figure 4.8 for various topics in the curriculum. Notice that the way choices are offered varies, but they do become more challenging as they go (Empson & Levi, 2011; Phelps, 2012).

Topic	Grade	Task
Subtraction	2–3	Eduardo had {12, 60, 121} marbles. He gave Erica {5, 15, 46} marbles. How many marbles does Eduardo have now?
Equal sharing	1–6	Isabelle has _____ pounds of popcorn and she is sharing it equally with her friends. Altogether _____ people are sharing the popcorn. If no popcorn is left over, how much popcorn will each person get? a. 4, 8 b. 2, 8 c. 6, 8 d. $4\frac{7}{8}$, 8 e. $4\frac{4}{5}$, 8
Fraction addition	4–7	Camila is making trail mix. She puts in some raisins and _____ cups of nuts. In total she has _____ cups of trail mix. How many cups of raisins did she put in the trail mix? a. $\frac{3}{4}$, $1\frac{1}{2}$ b. $\frac{5}{8}$, $2\frac{1}{4}$
Scientific notation	7–8	Select one of the following numbers and represent it in at least four ways. 3,500,000,000 0.0035

FIGURE 4.8 Sample choice tasks for various topics

Planning for Family Engagement

Learning Outcome 4.5 **Describe ways to engage families and use effective homework practices.**

As you have read throughout Part 1 of this book, effective mathematics teaching and learning must engage students in the Mathematical Practices, placing you (the teacher) in a position to support their learning through a before-during-after, problem-based lesson format that implements the effective Teaching Practices described in Chapter 3. Despite the fact that these recommendations have been around for more than a generation, most adults have experienced mathematics learning as memorizing algorithms presented by the teachers. Some parents/caregivers disliked such learning, while others liked it. Regardless, families know the importance of mathematics for their child's future. To partner with you in a student's learning, families must understand why you are approaching mathematics in the way that you are. Furthermore, they need to understand how important their positivity toward mathematics is for their child's success, because positive parent emotions lead to positive student emotions, and positive student emotions are connected to better performance (Else-Quest, Hyde, & Hejmadi, 2008).

Connecting with Families

Ways to engage families include math-specific messaging on back-to-school nights, inviting families to participate in your math lessons, letters explaining the mathematics approaches being used, and hosting family math nights.

Back-to-School Nights. Back-to-school nights are short opportunities to speak to families, giving them a first impression about what their child will experience with you in the upcoming year. Take this time to illustrate the importance of the Mathematical Practices! Here is what not to do: "This year, we will be working on Mathematical Practices, which means your child is going to be figuring out how to do the math on their own." This statement communicates several red-flag messages. The term *Mathematical Practices* may sound ominous; it is something new, which may mean it is experimental and may also sound like the children will be fending for themselves. Instead, do a quick and very open task with attendees. You could do a number talk or any of the other routines described earlier. Or,

you can simply pose some recognizable procedures that are important at your grade and that lend themselves to reasoning. For example, if you are a first- or second-grade teacher, you might pose these:

1. $98 + 37$
2. $49 + 48$
3. $302 - 294$
4. $456 - 360$

Give a card with these tasks on it and some think time. Then ask them to meet a new person in the room and pick one of these to share how they solved it. In a few sentences, you can share that being a good math thinker means you look at the numbers in the problem and then decide on an efficient way to solve it. That is what you will be working on all year with their children, and when students choose their methods, they demonstrate Mathematical Practices.

Family Math Events. With more time than a back-to-school night offers, a mathematics event for families provides an excellent opportunity to engage them in doing mathematics as a community of learners. A *math orientation workshop* introduces families to big ideas for an upcoming unit (Ernst & Ryan, 2014). Providing opportunities for parents or caregivers to learn about the mathematics prior to their children learning it can lead to increased relationships with these adults and increased achievement by their child (Knapp, Jefferson, & Landers, 2013). Certainly, it goes a long way in decreasing homework stress on both sides. Scheduling of these events is critical to their success, so schedule several workshops and invite families to attend any or all of them (Tobon & Hughes, 2020). And you can record sessions or offer webinars to increase convenience for families.

A family math event can also focus on what it means to do mathematics (the Mathematical Practices or strands of mathematical proficiency). You can use a pair of tasks, like the ones in Figure 4.6, or you can pose a single open task. Engage families in solving the task with their children. Debrief by highlighting the reasoning involved. For example, you can invite families to solve and compare two sets of tasks, as illustrated in Figure 4.9.

Pause & Reflect

What distinctions do you notice between the two tasks in Figure 4.6? What is valued as doing mathematics in both of the problems? ●

TASK 1

Find the answer to these equations. Use counters or draw pictures to show your work.

$1 + 5 =$ _____ $0 + 6 =$ _____ $5 + 1 =$ _____

$3 + 3 =$ _____ $2 + 4 =$ _____ $4 + 2 =$ _____

TASK 2

The parking lot has only blue and red cars. Six cars are parked. How many blue and how many red cars might be in the parking lot? Find as many ways as you can. Draw a picture and write an equation for each answer.

Extensions: Can you find all of the ways to have six red or blue cars in the parking lot? How many ways can you find to have a total of five cars? How many ways can you find to have a total of seven cars? Do you see a pattern?

FIGURE 4.9 Two tasks to explore at a kindergarten or first-grade family event.

Give participants time to do both tasks and discuss solution strategies (as you would with students); ask participants to consider the learning opportunities in the two contrasting tasks. Ask questions such as these:

- What do you notice about these two types of tasks?
- What content is learned in each? What thinking is involved in each?
- Which task would your child be most motivated to solve? Why?
- Which develops your child's reasoning skills?

Help parents or caregivers identify the important content. For example, in grades K–2, children are building important foundations of numbers and operations through algebraic reasoning—looking for patterns, reasoning, and generalizing. The NCTM book, *It's Elementary: A Parent's Guide to K-5 Mathematics* (Whitenack, Cavey, & Henney, 2015) is a wonderful resource that explains what math learning looks like and why. Consider having a copy for parents or caregivers to look at and even check out. To help you prepare for questions caregivers might have, see our Teacher Resource: Frequently Asked Questions, and personalize the ideas shared therein.

Pearson eText Teacher Resource: My Child's Math: Frequently Asked Questions

Beyond the learning opportunities, family math events must be low stress and fun! Play games that can be taken home and played again; invite students and their families to participate in estimation stations, wherein they may win if they land in a reasonable range; and offer food during the event.

Math Lesson Participation. Invite parents/caregivers into your mathematics lessons (elementary) or to view video clips from lessons (middle and elementary). Families can see the value of engaging students in the Mathematical Practices, learning new mathematics themselves, and reinforcing the kind of questioning and discussions you are modeling as they engage with their child at home. If videos are problematic, using photographs of students' work are excellent ways for families to see what is going on and notice the kind of thinking their child and other children are doing.

Newsletters. Face-to-face encounters or even webinars only reach some of your families. To build a strong relationship with every family, you need a variety of ways to communicate, and the way you share information must be culturally responsive (see Chapter 6). Newsletters can share key ideas for a mathematics unit or offer parents/caregivers suggestions for supporting their child in doing mathematics. Figure 4.10 is a sample newsletter that encourages families to engage in informal, sense-making fraction activities. It communicates the importance of fractions and the prevalence of fractions in the real world and offers ways to help build a strong foundation without having to do worksheets (and you can have it translated for multilingual families).

Pearson eText Teacher Resource: Sample Math Newsletter

Positive Notes Home. Every parent or caregiver wants to hear good news about their child! Catch your students doing something mathematically powerful or identify their math strengths and send that information home. You can create or buy note cards with a template to make it efficient for you to create the note. Attach the good thinking (the student's work or a picture of them doing the work). An email works, of course. But, it is very special when a child can pull a success card or math strength card out of their backpack or pocket. For families with a native language other than English, use a translation application.

Homework Tips

Research results on the value of mathematics homework have varied, but a recent meta-analysis of 30 years of research on the impact of homework on student achievement in mathematics found small positive gains on student achievement (Fan, Xu, Cai, He, & Fan, 2017). With homework, effort and parental involvement matter, not the amount of time doing homework. As you plan for experiences at home, consider the time caretakers may or may not have to

Gaining Traction with Fractions

Did you know that when you are sharing food, you are working with **fractions?** Making sense of fractions and feeling confident with fractions is a big idea for us this year. Besides fractions leading to success in Algebra and beyond, we use fractions all the time. To gain traction with fractions…try out these activities. Invite your child to **share** what you have done (or tell me yourself!).

Pose "Sharing" Problems. Sharing is the beginning of understanding fractions. An example: How much brownie do we each get if we both share 5 brownies? What if our family (of 4) equally shares 5 brownies? Keep going…equally sharing sandwiches, cookies, oranges, quesadillas, juice, playdough, injera (Ethiopian bread). Whatever you might share in your home!

Tell Stories. Share a math moment at dinner (or in the car). When have you used math today (shopping, laundry, budgets, etc.)? Think of the many things you might have estimated – how long it will take to get to work, or to run a series of errands). Take turns sharing stories. We will share family math moments in class!)

Chores. Yes, chores! If it takes $\frac{3}{4}$ of an hour to do a load of laundry, how long with 3 loads take? If you walk the dog for $\frac{1}{4}$ hours twice each day, how many hours is the dog walked in a week? A month? If you earn $5 an hour walking dogs, what might you earn in a week?

George photo cm/Shutterstock

Gaining Traction on the Road. Riding in the car can be more interesting if there are things to look for…consider challenging your child to look for fractions (or equivalencies like decimals on signs.). See if they can find a fraction equivalent to $\frac{1}{2}$ from a license plate in front of you (e.g., the Illinois plate has $\frac{1}{2}$, $\frac{6}{12}$, $\frac{8}{16}$, and 0.5 (if you include decimals). Be creative! For example, perhaps two numbers can be added together to form a numerator or denominator.

FIGURE 4.10 Sample Math Newsletter

help their children, remembering that students may have a single caregiver, other siblings in the household, and/or caregiver(s) with evening work hours. Be strategic with what you send home. Homework must be a positive experience for students and their families! To support homework experiences, implement a plan that parallels the lesson phases:

Before: Introduce and/or start the homework before it goes home.

During: Independent work done at home (alone or with parents/caregivers).

After: The next day. Far more than checking answers, this is a time to share and compare answers. This time is worthwhile when the focus is on comparing across problems (rather than on individual problems) and focusing on errors and difficulties (rather than on correct answers) (Otten, Cirillo, & Herbel-Eisenmann, 2015).

Following are a few tips for getting the most out of homework.

Help Parents/Caregivers Know How to Help. Caregivers want to help, so providing guidance on how they can best support their child is important. Many parents/caregivers have math anxiety from their own mathematics learning. When parents are more math anxious and they help their child with homework, their child learns significantly less math over the school year and have more math anxiety by the school year's end (Maloney, Ramirez, Gunderson, Levine, & Beilock, 2015). Yikes! We need to remove anxiety-inducing homework experiences and replace them with positive and meaningful interactions. Explicitly and kindly communicate this reality: that when they show anxiety or stress about math, it can have a negative impact on their child's learning. Then offer ways to engage positively and avoid stress!

Ask parents/caregivers to implement an ask-before-tell approach: Before explaining something, ask their child to explain how he or she did it (Kliman, 1999). The child may self-correct (a life skill), and if not, at least the listener can use what they heard from their child to provide targeted assistance. Send home bookmarks with strategies for homework assistance that both adult and child can use. The guidance and probing questions help the caregiver *really* help their child. Two possible notecards or bookmarks are illustrated in Figure 4.11.

Use Projects. Consider projects that engage families in gathering data about their home or community, such as a Counting with the Count (see Activity 7.5), in which students take turns taking home the Count (Sesame Street character) and selecting a numeracy project to do with their family (Muir, 2016). Projects provide opportunities to learn about students' cultural heritage. You can read *The Hundred Penny Box* (Mathis, 1975). The story describes a 100-year-old woman who remembers an important event in her life for every one of her 100 pennies she collected each year. Invite families to try to find a penny (or coin) for the years of the child's life, and create a memory booklet. Events students record start with their birthday and then might include first steps, accidents, vacations, pets, births of siblings, and so on. This can be a timeline, with coins fastened on. Or, it can be a photo project, wherein the coins are photographed and a poster created. What you learn about your students can be useful for future story problem writing. In middle school, students can record significant family events that happened before they were born, using negative numbers.

Limit Low-Level Practice. Rather than a long collection of low cognitive demand tasks, shorten the number of tasks and be sure they require higher-level thinking. Select homework that will be comprehensible to parents and other caregivers. It is a mistake to require students to use a new approach, unfamiliar to most adults (e.g., using arrays to multiply) on a homework assignment. Instead, give students a choice of approach (Bay-Williams, Duffett, & Griffith, 2016; Larson, 2016).

Use a Distributed-Content Approach. Homework can address content that has been taught earlier in the year as practice, that day's content as reinforcement, or upcoming content as groundwork. Interestingly, research has found that distributed homework that combines all three components is more effective in supporting student learning (Cooper, 2007).

In summary, no homework is better than anxiety-inducing homework. Providing tools to make homework a positive experience paired with limiting homework to meaningful and short experiences can have a positive impact on your students and on the participation of their parents or caregivers.

Math Fluency Card

I demonstrate fluency when I . . .

- Am **flexible** in how I solve problems
- Choose an **efficient** strategy for the problem
- Use that strategy accurately and get an **accurate** answer.

I will:

- Estimate
- Look at the numbers in the problem and choose a strategy. I will ask,
  - Can I do it in my head?
  - Do I know an efficient strategy?
- Try a different strategy if my first choice isn't going well
- Check to see if my answer is close to my estimate

Solving Story Problems

Read the problem and answer these three questions:

1. What is the problem about?
2. What are you being asked to find out?
3. What quantities do you need to answer that question?

When stuck, ask yourself:

- Can you imagine the problem?
- What do you know that can help you solve the problem?
- What words make sense? What words are confusing?
- What other problems have you solved that are similar to this one?
- What might you do first?
- What kind of visual might help with this problem (e.g., a drawing, table, or diagram)?

Tips for success:

- Draw a picture to represent the situation.
- Look for examples of similar problems.
- Take a mental break and then come back and try again.
- Try to put in words (verbally or in writing) what is difficult or confusing.

FIGURE 4.11 Sample notecards or bookmarks to help with homework

RESOURCES FOR CHAPTER 4

WRITING TO LEARN

1. Why use a three-phase lesson plan model? How does it compare to how you learned mathematics?
2. Imagine you are planning for a three-phase lesson on partitioning fractions (grades 1–2). Briefly describe what you will do prior to actually planning the lesson.
3. You find a task for partitioning fractions:

Using dotted grid paper, draw 4-by-4 squares. Then draw lines to show fourths. Find as many ways as you can. Draw each one you find on your grid paper.

How might you develop a three-phase lesson around this task?

4. Why are routines and centers valuable activities to implement as part of your mathematics instructional plan?
5. How are number talks and number strings alike and how are they different?
6. What does it mean to differentiate instruction? What is your favorite idea for how to differentiate instruction from this chapter?
7. Describe two important reasons to engage families in mathematics. Summarize two ways to engage families. Tell two actions to avoid that can limit family involvement or support.

RECOMMENDED READINGS

Articles

Barlow, A., Gerstenschlager, N. E., & Harmon, S. E. (2016). Learning from the unknown student. *Teaching Children Mathematics*, *23*(2), 75–81.

This article shares ideas and examples for using worked examples (both correct and incorrect) to improve student understanding as well as their ability to critique each other's reasoning.

Earnest, D., & Amador, J. M. (2018). Reflecting on standards when lesson planning. *Teaching Children Mathematics*, *24*(6), 344–346.

This quick read is a must for thinking about the relationship between standards and a single lesson plan. Short enough to read in a class session or workshop setting.

Tobon, G., & Hughes, M. T. (2020). Engaging Latino families as mathematicians. *Mathematics Teacher: Learning & Teaching PK–12*, *113*(3), 201–208.

This article describes a detailed plan for offering family math workshops, including many specific ideas for offering a culturally responsive family engagement plan.

Lambert, R. Imm, K., & Williams, D. A. (2017). Number strings: Daily computational fluency. *Teaching Children Mathematics*, *24*(1), 48–55.

Pragmatic guidance and many examples make this article an excellent resource for learning to implement a number strings routine effectively!

Books

Kobett, B., Miles, R. H., & Williams, L. A. (2018). *The mathematics lesson-planning handbook, Gr. K-2*. Corwin.

Miles, R. H., Kobett, B., & Williams, L. A., (2018). *The mathematics lesson-planning handbook, Gr. 3-5*. Corwin.

Williams, L. A., Kobett, B., & Miles, R. H. (2018). *The mathematics lesson-planning handbook, Gr. 6-8*. Corwin.

Effective planning leads to better learning for every student! This book provides in-depth planning ideas to meet the needs of all learners. Many examples and tasks are embedded throughout.

SanGiovanni, J. (2019). *Daily routines to jump start math class: Elementary*. Thousand Oaks: Corwin.

SanGiovanni, J., & Milou, E. (2018). *Daily routines to jump start math class: Middle School.* Thousand Oaks: Corwin.

Each book offers about 20 excellent routines that address a mix of mathematics (whole numbers, fractions, operations, geometry, etc.) and learning tools (number lines, pictures, graphic organizers, etc.). Instructions are straightforward, and many examples are provided in the book and online.

CHAPTER 5

Creating Assessments *for* Learning

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **5.1** Differentiate between formative and summative assessment.
- **5.2** Describe three methods for assessing students' mathematics understanding.
- **5.3** Analyze rubric types and uses.
- **5.4** Explain the value of students self-assessing.
- **5.5** Identify how tests can be used to enhance learning.
- **5.6** Explore ways to show evidence of and communicate about student learning that result in grades and more targeted instructional decisions.

What ideas about assessment come to mind from your personal experiences? Tests? Quizzes? Grades? Studying? Anxiety? All of these are common, shared memories. Now think about assessments you could use to help students learn, help you teach mathematics effectively, and avoid negative emotions and anxiety. How can assessments do these things? Read and find out.

Integrating Assessment into Instruction

Learning Outcome 5.1 **Differentiate between formative and summative assessment.**

A joint position statement on the formative assessment process from the Association of Mathematics Teacher Educators (AMTE) and the National Council of Supervisors of Mathematics (NCSM) (n.d.) stresses three important ideas: (1) Assessment should enhance students' learning, (2) assessment is a valuable tool for making instructional decisions, and (3) feedback should help learners progress. These ideas align with the distinction between assessment *of* learning, when students are only evaluated on what they know and don't know at a particular moment in time, and assessment *for* learning (AFL), when students are continually evaluated so that instruction can be targeted to their needs, and learning is improved over time (Hattie, 2015; Wiliam & Leahy, 2015).

Assessment is not separate from instruction. It must be aligned with content standards and Mathematical Practices (and processes) (Fennell, Kobett, & Wray, 2017). The typical end-of-chapter tests have value, but they occur too late in the learning process to help you

fine-tune instruction and support individual students' learning. Daro, Mosher, and Corcoran (2011) state that "the starting point is the mathematics and thinking the student brings to the lesson, not the deficit of mathematics they do not bring" (p. 48). NCTM's *Principles to Actions* (2014b) urges teachers in their Mathematics Teaching Practices to incorporate evidence of student thinking into an ongoing refinement of their instruction. This does not mean grouping students into labeled groups, but instead supporting them at whatever level they are to the next level and beyond. The NCTM *Catalyzing Change* series (2020) points to Stefanakis's (2010) reminder that the word *assess* comes from the Latin for "sit beside." The authors remind us of our role as teachers to sit beside as we assess to improve students' learning.

What Are the Main Assessment Types?

Assessments usually fall into two major categories: summative or formative. *Summative assessments* are cumulative evaluations that take place usually after instruction is completed (assessment *of* learning). They commonly generate a single score that measures overall progress toward content and practice/process standards, such as an end-of-unit test or a state standardized test. Although the scores from these assessments are important for schools and teachers, used individually, they often do not shape day-to-day teaching decisions.

On the other hand, the *formative assessment process* (assessment *for* learning) includes tools that check the status of students' development during instructional activities, pre-assess, or attempt to identify students' incomplete understandings related to the standards they are learning (Duckor, Holmberg, & Rossi Becker, 2017; Hattie, 2012; Wiliam & Leahy, 2015). When implemented well, the process of formative assessment is one of the most powerful influences on achievement (Hattie, 2009). It dramatically increases the speed and amount of student learning by providing targeted feedback to the student (Hattie & Clarke, 2018) and using the results and evidence collected to inform decision making about next steps in the learning progression. As Wiliam states, "To be formative, assessment must include a recipe for future action" (2010, slide 41).

Wiliam notes three key components of the formative assessment process:

1. Identify where learners are.
2. Identify the goal for the learners.
3. Identify paths to reach the goal.

Let's look at a case of this process in a third-grade class. To begin, students solve this word problem: "If Lindy has 34 shells in her collection and Jesse has 47, how many shells are in both collections?" Immediately, the teacher observes one student quickly jotting down a straight line on the paper, making a tick mark with a 34 underneath and using small arcs to jump up four times to 74, then making one jump of six and another jump of one and indicating 81 with a tick mark (see Figure 5.1). Another student selects from a collection of base-ten materials in the center of the table, models each number, groups the tens together, then combines the ones and trades 10 of the ones for an additional ten piece, getting 81 as an answer. Observing a different student, the teacher notes he uses counters but is counting by ones. First, he counts out 34, then the 47, and then recounts them all to reach 78 (miscounting along the way). The evidence gathered from observing these three students reveals different places in their addition fluency.

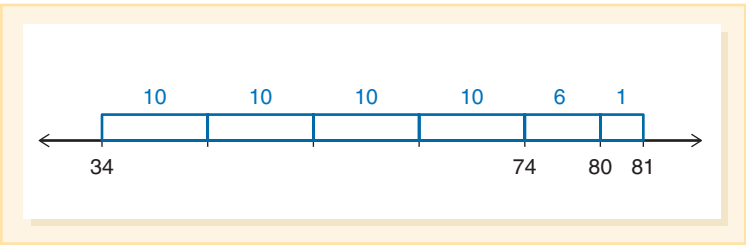


FIGURE 5.1 A student uses an open number line to add 34 + 47

This teacher is at the first step in Wiliam's three key processes, noting where students are in their learning. Moving to step 2, the teacher notes that one student should move to more challenging tasks, whereas the other two students need to move through more targeted instruction to the standard of representing and solving addition and subtraction problems within 100. In step 3, individualized lessons are designed that will reach each student's need. This may include a lesson that asks students to use the hundred chart or a number line to illustrate

how they find the sums. Additionally, the teacher may have students compare ways to use the number line to highlight commutativity and flexibility, including the idea that starting with the larger addend may be more efficient than starting with the smaller addend.

If summative assessment could be described as a digital snapshot, formative assessment is a streaming video. One is a picture of what a student knows captured in a single moment of time; the other is a moving picture that documents students' thinking and reasoning. In this chapter and throughout Part II of this book in Formative Assessment Notes, we will focus on several tools in the formative assessment process that can be used for evaluating students' understanding.

What Should Be Assessed?

Appropriate assessments must attend to all five strands of students' mathematical proficiency (National Research Council, 2001) (described in Chapter 2). Assessments are most effective when they reflect the full range of doing mathematics: concepts, skills, mathematical practices, and even students' dispositions.

Pearson eText
Video Example 5.1
In this video, Dylan Wiliam, an expert on formative assessment, discusses five strategies that support effective assessment.
https://www.youtube.com/watch?v=kPf0nQFfv50

Conceptual Understanding and Procedural Fluency. Assessments can include opportunities to demonstrate why something is true or why it works (conceptual understanding) as well as how to solve problems. Assessing conceptual understanding may include asking students to illustrate their approach using manipulatives or a visual representation, or explaining why a particular method works. Assessing fluency is different from assessing a skill. Assessing procedural fluency (including basic fact fluency) attends to efficiency, flexibility, and accuracy (see Figure 2.1 on p. 15). In essence, assessing fluency focuses on whether a student has several ways to solve a problem and chooses efficient options based on the numbers used in the problem. In other words, they show flexibility in their methods, knowing more than one way. If a student can compute with fractions yet has no idea of why a common denominator is needed for addition but not for multiplication, then the procedure is not learned to the extent required to reach procedural fluency. You can assess conceptual understanding and procedural fluency as students engage with a worthwhile task, observing as they discuss and justify. In this way, you gain information that provides insight into the nature of the student's understanding of that idea. You can also ask for more detail, something often not possible on tests.

Strategic Competence and Adaptive Reasoning. Truly doing mathematics is more than just assessing content knowledge. One way to communicate to students that Mathematical Practices are important is to assess them. Because the Mathematical Practices are actions, assessing them looks different. You can craft a list of descriptive statements related to these proficiencies. Here are several examples for use with individual work, group work, or mathematical discussions:

Problem Solving

- Works to make sense of and fully understand problems before beginning
- Perseveres to demonstrate a variety of strategies
- Evaluates the reasonableness of answers

Reasoning

- Justifies solution methods and results
- Recognizes and uses counterexamples
- Makes conjectures and/or constructs logical progressions of statements based on reasoning

Examples of rubric statements aligned with these practices should be discussed and explicitly modeled with your students to help them understand these valuable behaviors. Share weak and strong examples of anonymous student work with the class to help all students identify ways to improve (Barlow, Gerstenschlager, & Harmon, 2016). For example, evidence of adaptive reasoning may include student work where a student decided to abandon one method and pursue another method. Additionally, having students justify why a worked example is correct demonstrates their adaptive reasoning.

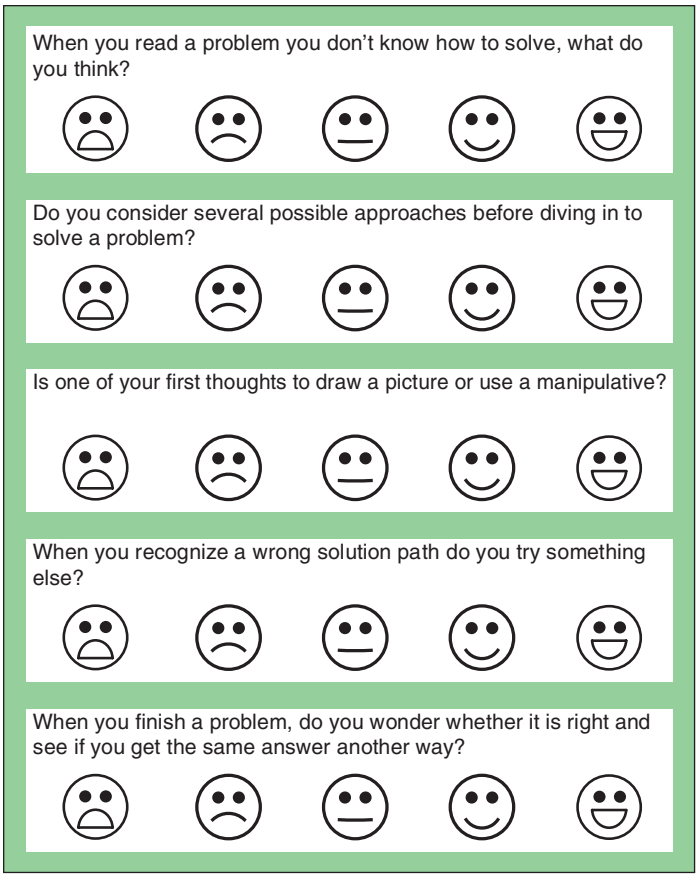

When you read a problem you don't know how to solve, what do you think?
Do you consider several possible approaches before diving in to solve a problem?
Is one of your first thoughts to draw a picture or use a manipulative?
When you recognize a wrong solution path do you try something else?
When you finish a problem, do you wonder whether it is right and see if you get the same answer another way?

FIGURE 5.2 Disposition Assessment Scale

Productive Disposition. Collecting data on students' ability to persevere, as well as their confidence and belief in their own mathematical abilities, is also important. This evidence is most often obtained with observation, students' self-reports, interviews, and writing. See also the self and peer assessment section later in this chapter. For example, students could self-assess their disposition by rating themselves on the following scale (see Figure 5.2)

Information on persistence and willingness to attempt new or novel problems is available to you every day when you use a problem-solving approach.

Pearson eText Teacher Resource:
Disposition Assessment Scale

Assessment Methods

Learning Outcome 5.2 **Describe three methods for assessing students' mathematics understanding.**

There are three basic approaches in the formative assessment process that can be used to evaluate students' mathematics proficiency: observations, interviews, and performance tasks.

Observations

All teachers learn useful bits of information about their students every day, but with intentional efforts to gather observation data, much more information can be obtained. When the three-phase lesson format suggested in Chapter 4 is used, the flow of evidence about student performance increases dramatically, especially in the *during* and *after* portions of lessons. Gathering evidence during classroom instruction has at least two valuable outcomes. First, student thinking that may have gone unnoticed is suddenly visible and can be highlighted in the *after* phase. Second, the data can be used beyond the lesson for these purposes: planning future lessons, providing student feedback, conducting parent conferences, and determining grades.

Depending on what evidence you are trying to gather, several days to two weeks may be required to complete observations on how each student is progressing on a standard. Shorter periods of observation will focus on a particular cluster of concepts or skills or on particular students. Over longer periods, you can note growth in mathematical proficiencies, such as strategic competence or productive disposition. To use observation effectively, consider the following maxim: *Only try to collect data on a reasonable number of students in a single class period.*

Anecdotal Notes. The act of *professional noticing* is a process by which you observe learners through a focus on three phases: (1) attending, (2) interpreting, and (3) deciding (Jacobs, Lamb, & Philipp, 2010). You want to collect anecdotal notes to understand learners' strategies and interpret their thinking as you plan next steps. That means you *attend* to everything such as if the child is nodding his head, using fingers to count, creating appropriate models or using strategies that are clearly described and defended. Then you *interpret* those gestures, comments, drawings, and actions by making notes of students' strengths and the level of sophistication of their conceptual understanding. The last step is noting *decisions* for subsequent instructional actions.

For example, first graders may play a game in which each child draws a domino, adds the dots on the two halves and compares their total to another child's. As students play, the teacher observes the way in which students are adding the two domino parts and how they use number sense and records their methods on an observation table. Some students count every dot on the domino, and the teacher notes, "counts all" on her chart. Others use a counting-on strategy

(the student sees four dots on one half and counts on from four to tally the total number). The teacher notes, "counts on" for these students. Some students just knew how many dots without counting. The teacher notes, "recognizes quantities." Data gathered from asking questions about and listening to a pair of students work on an activity or an extended project provide significant insights into students' thinking (Petit & Zawojewski, 2010).

One system for recording observations is to write these anecdotal notes on an electronic tablet and store them in a spreadsheet or in a multicolumn table that documents such things as students' use of mathematical practices. Focus your observations on about five students a day. The students selected may be members of one or two cooperative groups or a group identified as needing additional experiences or support.

Pearson eText Teacher Resource: Multicolumn Table

Look-for Tools. To focus your attention, a look-for tool with several mathematical practices or content objectives can be devised (see Figures 5.3 and 5.4). As you can see, there is a place for comments that should concentrate on students' achievement of big ideas and conceptual understandings. For example, you will probably find a note such as "is recognizing how multiplication facts can be related, such as using 10×7 to think about 9×7" more helpful than "knows the easy multiplication facts but not the hard ones" when assessing students on the mathematical practice of "reasons abstractly and quantitatively."

Pearson eText Teacher Resource: Mathematical Practices Look-For Tool

Another observation look-for tool lists all students in a class on one to three pages (see Figure 5.4). In the items to observe across the top of the tool, include specific abilities or common misconceptions based on a sequence such as a learning progression. Symbols such as checks or codes can be entered in the cells and used to identify patterns of student performance (Accardo & Kuder, 2017). A full-class look-for tool is more likely to be used for long-term objectives such as problem solving strategies, strategic use of representations or tools, and such skill areas as basic fact fluency or computational estimation. Dating entries and noting specifics about observed performance are also helpful.

Pearson eText Teacher Resource: Observation Look-for Tool

Interviews

Interviews are a blending of observations with questioning. "An assessment system designed to help steer the instruction system must give good information about direction as well as distance to travel. A system that keeps telling us we are not there yet is like a kid in the back seat whining, 'Are we there yet?'" (Daro et al., 2011, p. 51).

Quick Checks. This type of interview can be completed in a minute or less as you walk around the room. For example, imagine you are working with second graders on strategies for subtracting within 20. You might have three cards, each with a task on it: 15 – 6; 12 – 9; and 14 – 7. You sit or squat near a student and ask them to use think-aloud to tell you the answers. From these three tasks you can determine whether students are counting back, using a think-addition approach, or applying a subtraction strategy, or that they know the answers.

NAME:				
	Good Start	Got it	Wow	Evidence of Proficiency
Mathematical Practices				
Makes sense of problems and perseveres				
Reasons abstractly and quantitatively				
Constructs viable arguments, critiques others' reasoning				
Models with mathematics				
Uses appropriate tools				
Attends to precision				
Looks for and makes use of structure				
Look for and express regularity in repeated reasoning				

FIGURE 5.3 Look-for tool and rubric for each student

Topic *Computation* *Adding 2-digit numbers* **Names**	Developing *Uses one strategy*	On Target *Has at least two strategies*	Excellent *Used different methods based on the different numbers in the problem*	Comments
Lalie		✓ 3-18-2022 3-21-2022		
Pete	✓3-20-2022	✓ 3-24-2022		*Difficulty with problems using 8 and 9 as addends*
Syd			✓3-20-2022	*Flexible approaches used including mental math*
Lakeshia		✓		*Counts by tens, then adds one, also used compensation*
Zuma		✓		
Pam	✓			*Beginning to add the group of tens first*
Owen		✓ 3-24-2022		*Using a posted hundred chart and counting on*

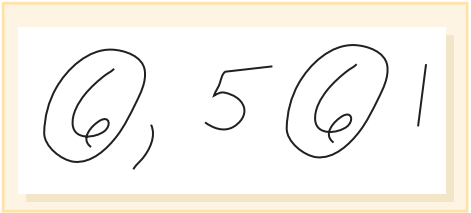
FIGURE 5.4 A class observation look-for tool

Diagnostic Interviews. A diagnostic interview is a one-on-one investigation lasting 3 to 10 minutes with the purpose of learning where the student needs to be (learning goal), where they are (learning location), and what is the next step on the path to reach the goal (learning direction). Diagnostic interviews are assessment opportunities, not tutoring opportunities. It can be hard to watch a student make errors and not respond with immediate assistance to correct it. Instead, use the interview to listen to the student's explanations of their strategies and probe their understanding with the purpose of discovering their strengths and areas that they need more experience with and the precise bridges to move them from current to desired learning.

To design a diagnostic interview, select a problem that matches a standard your students are or will be studying and have paper, pencils, and a variety of materials available (particularly those models and materials used during previous instruction). Use learning progressions, such as those in this book, to provide guidance for creating a conceptual connection (Ebby & Petit, 2017). Also, prepare notes about what you anticipate might be emerging understandings, common methods, or common barriers. Ask the student to solve the problem using think-aloud to share their reasoning. Encourage multiple representations by asking the student to demonstrate their thinking, using materials or drawings. Fennell, Kobett, and Wray (2015) called this method the show-me approach.

Diagnostic interviews depend on students' needs and could include the following examples of starting points:

- Which is greater, $\frac{7}{8}$ or $\frac{7}{7}$?
- Are the two numbers circled the same?



- What is the unit rate (cost per cap) if we paid $102.00 for 17 caps?
- Solve $6\frac{1}{4} \times 2\frac{3}{4} =$ _____. Draw a picture to illustrate the solution.

In each case, the initial problem should be aligned with recent work or your attempts to pinpoint underlying foundational gaps in understanding. Also, see sample interview options

Pearson eText
Video Example 5.2
In this video, consider this diagnostic interview to probe a student's understanding of subtraction of decimals. What do you learn from her use of drawings to explain her thinking? For example, why doesn't she draw the tenths on each whole? What does that tell you about her strengths?

online, including a Sample Interview for Primary Grades, a Sample Interview for Intermediate Grades, a Sample Interview for Middle Grades, and Student Observation and Interview: Learning through Problems.

Pearson eText Teacher Resource: Sample Interview for Primary Grades
Pearson eText Teacher Resource: Sample Interview for Intermediate Grades
Pearson eText Teacher Resource: Sample Interview for Middle Grades
Pearson eText Teacher Resource: Student Observation and Interview: Learning through Problems

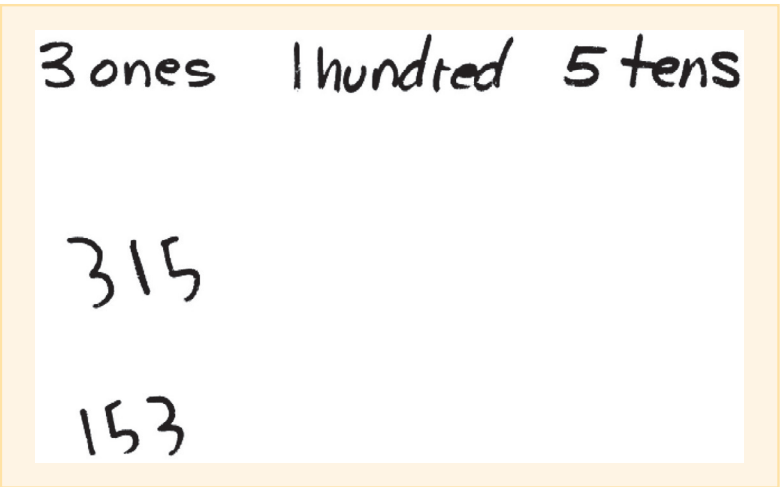
Sometimes students self-correct a mistake during an interview, but more frequently, teachers unearth a student's level of understanding or reveal what strategies students have learned. When you focus on exploring errors and identify common pitfalls, you can use the assessment results to build greater sophistication in students' conceptual understanding (Bray & Santagata, 2014).

Interviews can be time intensive, but they have the potential to provide evidence that you simply cannot get any other way. So, how can you accomplish these interviews? Think of them as tools to be used for only a few students, not for every student in the class. Briefly interview a single student while others work on a task or are in learning centers. You can use a digital tablet or screen-casting app (e.g., Explain Everything, www.explaineverything.com) to record the whole conversation and written work.

After examining hundreds of research studies, Hattie (2009) found that the feedback that teachers received from students on what they knew and did not know was critical in improving students' performance. That is precisely what diagnostic interviews are designed to do! For example, are you certain that your students have a good understanding of place value, or are they just using rote procedures? Instruction will be more successful if you can pinpoint why a student is having difficulty before you try to fix the problem.

Let's look at this vignette of a diagnostic interview.

Ms. Marsal was working with George, a student with special needs in mathematics who was displaying difficulty calculating multi-digit numbers. George also made unreasonable estimates to computation problems, in some cases thinking the answer would result in a number in the tens when it was actually in the hundreds. To get feedback from George that could reveal what he understood and where any confusion may exist, Ms. Marsal adapted a task (Griffin & Lavelle, 2010) for the interview. She asked George to write the number that goes with 3 ones, 1 hundred, and 5 tens (see the top of the student's work). Although base-ten materials were available, they went untouched. In response, George wrote 315, writing the number in the left-to-right order he saw the numerals, without attention to place value. Ms. Marsal resisted the temptation to correct him immediately and, instead, probed further by asking George to take out the amount shown using base-ten materials. Showing fluency with the values of base-ten materials, he selected the correct amounts and placed them on the table, again in the order given in the problem, not in place value order. When offered a place value mat, George said, "I get it," and placed the materials in the correct positions on the mat. When asked to write the number that corresponded with the materials, George wrote 153 as shown below. Ms. Marsal asked why he got two different answers and which number he thought was correct. George pondered and pointed to the second number and said, "This one is right; I think you were trying to trick me."



Although this interview revealed that George had a solid grasp of the value of the base-ten materials, it also revealed he will benefit from more experiences to understand place-value. This case reveals an actual learning experience where George figured it out on the spot. Notice that the teacher linked the task to previous learning through the use of concrete materials and the structured semi-concrete support of the place-value mat. These connections provided George with a way to think about the actual size of the number rather than just the individual digits. The cognitive dissonance caused by getting two different numerical outcomes, one responding to just the written numbers and words and the other using corresponding concrete materials, enabled more connected ideas to emerge. Planning could then begin for future instructional experiences for George, based on actual evidence.

There is no one right way to plan or structure a diagnostic interview. In fact, flexibility is a key ingredient. However, you should have an overall plan that includes an easier task and a more challenging task in case you have misjudged the starting point. Also, did you notice that the teacher in the vignette had instructional materials ready for George to use? Be sure you have materials available that have the potential to provide insights into students' understanding. Then probing student thinking through questioning provides useful data and insights that could inform instruction. Also, your use of questions helps students get past being stuck and promotes their ability to think for themselves—like George did. Be prepared to probe students' thinking with questions like these:

- Can you explain what you just did?
- How would you explain this idea to a friend?
- Can you draw a picture to help you think about this problem?
- What does this [point to something on the paper] stand for?
- Why did you solve it that way?
- Can you show me what you are thinking with the [materials such as fraction pieces, counters, hundred chart, and so on]?
- Why do you think you got two different answers? Which one do you think is correct?
- If you tried to do this problem again, which approach would you try first?

Pearson eText Teacher Resource: Question Probes

In each case, it is important to explore whether students (1) understand what they did, and (2) can use models to connect actions to what they wrote or explained earlier.

Remember to consider these suggestions as you implement diagnostic interviews:

- *Avoid revealing whether a student's answer is right or wrong*. Often facial expressions, tone of voice, or body language can give students an idea as to whether their answer is correct or incorrect. Instead, use a response such as "Can you tell me more?"
- *Wait silently for the student to answer*. Give ample time to allow the student to think and respond. Only then should you move to rephrasing the question or probing for a better understanding of the student's thoughts. After the student gives a response (whether accurate or not), wait again! This second wait time is even more important because it encourages the student to elaborate on his or her initial thought and provide more information.
- *Avoid interjecting clues or teaching*. The temptation to help is sometimes overwhelming. Watch and listen. Your goal is to use the interview not to teach but to find out where the student is in terms of conceptual understanding and procedural fluency.
- *Give opportunities for students to share their thinking without interruption*. Encourage students to use their own words and ways of writing things down. Correcting language or spelling can sidetrack the flow of students' explanations.

The benefits of using diagnostic interviews become evident as you plan instruction that capitalizes on students' strengths while recognizing possible needs for additional foundational experiences. Also, unlike summative tests, you can always ask another question to find out why

a student is taking an unexpected path. You may also want to discuss results of interviews with colleagues to gain shared insights (Stephan, McManus, & Dehlinger, 2014).

Tasks for Assessment

Tasks for assessment include products, which can be problem-based tasks, writing tasks, and students' self-assessments. High-quality assessment tasks permit each and every student in the class to demonstrate understanding.

Problem-Based Tasks. *Problem-based tasks* used for assessment are tasks connected to actual problem-solving activities used in instruction. High-quality tasks permit every student to demonstrate their abilities (Rigelman & Petrick, 2014; Smith & Stein, 2011). They also ideally include real-world or authentic contexts that interest students or relate to recent classroom events. Of course, be mindful that students who are emergent bilinguals may need support with contexts to bring their mathematical abilities to light.

Notice that the following examples of problem-based assessment tasks are not elaborate, yet they provide an opportunity for students to demonstrate several of the five strands of mathematical proficiencies described earlier. What mathematical ideas and practices are required to respond successfully to each of these tasks? Will the task help you determine how well students understand the ideas?

Pearson eText Teacher Resource: Problem-Based Assessment Tasks

Shares **(Grades K–2)**

Learning Targets: (1) Solve multistep problems involving the operations. (2) Use manipulatives and words to describe a solution.

Leila has 6 gumdrops, Darlene has 2, and Melissa has 4. They want to share them equally. How will they do it? Draw a picture to help explain your answer.

At second grade, the numbers in the Shares task should be larger. What additional concepts would be involved if the task were about sharing cookies and the total number of cookies was 34?

The Whole Set **(Grades 3–4)**

Learning Targets: (1) Determine a whole, given a fractional part (using a set model). (2) Make sense of quantities and their relationships in a context.

Mary counted 15 cupcakes left from the whole batch that her father made for the picnic. "We've already eaten two-fifths," she noticed. How many cupcakes did her father bake?

In the following task, students are asked to think about the thinking of other students. Analysis of other students' performances is a good way to create tasks.

Decimals **(Grades 4–5)**

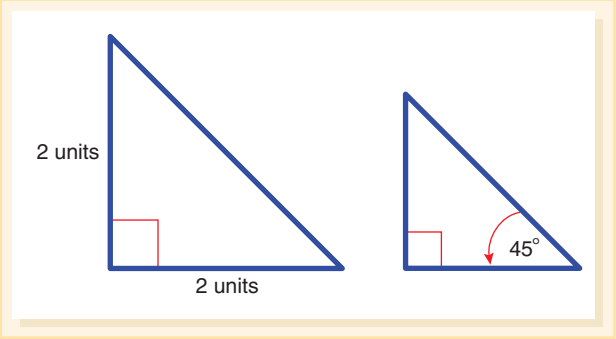
Learning Targets: (1) Compare two decimals by reasoning about their size. (2) Analyze and critique the reasoning of others.

Alan tried to make a decimal number as close to 50 as he could, using the digits 1, 4, 5, and 9. He arranged them in this order: 51.49. Jerry thinks he can arrange the same digits to get a number that is even closer to 50. Do you agree or disagree with Jerry? Explain.

This next task is a good example of an open-ended assessment. Consider how much more valuable this task is than asking for an angle measure in the triangle.

Two Triangles **(Grades 7–8)**

Learning Targets: (1) Classify two-dimensional shapes into categories based on their properties. (2) Attend to precision by applying definitions to define categories.



Tell everything you can about these two triangles. Given what you wrote about the two triangles, determine which of the following statements are true: The large triangle is an isosceles triangle; the small triangle is an isosceles triangle; the large triangle has an area of 2 square units; the small triangle has an area of 1 square unit; the large triangle has at least one angle that measures 45 degrees; the small triangle has at least one angle that measures 30 degrees; the two triangles are similar. Explain your thinking.

Algebra: Graphing Functions **(Grade 8)**

Learning Targets: (1) Compare and analyze quadratic functions. (2) Build a logical argument for a conjecture, using logical reasoning.

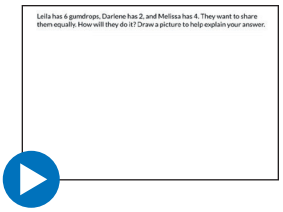
Does the graph of $y = x^2$ ever intersect the graph of $y = x^2 + 2$? What are some ways that you could test your conjecture? Would your conjecture hold true for other equations in the form of $y = x^2 + b$? Within all quadratic functions of the form $y = ax^2 + b$, when would your conjecture hold true?

Just because a task is used for assessment purposes, it does not mean you can't still engage students in a mathematical discussion about the tasks. More can be learned about students' understanding in a discussion that follows their individual efforts on a task.

Translation Tasks. One important assessment option is what we call a *translation task*, which requires students to represent an idea in multiple formats. For example, students are asked to use four possible representations for a concept, using words, models, numbers, and word problems. Look back to Figure 2.11 to see the Web of Representations to Demonstrate Understanding that this task is aligned to. As students move flexibly between these representations, there is a better chance that a concept will be integrated into a rich web of ideas.

So, what is a good way of structuring a translation task? Using a 2-by-2 grid, inspired by the classic Frayer Model (Frayer, Fredrick, & Klausmeier, 1969) (see Figure 5.5), you can give students an equation and ask them to:

- Tell a story or write a word problem that matches the equation.
- Illustrate the equation with materials, models, or drawings.
- Explain their thinking about the process of arriving at an answer or describe the meaning of the operation.


Leila has 6 gumdrops, Darlene has 2, and Melissa has 4. They want to share them equally. How will they do it? Draw a picture to help explain your answer.

Pearson eText
Video Example 5.3
In this video, consider how a student solves a sharing problem. What do you learn from this diagnostic interview about the student's strengths and areas for growth? What might you teach next?

Pearson eText Activity Page: Translation Task for Grades K–2
Pearson eText Activity Page: Translation Task for Grades 3–5
Pearson eText Activity Page: Translation Task for Grades 6–8

Translation tasks can be used for whole-class lessons or for individual or small groups. For example, fourth-grade children may be given an equation such as 16 × 42 = ? They could draw a model, say, of an open array in the "Model/Illustration" area (or use manipulatives), describe a story problem or real-world situation in the upper right corner and explain to another person in writing (or scripted or audio recorded when appropriate) how they solved the multiplication in the fourth area marked "Explain Your Thinking/Describe the Meaning of the Operation." The translation task can start in any section of the template and be used across content topics. For example, in the "Word Problem" area, you could write "One side of the rectangle is 6 cm. The area is 48 cm^2. How long is the other side?"

Translation tasks are effective assessment tools because they allow us the chance to notice students' conceptual understanding and their representations. Studying these data with a grade-level partner or a math coach can help to highlight student strengths as well as decipher common errors, categorize student strategy use, and determine targeted instruction (Kazemi, Gibbons, Lomax, & Franke, 2016; Morrow-Leong, 2016). Working as a team to create, implement, and analyze assessments will enrich your ability to select and administer meaningful performance-based questions or tasks and enhance your professional judgment by either questioning or confirming your thinking.

Equation/Written Symbols	Word Problem/Real-World Situation
36 + 49 =	
Manipulatives/Illustration	**Explain Your Thinking**

FIGURE 5.5 Translation task template with example task

Writing. As an assessment tool, writing in journals, exit tasks, or other formats about tasks provides a unique window to students' perceptions and ways that they are thinking about an idea. Students' writing can make sense of problems, express early ideas about concepts, unearth confusion, connect representations, or even clarify strategy use (Casa et al., 2016; Powell, Hebert, & Hughes, 2020). As discussed in Chapter 3, writing prior to a classroom discussion can serve as a rehearsal for the classroom conversation, increasing participation. From an assessment

Pearson eText
Video Example 5.4
In this video, consider how the class engages in a formative assessment activity where they are writing a letter to another student about how to decide how large an area in a fictitious store is needed to put shopping carts.

standpoint, that prewriting is called rough-draft talk (Jansen et al., 2016/2017; Jansen, 2020). Following a discussion, students can return to their writing and revise it by adding notes or ideas that they gleaned from the discussion.

Students can also write to explain a problem, summarize the big ideas of a lesson, or reflect on their own learning through such prompts as:

- "I think the answer is… I think this because…"
- Write an explanation for a new student of why the answer to 4×7 is the same as 7×4 and whether this relationship works for 6×49 and 49×6. If so, why?
- Write an explanation for a new student of when the ratio of 3:6 means the same thing as 4:8 and when they mean something different.
- Explain to a student who was absent from class what you learned about decimals.
- If you got stuck today in solving a problem, where in the problem did you have trouble?
- After you got the answer to the problem, what did you do so that you were convinced your answer was reasonable? How sure are you that you got the correct answer?
- Write a story/word problem that goes with this (equation, graph, diagram, picture, model).

In the early grades, rather than writing prompts like these, use a language experience approach. After an activity or task, write the words *Giant Journal* and a topic or prompt on a large flipchart or interactive whiteboard. As students respond verbally, write down their thoughts, including the contributor's name. Include drawings when appropriate (see Figure 5.6).

Sometimes you can assess students rapidly by using an exit task. Exit tasks contain a quick question or two, are administered when students are concluding the instructional period, and can be completed using pencil/paper or a digital tool (Wieser, 2008) (See Table 5.1).

Other exit tasks can be captured through tools involving technology (see Figure 5.7) including Desmos Classroom Activities (teacher.desmos.com), Formative (goformative.com), Google Forms (docs.google.com/forms), Google Jamboard (jamboard.google.com), Kahoot (getkahoot.com), Padlet (www.padlet.com), Pear Deck (www.peardeck.com), and Plickers (get.plickers.com).

An unusual approach to assessing is having students be the assessors, as in the Mathematical Practice of critiquing the reasoning of others. Mathematically proficient students are able to "justify their conclusions, communicate them to others, and respond to the arguments of others" (NGA & CCSSO, 2010, pp. 6–7). By showing exemplars and counterexamples from real or fictitious peers and having students assess it and write comments "like the teacher," children begin to identify elements of strong strategy use, a sound argument, and cohesive communication. Focusing on their

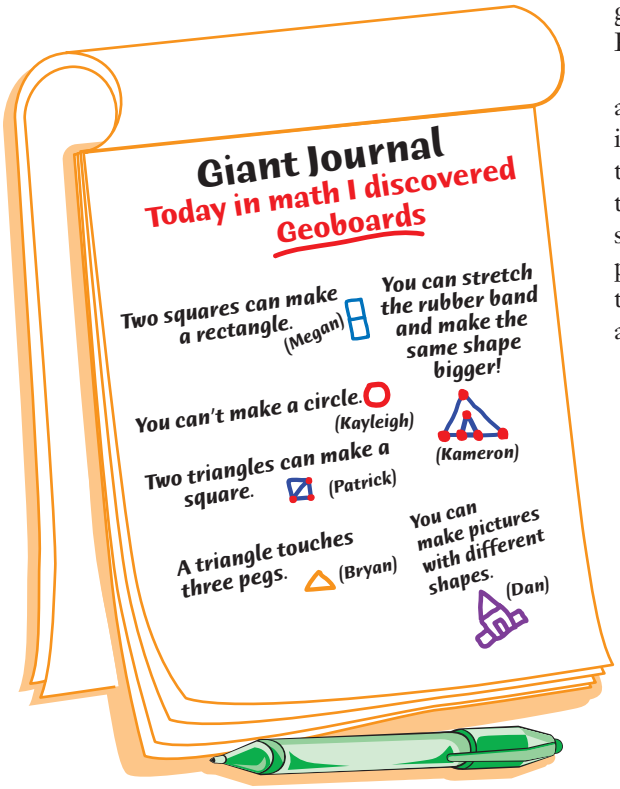

Giant Journal
Today in math I discovered
Geoboards
Two squares can make a rectangle. (Megan)
You can stretch the rubber band and make the same shape bigger!
You can't make a circle. (Kayleigh)
(Kameron)
Two triangles can make a square. (Patrick)
A triangle touches three pegs. (Bryan)
You can make pictures with different shapes. (Dan)

FIGURE 5.6 Writing in pre-K or kindergarten may be a class product on a flipchart or whiteboard.

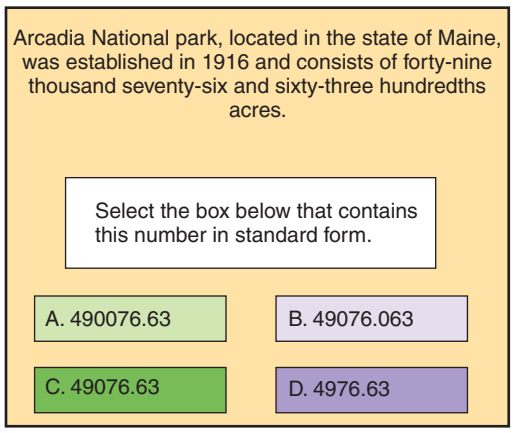

Arcadia National park, located in the state of Maine, was established in 1916 and consists of forty-nine thousand seventy-six and sixty-three hundredths acres.
Select the box below that contains this number in standard form.
A. 490076.63
B. 49076.063
C. 49076.63
D. 4976.63

FIGURE 5.7 Sample Exit Task using technology

TABLE 5.1 POSSIBLE EXIT TASKS BASED ON THE NCTM *PRINCIPLES TO ACTIONS*

Assessment Goal	Sample Exit Tasks
Information gathering	What does the equal sign mean? Share an example. What are the properties of a trapezoid?
Student thinking	What does it mean that the operations of multiplication and division have an inverse relationship? How does your equation fit the situation in the word problem?
Mathematical structures, connections, and relationships	Are $10r$ and $5 \times 2 \times r$ equivalent expressions? How do you know? What is the maximum number of digits in a sum when you add two 3-digit numbers? Why?
Reflection and justification	How are the two-solution strategies you used to solve the problem today alike? Different? How can you show that all squares are rectangles?

ability to analyze and assess other students' errors can showcase their strengths, point to needs for more experiences, increase understanding, and expand learning opportunities (Bray, 2011; Pace & Ortiz, 2016). Assessing can foster learning in multiple ways.

Finally, student writing is an excellent form of communication with parents during conferences. Student writing shows evidence of their child's thinking, providing families with much more than any grade or test score.

Rubrics and Their Uses

Learning Outcome 5.3 Analyze rubric types and uses.

Problem-based assessment tasks may tell us a great deal about what students know, but how do we analyze and use this information? These assessments yield an enormous amount of evidence that must be evaluated by examining more than just a simple count of correct answers. One step in the process of moving from teaching tasks to assessment tasks involves the addition of a rubric. A *rubric* is a scale based on predetermined criteria with two important functions: (1) It permits students to see what is central to excellent performance, and (2) it provides the teacher with scoring guidelines that support the equitable analysis of students' work. The evidence collected from rubrics helps teachers target follow-up instruction, building on students' strengths and responding to any incomplete understandings.

In a teaching-through-problem-solving approach, you will often include criteria and performance indicators on your rubrics such as the following:

- Solved the problem(s) accurately and effectively
- Persevered and demonstrated resilience when facing a challenging problem
- Explained strategies used or justified their answer
- Used logical reasoning
- Expressed a grasp of numerical relationships and/or mathematical structure
- Incorporated multiple representations and/or multiple strategies
- Demonstrated an ability to select and use tools and manipulatives (including virtual manipulatives) appropriately
- Communicated with precise language and accurate units
- Identified general patterns of ideas that repeat, making connections from one big idea to another

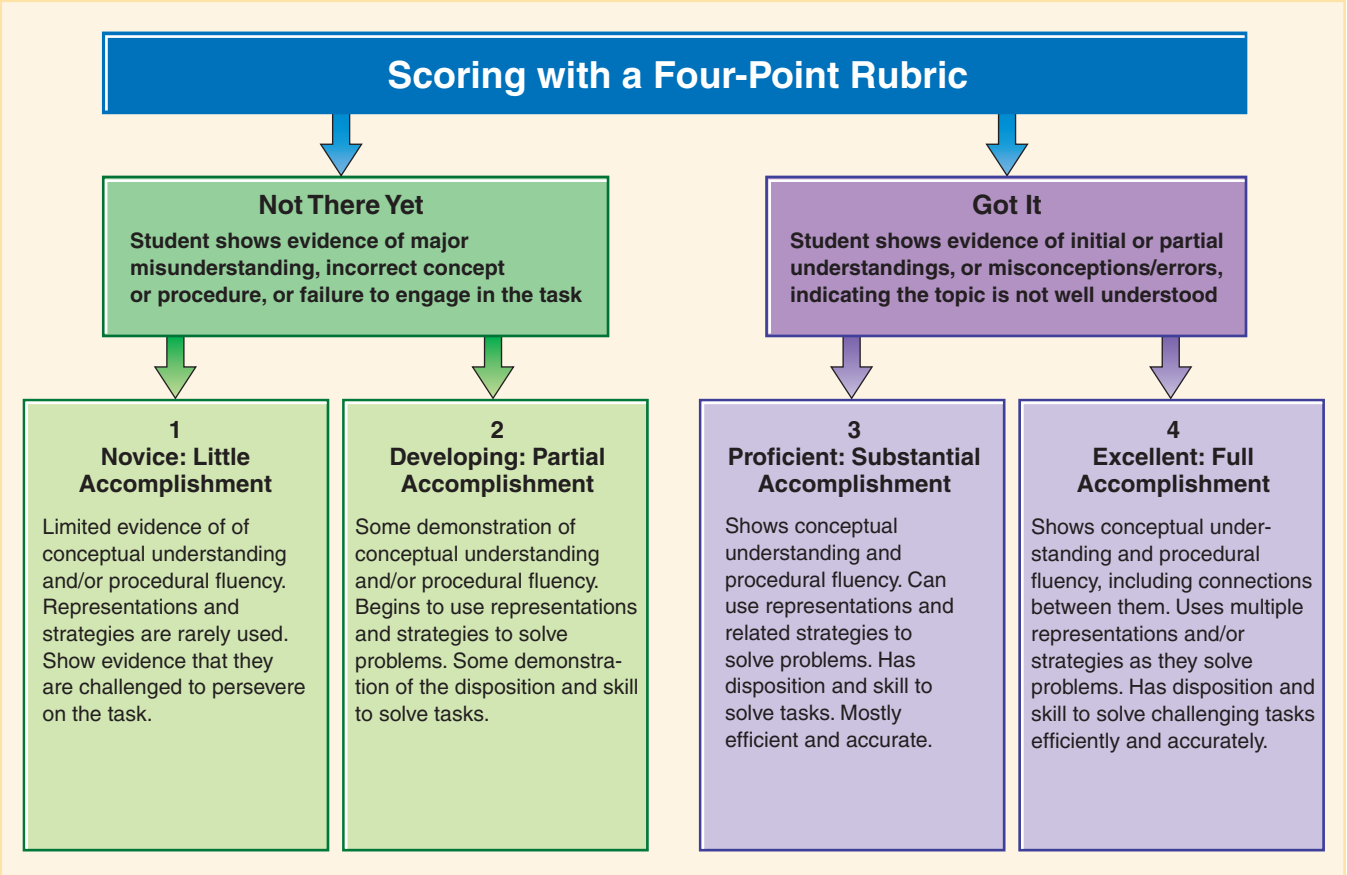

Scoring with a Four-Point Rubric
Not There Yet
Student shows evidence of major misunderstanding, incorrect concept or procedure, or failure to engage in the task
Got It
Student shows evidence of initial or partial understandings, or misconceptions/errors, indicating the topic is not well understood
1
Novice: Little Accomplishment
Limited evidence of of conceptual understanding and/or procedural fluency. Representations and strategies are rarely used. Show evidence that they are challenged to persevere on the task.
2
Developing: Partial Accomplishment
Some demonstration of conceptual understanding and/or procedural fluency. Begins to use representations and strategies to solve problems. Some demonstration of the disposition and skill to solve tasks.
3
Proficient: Substantial Accomplishment
Shows conceptual understanding and procedural fluency. Can use representations and related strategies to solve problems. Has disposition and skill to solve tasks. Mostly efficient and accurate.
4
Excellent: Full Accomplishment
Shows conceptual understanding and procedural fluency, including connections between them. Uses multiple representations and/or strategies as they solve problems. Has disposition and skill to solve challenging tasks efficiently and accurately.

FIGURE 5.8 Four-point generic rubric

Generic Rubrics

Generic rubrics identify broad categories of performance and therefore can be used for multiple assignments. The generic rubric allows a teacher to score performances by first sorting into two broad categories, as illustrated in the four-point rubric shown in Figure 5.8. Then student work is separated into two additional levels as shown. A rating of 0 is given for no response, no effort, or for answers that are off task. The advantage of the four-point scale is the relatively easy initial sort into Got It or Not There Yet.

Another possibility is to use your three- or four-point generic rubric as in Figure 5.8 on a reusable form (see Four-Point Rubric). With a four-point generic rubric, performances are first sorted into two categories. Each performance is then considered again and assigned to a point on the scale. This method is especially useful for planning purposes. But there are times when the generic rubrics do not give enough definition of the specific criteria for a particular task. For those instances, try a task-specific rubric.

Pearson eText Teacher Resource: Four-Point Rubric

Task-Specific Rubrics

Task-specific rubrics include specific statements, also known as *performance indicators*, that describe what students' work should look like at each rubric level and, in so doing, establish criteria for acceptable performance on that particular task (see Figure 5.9 and Anecdotal

Note Rubric). Initially, it may be difficult to predict what student performance at different levels should look like, but your criteria depend on your knowledge and experience with students at that grade level and your insights about the task or mathematical concept or procedure. One important part of setting performance levels is predicting students' common misconceptions as identified through research or the expected approaches to similar problems.

Pearson eText Teacher Resource: Anecdotal Note Rubric

To facilitate developing performance levels, write out indicators of "proficient" or "on target" performances before using the task. This self-check ensures that the task is likely to accomplish your purpose. If you find you are writing performance indicators in terms of the number of correct responses, you are most likely looking at drill or practice exercises, not the problem-based assessment tasks for which a task-specific rubric is appropriate. Like athletes who continually strive for better performances rather than good-enough, students should always recognize opportunities to excel. When you take into account the total performance (processes, practices, answers, justifications, extension, and so on), it is always possible to go above and beyond.

Early in the year, discuss your rubric (such as Figure 5.8) with the class and post it prominently. Make it a habit to discuss students' performance on tasks in terms of the rubric language. For example, if you are using the anecdotal note rubric, rubric language can be used informally: "Amelia, the rubric states to get an Excellent, you need to solve the problem with two different representations and explain your thinking. Is that what you did?" This approach lets students know how well they are doing and encourages them to persevere by giving specific areas for improvement.

Observation Rubric

Partition Regions into Equal Shares 3/19

Excellent Demonstrates conceptual and procedural knowledge of topic and extends that understanding to create or model connections. Partitions rectangles and circles into two, four, and eight equal shares and uses the language of halves, fourths, etc. Is able to draw equal shares that are not the same shape. Explains that partitioning the same wholes into more shares makes smaller shares, noticing that fourths, for example, are half the size of halves.	Calliope Latania Greg Zal
On Target Demonstrates conceptual and procedural knowledge of topic. Partitions rectangles and circles into two, four, and eight equal shares and describes them as "halves," and "fourths." Recognizes that equal shares are not always the same shape. Explains that partitioning the same wholes into more shares makes smaller shares.	Lavant Tanisha Julie Lee Moses J.B. Arya John N.
Developing Demonstrates some conceptual and/or procedural knowledge but it is not connected or complete. Partitions into two and four parts but may not understand they must be the same size OR recognizes partitions must be the same size but also thinks the parts need to be the same shape.	Silas Edla

FIGURE 5.9 Record names in a rubric during an activity or for a single content standard over several days.

Pause & Reflect

Consider the task titled "The Whole Set" on page 91. Assume you are creating a task-specific four-point rubric to share with fourth graders. What indicators would you use for the different levels of performance? Before reading further, start by thinking what an on-target level 3 performance would look like. Then read the following. ●

Determining performance indicators is always a subjective process based on professional judgment. Here is one possible set of indicators for the Whole Set task:

4. Determines the total number of cupcakes baked and the amount of each equal share, using words, pictures, and numbers to explain and justify the result and how it was obtained. Demonstrates knowledge of fractional parts and the relation to the whole.
3. Determines the correct answer or uses an approach that would yield a correct answer if not for minor errors. The drawing or the explanation does not fully describe the sharing process. Giving a correct result and reasoning for the number eaten but an incorrect result for the total baked would also be a level 3 performance.

2. Uses some aspect of fractions appropriately (e.g., divides the 15 into 5 groups instead of 3) but fails to illustrate an understanding of how to determine the whole. Shows evidence that they don't understand a fraction is a number (students may suggest a fraction is two whole numbers).
1. Shows some effort but little or no understanding of what makes the whole set of cupcakes or how to make equal shares.

Unexpected methods and solutions that you didn't predict will happen. Don't limit students to demonstrating their understanding only as you thought they would when there is evidence that they are accomplishing your objectives in different ways. Such occurrences can help you revise or refine your rubric for future use.

Indicators such as these should be shared ahead of time with students. Sharing indicators before working on a task clearly conveys what is valued and expected. In addition, when you return papers, review the indicators with students, including examples of correct answers and successful explanations. This debriefing will help students understand how they may have done better.

Pause & Reflect

How can having students assess peers' work (both strong and weak responses) support their ability to generate more in-depth answers? ●

Often it is useful to show anonymous students' work and discuss it as a group (including responses that would get full credit). Have students share what score they would give and the reasoning behind their choice. You might also have students use the rubric to evaluate a peer's work, having them discuss whether the classmate is clearly communicating. Such rubric rating activities of peers have helped middle schoolers reap long-term benefits in effectively expressing mathematical ideas (Lepak, 2014).

Student Self-Assessment

Learning Outcome 5.4 **Explain the value of students self-assessing.**

Wiliam and Leahy (2015) stress that a key strategy in effectively incorporating the formative assessment process into instruction is activating students as "owners of their own learning" (p. 169). Students' abilities to self-regulate as involved participants in their own learning process is a predictor of mathematics achievement in the intermediate elementary grades (Ocak & Yamac, 2013; Yüksel, 2013). Student self-assessments should not be the only measure of their learning or confidence but, rather, a record of how they perceive their strengths and weaknesses as they begin to take responsibility for their learning. This assessment of their own progress and growth is an important lifelong learning skill. With peer and student self-assessment, we add to the assessment *of* learning and the assessment *for* learning with assessment *as* learning, where students direct their own mathematical growth (Mok, 2011).

You can gather student self-assessment data in different ways, including pre-assessments that catch areas of confusion or identify points of confidence prior to assessing students or exit tasks formally (discussed previously in this chapter). Exit tasks can include such self-assessment questions as, "Are you confident that you can do more fraction multiplication on your own?" or "What are the ideas you learned about perimeter today?" Younger students can post a sticky note with their name on a chart under Got It or Not There Yet as a way to have them describe where they are in the learning process. For another quick and useful approach that can be used at any grade, consider a stoplight of green (I understand), yellow (I have questions), or red (I need more assistance).

As you plan for student self-assessment, consider what you need to know to find better instructional strategies and revised learning targets. Convey to your students why you are having them do these activities—that they must play a role in their mastery of mathematics

rather than just focus on completing a task. Encourage them to be honest and candid. Use open-ended prompts such as:

- How well do you think you understand the work we have been doing on fractions during the last few days? What is still causing you difficulty?
- Write two of the important things you learned in class this week.
- Which problem(s) on the activity sheet/quiz did you find the most challenging? Which did you need help on? Which were the easiest?
- What are your strengths in learning mathematics this week? What did you feel you understood well?

Discussions of how students can improve start when they analyze their own mistakes or have discussions with other students about which answers they think are correct. When students get back a test, have them use the feedback or revisit any errors and confirm that they understand what they need to learn next or how to revise. This attention to using feedback and mistakes to improve one's understanding moves students from a performance orientation to a mastery orientation (Pintrich, 2003).

Peer assessment is useful in that the students are working on the same topics and are likely to have had similar learning experiences. You can start with collaborative peer review where students share their papers as partners and then communicate ideas for improving their work before they turn it in. Many times, students make more sense to one another, and their feedback is often more easily interpreted. Remind students to listen to their peers, decide which comments they will use in their revisions, and thank their peers for the assistance. Consider asking students to give feedback on the process in rating scales that give them more agency in the learning process such as the Peer Assessment Recording Sheet (see Figure 5.10), where students put their initials where they think they are on each line. By marking the location on the arrow, students indicate the degree to which they believe they exhibit those characteristics.

Pearson eText Teacher Resource: Peer Assessment Recording Sheet

Although it may take additional time to infuse peer and students' self-assessments into the daily schedule, allowing students to take part in the formative assessment process is motivating and encourages them to monitor and adapt their approaches to learning. Remember, start the process of incorporating ideas in this chapter over time, building strategy by strategy and growing your ability to assess students effectively (Petit & Bouck, 2015).

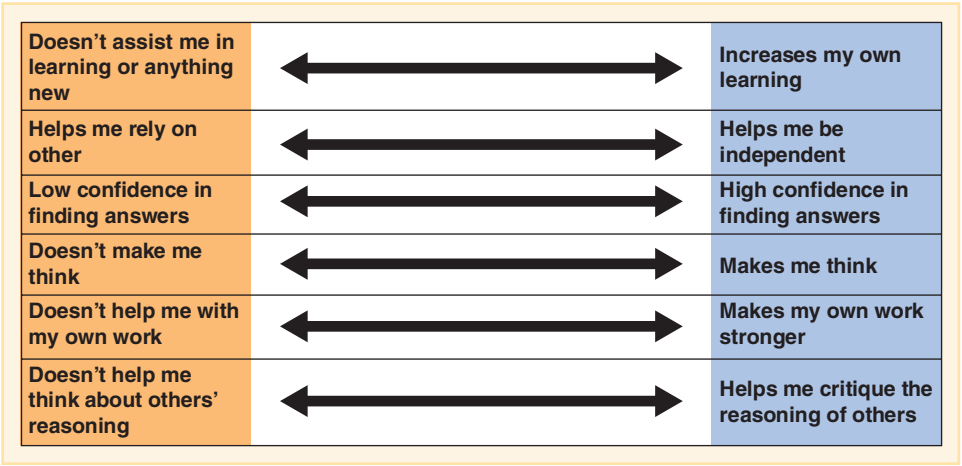

Doesn't assist me in learning or anything new
Increases my own learning
Helps me rely on other
Helps me be independent
Low confidence in finding answers
High confidence in finding answers
Doesn't make me think
Makes me think
Doesn't help me with my own work
Makes my own work stronger
Doesn't help me think about others' reasoning
Helps me critique the reasoning of others

FIGURE 5.10 Sliding scale peer assessment recording sheet

Tests

Learning Outcome 5.5 **Identify how tests can be used to enhance learning.**

Tests will always be a part of assessment and evaluation, and like all other forms of assessment, they should match your instructional goals, which attend to the five strands of mathematical proficiency described earlier and in Chapter 2. Tests should go beyond just showing how to perform an algorithm and expect students to demonstrate conceptual understanding. Additionally, tests should include application of ideas to new situations (strategic competence), and flexibility (adaptive reasoning). The following examples of open response items will illustrate these ideas.

1. Write a multiplication problem that has a product that falls between the answers to these two problems:

$$\begin{array}{rr} 49 & 45 \\ \times 25 & \times 30 \end{array}$$

2. **a.** In this division exercise, what number tells how many tens were shared among the 6 sets?

$$6\overline{)296} \qquad 6\overline{)296}$$

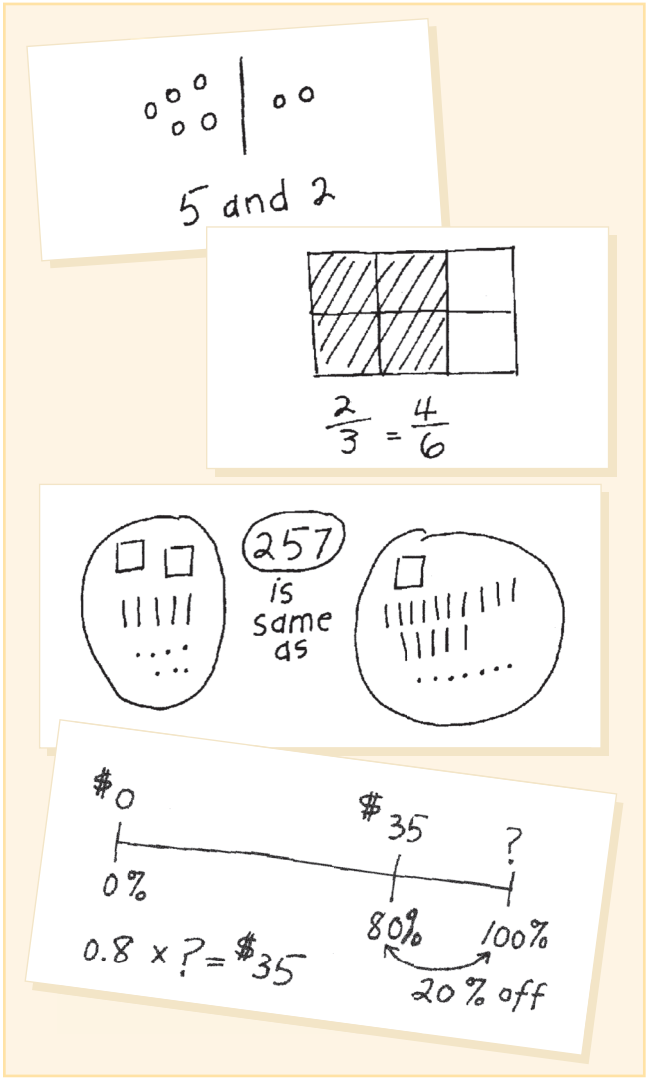


FIGURE 5.11 Students can use drawings to illustrate concepts on tests

b. Instead of writing the remainder as R2, Elaine writes "$\frac{1}{3}$." Explain the difference between these two ways of recording the leftover part.

3. On a grid, draw two figures that have the same area but different perimeters. List the area and perimeter of each.
4. For each subtraction fact, write an addition fact that helps you think of the answer to the subtraction fact.

$$\begin{array}{rrrr} 12 & 9 & 9 & 14 \\ -3 & +3 & -4 & -7 \\ \mathbf{9} & \mathbf{12} & & \end{array}$$

5. On a number line, draw pictures, using arrows to show why $^{-}3 + {}^{-}4$ is the same as $^{-}3 - {}^{+}4$.

Expanding the Usefulness of Tests

When a test is well constructed, much more information can be gathered than simply the number of correct or incorrect answers. The following considerations can help maximize your mathematics tests:

1. *Encourage students to use models, manipulatives, and drawings.* Students can use models to work on test questions when those same models have been used during instruction. Simple drawings can be used to represent counters, base-ten pieces, fraction pieces, and the like (see Figure 5.11). Provide examples in class of how to draw the models before you ask students to draw on a test.
2. *Include opportunities for explanations.* Give students the time and space to describe their thinking and their use of strategies. This component also can reveal strong or faulty reasoning (Fagan, Tobey, & Brodesky, 2016).

3. *Use open-ended questions*. Test questions that only have one correct answer can limit what you learn about what students know and can do and what they are ready to know and do!
4. *Use worked examples*. Let students evaluate and correct other students' mistakes especially of commonly made errors and confusions in algorithms, solution strategies, and reasoning (Barlow et al., 2018).
5. *Permit students to use technology when appropriate*. For example, except for tests of computational skills, calculators can allow students to focus on what you really want to test.

Consider quizzes that can be done in collaboration with a peer. These partner quizzes (Danielson & Luke, 2006) allow for more complex problems, so students can benefit from a group effort (no talking to others—only your partner). Although some partners will work individually first and then discuss the problems, others will discuss ideas along the way. You should evaluate only one of the team's quizzes and then give partners an opportunity to revise and resubmit. You can also use a digital tool such as Flipgrid (https://info.flipgrid.com) as an effective method for engaging students in video feedback as part of the formative assessment process. The increased student dialogue and discussion of answers links well to the practice of analyzing the reasoning of others!

Improving Performance on Large-Scale Assessments

While the formative assessment strategies in this book are designed to be responsive to the individual needs of students—and to provide multiple ways to view student success—students will nonetheless be required to participate in large-scale assessments commonly at every grade from 3 through 8. Large-scale assessments are used to assess the quality of programs or curriculum, monitor the school or district performance over time, and place students. But be aware that some large-scale assessments, even kindergarten readiness tests, result in the sorting and labeling of students, "resulting in segregation, marginalization, or privileging that is strongly correlated with race, language, class, and ability status" (NCTM, 2020 p. 26). To counteract some of the negative and biased large-scale assessments, we suggest ways throughout this book to develop equitable classroom assessments.

Although classroom instructors can perform assessments that tease out students' solution pathways, large-scale tests cannot. They generally focus on determining whether students can accurately perform procedures and recall facts related to their grade level standards. Summative assessment scores on high-stakes tests are often of little utility in creating instructional next steps to help students progress (Daro, Mosher, & Corcoran, 2011). But the formative assessment process described throughout this book can help.

Whatever the specific purposes or formats of the large-scale testing in your particular location, these external tests can impose significant pressure on principals, teachers, parents, and, of course, students. These tests make the pressures of testing significant for both students (Will I pass? Will my parents be upset with my score?) and teachers (Will my class meet proficiency levels?). These pressures certainly have an effect on instruction. It is unlikely you will be able to avoid the pressures of these tests, so the question becomes, "How will you respond?" Here are some tips:

- Teach the big ideas in the mathematics curriculum that are aligned with your required state standards.
- Focus on conceptual foundations. Students who have a relational understanding and who understand the procedures they are using are more likely to solve problems accurately.
- Emphasize the Mathematical Practices and doing mathematics. This supports a productive disposition, which will certainly lead to better results on a test.
- Don't make the test the purpose of learning the mathematics.

In short, maintain a problem-based approach and focus on learning mathematics because it is useful.

Communicating Grades and Shaping Instruction

Learning Outcome 5.6 **Explore ways to show evidence of and communicate about student learning that result in grades and more targeted instructional decisions.**

Assessment and evaluation go hand in hand. Although we've described the process of using tools to identify students' strengths and needs for additional experiences to provide additional challenges or targeting supports for learning, assessment is also used as a way to assign grades and communicate important results to stakeholders, particularly families.

Grading

A grade is a statistic used to communicate to others the achievement level that a student has attained in a particular area of study. The accuracy or validity of the grade depends on the evidence used in generating the grade, the teacher's professional judgment, and the level of alignment of the assessments with standards. Determining a grade requires using results of formal assessments along with other evidence about a student's work—it is not merely averaging available scores.

For effective use of the assessment information gathered from problems, tasks, and other appropriate methods to assign grades, some hard decisions are inevitable. Some are philosophical, some depend on school policies about grades, and all require us to examine what we value and the objectives we communicate to students and their families.

Among the many components of the grading process, one truth is undeniable: *What gets graded by teachers is what gets valued by students*. Using observations, interviews, exit tasks, journals, and rubric scores to provide feedback and encourage a pursuit of excellence must also relate to grades (NCTMb, 2014). However, mistakenly converting four out of five on a rubric score to a percentage or suggesting a three out of four on a rubric is a grade of C can focus the attention away from an emphasis on learning and striving for excellence (Kulm, 1994). Instead, grading must be partially based on the performances for which you assigned rubric ratings; otherwise, students will soon realize that these are not important scores. The grade at the end of the marking period should reflect a holistic view of where the student is relative to grade-level mathematics standards.

Shaping Instruction

For assessments to be useful, teachers must act on the evidence revealed in the assessments to address students' learning needs (Heritage, Kim, Vendlinski, & Herman, 2009). This process includes shifting from one approach or strategy development to another, pointing out examples or counterexamples to students, or using different materials and prompts. Knowing how to shape next steps in instruction for an individual is critical if you are going to avoid covering topics and move toward student growth and progress. If instead you just move on without some students grasping the learning objectives, "students accumulate debts of knowledge (knowledge owed to them)" (Daro, Mosher, & Corcoran, 2011, p. 48). Interspersed in Part II of this book are formative assessment notes that suggest ways to assess important content areas, including, in some cases, specific activities as follow-up lessons. As you learn more about your students through assessment, you will be able to target instruction through the learning supports provided in each content-based chapter.

RESOURCES FOR CHAPTER 5

WRITING TO LEARN

Use these questions to assess your understanding and application of chapter content.

1. What is the difference between formative and summative assessment? Give examples of each.
2. Describe the essential features of a rubric. Give three examples of performance indicators.
3. How can you incorporate observational assessments into your daily lessons? What is one method of recording observations that you would try?
4. How can students with limited writing skills "write" in mathematics class?
5. How do diagnostic interviews help capture student thinking?

RECOMMENDED READINGS

Article

Duker, B., Holmberg, C., & Rossi Becker, J. (2017). Making moves: Formative assessment in mathematics. *Mathematics Teaching in the Middle School*, *22*(6), 334–342.

The authors describe a framework of seven moves that increase students' higher-level thinking through engaging class discussions. Tasks, examples of the moves, and insights about student learning are shared.

Books

Fennell, F., Kobett, B., & Wray, J. (2017). *The formative 5: Everyday assessment techniques for every math classroom*. Thousand Oaks, CA: Corwin Press.

The focus is on formative assessments for use in every grade for every learner. Emphasizing the strategies of observations, interviews, show-me, hinge questions, and exit tasks, this valuable guide includes actual stories from classrooms.

Keeley, P., & Tobey, C. R. (2016). *Mathematics formative assessment: 50 more practical strategies for linking assessment, instruction and learning*. Thousand Oaks, CA: Corwin.

If you are looking for a strong collection of assessment prompts from across the grades, you've found what you need. Two veteran leaders in mathematics teaching pulled together tried and true ideas that help teachers know what next instructional steps are needed.

SanGiovanni, J., & Novak, J. (2017). *Mine the gap for mathematical understanding, grades 6-8: Common holes and misconceptions and what to do about them*. Thousand Oaks, CA: Corwin.

One of a three part series (also K–2 and 3–5), this book helps teachers know how to use student work to figure out, what they should do next.

Wright, R., & Ellemor-Collins, D. (2018). *The learning framework in number: Pedagogical tools for assessment and instruction (math recovery)*. Thousand Oaks, CA: Sage Publications, Inc.

This book includes assessment tools, ways to summarize the assessment by grade level expectations, and teaching progressions to guide where the student is on assessment for the four basic whole-number operations.

CHAPTER

6

Teaching Mathematics Equitably to All Students

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **6.1** Connect Mathematics Teaching Practices to equitable mathematics teaching.
- **6.2** Describe the components of a multitiered system of support.
- **6.3** Identify successful interventions for students who struggle and students with special needs in math.
- **6.4** Explain mathematics language demands and strategies to support multilingual learners.
- **6.5** Apply knowledge of advanced education and talent development.

Mathematics teaching must improve. As a nation, the United States is far from ensuring that each and every child has access to high-quality mathematics learning, despite this being the focus of mathematics educational reform for decades (NCTM, 1989; NCTM, 2000; NCTM, 2014b). As you read in Chapter 1, far fewer than half of fourth graders and eighth graders are reaching mathematical proficiency (National Center for Education Statistics (NCES), 2019). Mathematics achievement is not the only way in which mathematics teaching is failing U.S. students. Far too many students lack confidence, do not like, or do not feel like they can do mathematics. To ignite the needed and significant changes in mathematics teaching, NCTM published a *Catalyzing Change* (NCTM, 2020) series (for each grade band). Additionally, many researchers have provided evidence of what it takes really to prepare all students to become mathematically proficient (for example, Bartell, Wager, Edwards, Battey, Foote, & Spencer, 2017a). The consensus of these researchers is that students often do not have access to equitable mathematics instruction.

All students can and must reach mathematical proficiency regardless of race, ethnicity, linguistic, sex, gender, and socioeconomic status (NCTM, 2014b). Teaching for diverse learners includes providing equitable mathematics teaching, combined with specific strategies to support individual students, including:

- Students who have special needs in math.
- Students who are multilingual learners (sometimes called English learners).
- Students who would benefit from advanced education or talent development.

This chapter begins with a section on equitable mathematics instruction, followed by sections that focus on research-based instructional strategies for specific learners.

Equitable Mathematics Instruction

Learning Outcome 6.1 **Connect Mathematics Teaching Practices to equitable mathematics teaching.**

Equitable mathematics instruction takes into consideration content, relationships, cultural knowledge, flexibility in approaches, use of familiar or interesting learning contexts, a responsive learning community, and working in cross-cultural partnerships (Aguirre & del Rosario Zavala, 2013). Equitable mathematics instruction can improve the performance of all students, narrow the academic performance gap, and increase students' mathematics course–taking (Boaler, 2008; Kisker et al., 2012; Thompson, 2017). To be clear, equitable instruction abandons actions based in *deficit perspectives*. Here are a few examples of what to avoid:

"My students can't do story problems because they barely speak English."

"These problems are too hard for my low kids."

"We are working to close the achievement gap for our students of color."

Each of these statements communicates that certain students are not as capable at doing mathematics. Over time, this thinking may lead to teachers having different expectations for students from some racial or ethnic groups, immigrants, and students with special needs in math. These students are thereby denied access to a challenging, quality mathematics program. Instead, equitable instruction seeks to highlight the power of mathematical thinking from children with diverse academic, cultural, linguistic, and socioeconomic backgrounds. As stated in *Catalyzing Change* (2020):

> **We must openly challenge deficit labels and the institutional tools and practices that perpetuate static views about children's mathematical abilities and about who is or is not ready to learn. Each and every child is always ready and eager to learn more about their mathematical world. It is the adults who must reexamine their beliefs about readiness and learn to notice and support children's ever-evolving mathematical strengths. Confronting our beliefs is the first step toward more equitable mathematics learning opportunities (p. 32).**

Rather than focus on deficits, equitable mathematics instruction focuses on the strengths students bring to their learning. You instead connect to student strengths when your lessons incorporate children's questions, interests, and lives. This approach includes attending to students' funds of knowledge (Bartell et al., 2017; Civil, 2016; González, Moll, & Amanti 2005). Students' *funds of knowledge* include their home, culture, language, and community-based knowledge and experiences. Valuing a person's background is demonstrated through actions that communicate to the student, "I want to know about you, I want you to see mathematics as part of your life, and I expect that you can do high-level mathematics." For example, if you are teaching the topic of area and working in an urban setting, a story such as *City Green* (Disalvo-Ryan, 1994) may connect to students' community knowledge of how a land plot in an urban community is shared among neighbors. Telling relatable stories about your life, or asking students to tell stories related to their experiences, makes the mathematics relevant and can raise student achievement (Turner et al., 2009). Exploring the work of diverse mathematicians is also of interest (https://uploads.desmos.com/assets/desbook/desmos-mathematician-names.pdf). Beyond telling stories, you can incorporate literature that reflects diversity. Table 6.1 offers a list of books that are excellent choices for their stories as well as for their connections to mathematics learning. You can find a variety of YouTube videos and other online sources where these books are read aloud.

TABLE 6.1 CULTURALLY DIVERSE MATHEMATICS LITERATURE LIBRARY

George Rudy/ Shutterstock

Title	Author, Date & Age Level	Story Line
Counting on Katherine: How Katherine Johnson Saved Apollo 13	Helaine Becker 2018, 6–8	A Black mathematics expert supports the NASA space missions.
Blockhead: The Life of Fibonacci	Joseph D'Agnese 2010, 6–11	A daydreamer identifies a beautiful mathematical pattern.
Odd Boy Out: Young Albert Einstein	Don Brown 2008, 4-7	Initially thought to be odd, Einstein becomes one of the great thinkers of all time.
Nothing Stopped Sophie: The Story of Unshakable Mathematician Sophie Germain	Cheryl Bardoe 2018, 4–8	Sophie was self-taught and persevered beyond challenges to develop an important formula.
The Girl with a Mind for Math: The Story of Raye Montague	Julia Finley-Mosca 2018, 5–10	Sometimes keeping her accomplishments a secret, this Black woman becomes a brilliant engineer.
The World Is Not a Rectangle: A Portrait of Architect Zaha Hadid	Jeanette Winter 2017, 5–11	An Iranian architect faces obstacles in her career, but eventually her love of math leads her to become the first woman to win the biggest prize in her field.
The King's Chessboard	David Birch 1993, 6–adult	Set in India, a wise man uses mathematics to outwit a king.
The Librarian Who Measured the Earth	Kathryn Lasky 1994, 5–adult	The Greek librarian and geographer Eratosthenes invents a way to measure the earth's circumference.
One Hen: How One Small Loan Made a Big Difference	Katie Smith Milway 2020, 8–12	A boy from Ghana buys a hen with a microloan and a year later has 25 hens. He ends up with a thriving farm.
Hidden Figures: The True Story of Four Black Women and the Space Race	Margot Lee Shetterly 2016, 8–12	A group of Black female mathematicians help the U.S. achieve in space.
Ticktock Banneker's Clock	Shana Keller 2016, 6–9	A self-taught Black inventor uses perseverance and engineering skills to make a striking clock.
Wilma Unlimited: How Wilma Rudolph Became the World's Fastest Woman	Kathleen Krull 2000, 4–7	Although afflicted with polio as a child, Wilma is the first American woman to win three gold medals in a single Olympiad.
Ron's Big Mission	Rose Blue 2009, 6–8	One of the first Black astronauts uses peaceful resistance as a young boy to desegregate his library.
The Boy Who Dreamed of Infinity: a Tale of the Genius Ramanujan	Amy Alznauer 2020, 4-7	Independent learner who started as a clerk in India and became one of the great mathematical thinkers of all time.
The Girl Who Thought in Pictures: The Story of Dr. Temple Grandin	Julia Finley Mosca 2017, 4–11	A learner on the autism spectrum becomes a passionate scientist working with animals. The message: learning differences can be a benefit!

Pearson eText Teacher Resource: Culturally Diverse Mathematics Literature Library

Equitable mathematics instruction means implementing the eight Effective Teaching Practices while also nurturing children's mathematics identity and agency (NCTM, 2020). The eight Teaching Practices were the focus of Chapter 3. Equitable mathematics instruction attends to three aspects: (1) focusing on a deep understanding of mathematics, (2) developing a positive mathematics identity, and (3) nurturing a sense of mathematical agency. The relationship of these three aspects to the Mathematics Teaching Practices (NCTM, 2014b) is illustrated in Figure 6.1. The arrows in the framework are a reminder that all eight Mathematics Teaching Practices contribute to mathematics learning, and therefore, each of the eight Teaching Practices influences the three aspects of equitable mathematics instruction in indirect ways.

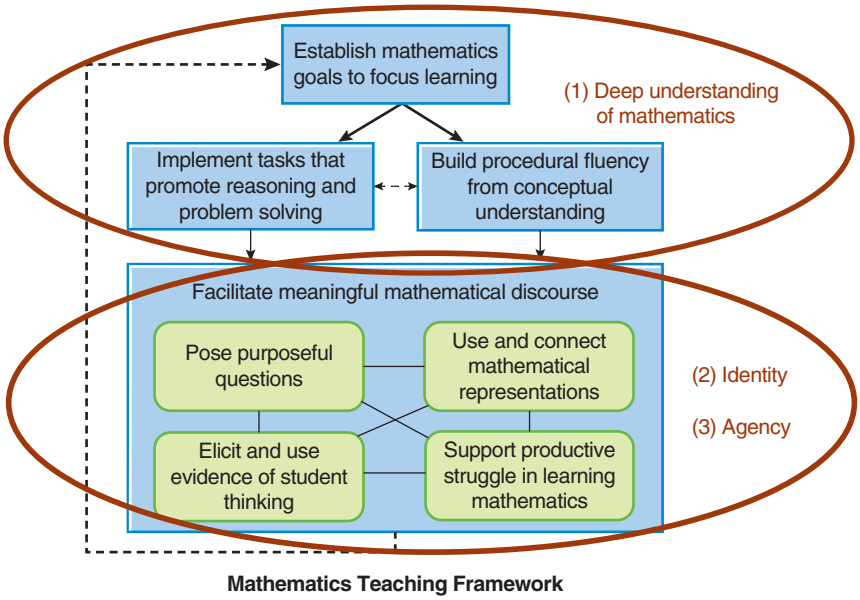

Establish mathematics goals to focus learning
(1) Deep understanding of mathematics
Implement tasks that promote reasoning and problem solving
Build procedural fluency from conceptual understanding
Facilitate meaningful mathematical discourse
Pose purposeful questions
Use and connect mathematical representations
Elicit and use evidence of student thinking
Support productive struggle in learning mathematics
(2) Identity
(3) Agency
Mathematics Teaching Framework

FIGURE 6.1 Aspects of equitable mathematics teaching connected to the Mathematics Teaching Framework. From D. Huinker & V. Bill. (2017). *Taking Action: Implementing Effective Mathematics Teaching Practices in Kindergarten–Grade 5*. Taking Action Series. NCTM.

Deep Understanding of Mathematics

Equitable instruction focuses on the big ideas of mathematics (i.e., is based on state standards), with a focus on conceptual understanding and procedural fluency. Equitable instruction also attends to Mathematical Practices (Bartell et al., 2017; NCTM, 2020; Wood et al. 2019). Recall that these are the eight practices that define a person who is mathematically proficient (see Table 1.1 and Appendix A). A mathematics lesson, therefore, must establish learning goals that focus on both mathematics content and Mathematical Practices, and it must go deeper than just naming a mathematical practice. Objectives should connect the content and the selected practice. That is a start to ensure that students develop mathematical proficiency. Deep understanding also requires using tasks that focus on reasoning as well as attention to procedural fluency. Compare a before-and-after look at the first parts of a lesson plan, revised to focus on deep understanding.

Grade 2 Standard: ***Fluently add and subtract within 100, using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.*** **[2.NBT.B.5] (NGA & CCSSI, 2010]**

Lesson A: *Shallow Understanding*

Objective

1. Students will add using a partial sums strategy.

Task: Cut out cards with addition problems. Students cut and paste each in their journal and solve each using the partial sums method.

This plan only attends to content, and the content goal itself does not address *understanding* the meaning of addition or procedural fluency. The task, while tactile, still lacks opportunities for students to reason or develop their fluency (recall that fluency includes flexibility). So, how can this lesson be revised to focus on these aspects of equitable instruction? Increase the rigor of the objective, add a focus on one Mathematical Practice, and increase the opportunities for reasoning in the task.

Lesson B: *Deep Understanding*

Objectives

1. Students will select and implement strategies for adding.
2. Students will justify their strategy, using manipulatives, semi-concrete representations (e.g., hundred chart or number lines), or words. [MP 2-Reasoning abstractly and quantitatively]

Task: Cut out cards with addition problems. Students sort the cards based on the strategy they will use to solve them. Then they glue them in their journal and prepare an illustration or explanation of the strategy used.

Lesson B not only reflects the three Teaching Practices identified at the top of the framework, it also incorporates students' funds of knowledge, because students are invited to use their own strategies to represent and solve the problems.

Lesson B also sets up the opportunity for facilitating discourse (the rest of the Mathematics Teaching Practices) because students can engage in discussions about which strategies fit which problems (in small groups and in whole class discussions). Using tasks worthy of group work, emphasizing multiple representations, and incorporating student justifications and presentations are features of classrooms that support equitable opportunities to learn mathematics (Cabana, Shreve, Woodbury, & Louie, 2014; Dunleavy, 2015). How that discourse is facilitated has a significant influence on students' mathematics identities and sense of agency (Berry, 2019).

Mathematics Identity

A student's *mathematics identity* is a sense of themself as a doer of mathematics and includes their disposition toward mathematics and sense of competence as learner and contributor in the mathematics classroom (Cobb, Gresalfi, & Hodge, 2009). Attending to a student's identity has a powerful impact on your development of equitable Teaching Practices (Aguirre, Mayfield-Ingram, & Martin, 2013). Whether intentional or not, all teaching is identity work, because students are constantly adapting and redefining themselves based on their experiences (Gutierrez, 2015). When a teacher (or family member) says, "I was never good at math" or "Math is hard," it can instill a false perception that only some people are good at mathematics and that it is inherently difficult. This can lead students to think they are not capable of doing mathematics—a negative mathematics identity. Our goal is to develop positive mathematics identities in each and every student. (This should remind you of the productive dispositions described as part of mathematical proficiency in Chapter 2!)

There are a number of ways that teachers shape students' mathematics identities. One way to do that is positioning students as capable (Bartell et al. 2017; Chval, Smith, Trigos-Carrillo, & Pinnow, 2021). You can assign competence: attribute ideas to individuals, saying something like, "That idea connects to a strategy Nicolas used yesterday." This approach assigns

competence to Nicolas as capable in mathematics, influencing how he perceives himself, as well as how other classmates might perceive him (Boaler & Staples, 2014; Featherston et al., 2011). Another technique is to establish a norm of the "… yet" prompt: students are expected to change statements like, "I don't know how to do this" to "I don't know how to do this *yet*" (Wood et al., 2019). Students do get stuck as they solve complex tasks. Showing students how to solve the problem may get them to an answer, but it does not position them as capable problem solvers. An alternative is to say something that helps students continue thinking. Two effective teacher moves to support students engaged in productive struggle are: "You have an idea. What is it?" and "What could you do?" (Munson, 2018). Finally, a great way to position students as capable is to expect students to use one another as mathematical resources (Dunleavy, 2015). Over time, these teacher moves communicate to students that they are capable doers of mathematics.

You also influence identities when you (1) provide the tools and mathematical representations that support mathematical reasoning, (2) support productive struggle, (3) elicit student mathematical thinking, and (4) pose purposeful questions. Note that these are the four Teaching Practices in the Facilitating Discourse box of the Mathematics Teaching Framework. One of the most important things you do is display a positive mathematics identity yourself and make sure you are focusing on developing a positive mathematics identity for *each and every* student, not just some students. Conversely, you cannot display your own math anxiety to students if you experience it. Students can develop or experience a worsening of their own mathematics anxiety by observing math anxiety in their teachers (or parents/families) (Beilock & Willingham, 2014). Ways to reduce student anxiety include telling students how well they are doing in mathematics class, giving students feedback on their strengths, and telling students what they need to do to become better in mathematics (OECD, 2016).

Each student is different, so supporting the development of each student's positive mathematics identity is one of the great challenges of teaching. Imagine the student who is reluctant to share in class. Why might that be? It could be fear of being wrong, which actually shuts down problem-solving activity in the brain (Boaler, 2019; Terada, 2020). So, establishing a classroom norm that focuses on ideas, and acknowledges that doing mathematics naturally involves making mistakes, can invite participation. Use activities that invite participation from every student and are open to many ways of thinking, such as the routines described in Chapter 4. Conversely, avoid activities that create anxiety, such as competitive activities and timed tests. Other students may not participate because they do not relate to the contexts you are using. Rather than just assume what students' interests are, find out what they are and connect those interests to your math lessons. Some students may think they are not good at mathematics because they are not fast at answering. Frequent noticing of students who are persevering, making multiple attempts at a problem, and using an interesting approach are ways to communicate what someone good at math looks like. Avoid saying, "Wow, that was fast," or calling on the first hands up, or doing any kind of competitive races. All of these actions communicate that being good at math is being fast, and that is simply not true (and such an emphasis results in many students developing negative mathematics identities).

Mathematical Agency

A student with a positive mathematics identity is confident that they can eventually reach a solution to a problem. These students want to figure out problems themselves, not be shown how to do it; they want to solve challenging problems, not easy ones; they continue to try new pathways, not give up when a first approach doesn't work. They have agency. Mathematical agency is the behavioral side of a person's mathematics identity, or their identity-in-action. When a person is able to participate actively and perform effectively in mathematics contexts, they have mathematical agency (Aguirre, Mayfield-Ingram, & Martin, 2013). Students with mathematical agency exhibit the five strands of mathematical proficiency, demonstrating conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and a productive disposition (described in Chapter 2).

It is through teacher actions in advancing an equitable mathematics classroom that students' mathematical agency develops. It begins with creating a positive classroom environment, one in which all students feel as if they can participate and learn. One way to support mathematical agency is *not to tell* students whether their answer is right or wrong. Students themselves can confirm whether an

answer is correct through their own reasoning. Another way is to expect students to determine their own solution pathway and give students choices for how they will engage with and solve the problem (like Lesson B described earlier). Mathematical agency can also apply to a classroom environment. A classroom where students feel comfortable taking risks, are not worried about making mistakes, and value reasoning and each person's capacity to contribute illustrates a collective mathematical agency (Aguirre, Mayfield-Ingram, & Martin, 2013). Teachers support such a community by setting norms for participation, encouraging risk-taking, and valuing productive struggle.

The way that you assign groups has an impact on both identity and agency. Although unintentional, the unfortunate outcome of grouping students in low-, middle-, and high-ability groups results in students making decisions about themselves as doers of mathematics (identity). The groups described as low tend to receive more help and engage with easier tasks, which can lead to negative mathematics identities and interference with students' positive mathematical agency because they are denied access to tasks that are challenging and involve reasoning (Boaler 2016). The result is logical and disturbing: children placed in low-ability tracks stay there, their curriculum focuses on skills, not deep mathematics, and the outcome is a trajectory of lower achievement in mathematics (Fernandes, Crespo, & Civil, 2017; Stein, Kaufman, Sherman, & Hillen, 2011). As stated in *Catalyzing Change*, educators must create equitable structures, that we "dismantle inequitable structures, including ability grouping and tracking" (NCTM, 2020, p. 9). Mixed-ability groups are necessary to ensure that every child develops a positive mathematics identity and sense of mathematical agency.

Table 6.2 lists three aspects of equitable mathematics instruction along with some reflection questions. Combined with the eight Mathematics Teaching Practices actions, these questions can help implement equitable instruction.

TABLE 6.2 ASPECTS OF EQUITABLE MATHEMATICS INSTRUCTION

Aspect	Teacher Reflection Questions
1. Deep understanding of mathematics	• Do learning goals focus on understanding big ideas in mathematics? • Do learning goals explicitly attend to one or more Mathematical Practices? • To what extent does the task support student mathematical reasoning? • What potential does the task have for students to connect mathematical representations? Justify their strategies and answers? • To what extent does the task engage students in developing procedural fluency, conceptual understanding, and connections between conceptual and procedural knowledge? • To what extent will the task connect to (or have potential to connect to) students' funds of knowledge?
2. Positive mathematics identities	• Is mathematics talked about in positive ways and as a domain that is accessible to everyone? • Is a productive disposition made visible to students and connected with being a mathematician? • Are students positioned as capable? • Is productive struggle valued and supported? • Are each student's ideas presented and showcased so that each student sees their ideas as important to the teacher and their peers?
3. Sense of mathematical agency	• Is risk-taking encouraged? • Are students given a choice in how they solve a problem? • Are multiple representations used and valued (e.g., visuals, explanations, equations, models)? • Are students justifying solutions? • Are norms established for participation so that every student feels that they are contributing to and learning from other members of the class? • Are students invited (expected) to engage in whole-class discussions in which they share ideas and respond to one another's ideas?

Teacher Resource: Reflection Questions to Guide Equitable Mathematics Teaching

Implicit Bias and Structural Barriers in Mathematics Learning

Implicit biases are "attitudes or stereotypes that affect our understanding, actions, and decisions in an unconscious manner" (Staats et al. 2016, p. 14). We all have implicit biases. As Herbel-Eisenmann and Shah (2019) state, "Teachers are some of the best-intentioned people we have in society. And yet, they are not immune to the influence of implicit bias" (p. 284). While we say all children can learn, we sometimes can inadvertently do things that lower expectations (and the performance) for some learners. For example, let's consider participation. In an attempt to engage all students, teachers try to call on different students, but years of studies have found that the types of questions asked to different students uncover teachers' implicit biases (Herbel-Eisenmann & Shah, 2019; McAfee, 2014; Sadker, Sadker, & Zittleman, 2009). For example, boys get asked more high-level thinking questions than girls do, and similarly, students of color get asked fewer high-level questions than their White peers. When students are working in small groups, White males tend to dominate the conversation over students of color and females, which shapes each student's mathematics identity (Esmonde, Brodie, Dookie, & Takeuchi, 2009). How will you ensure that you are equitably asking questions to your students? One way is to gather data on your questioning intentionally, for example, when being observed (or by videotaping yourself and reflecting).

Catalyzing Change (NCTM, 2020) identifies four structural barriers that exclude or limit students' access and opportunity to learn mathematics:

1. Ability grouping
2. Tracking
3. High-stakes assessments and readiness measures
4. Inflexible curriculum implementation

The first two on the list lead to differential learning opportunities that (1) widen achievement gaps and (2) negatively affect students' mathematics identities. The third item leads to the labeling and sorting of students and has historically led to segregation and marginalization related to race, language, class, and ability status. Finally, an inflexible curriculum does not allow for students who need more time on a topic, or more enrichment, to receive the instruction they need.

Addressing the Needs of Specific Learners

The goal of equitable mathematics instruction is that each and every student has access to developing a deep mathematics understanding, positive mathematics identity, and sense of mathematical agency. The instructional strategies described so far refer to all students. However, inequities can still exist, even if unintentionally, if we do not attend to specific resources and needs of specific learners. For example, if teachers do not build in opportunities for student-to-student interaction in a lesson, they may not be addressing the needs of multilingual learners, who benefit from opportunities to speak, listen, and write in small-group situations. It takes more than a desire to be fair or equitable; it takes knowing the strategies that support each learner and making every effort to incorporate those strategies into your teaching. In the sections that follow, we outline specific strategies to ensure that students have access and opportunity to develop deep mathematical understanding, positive mathematics identities, and mathematical agency.

Pearson eText
Video Example 6.1
In this video late author John Van de Walle discusses equitable instruction through access to tasks.

Multitiered System of Support (MTSS)

Learning Outcome 6.2 **Describe the components of a multitiered system of support.**

In many schools, a systematic process for achieving higher levels of performance for all students includes a multitiered system of support (MTSS), one component is known as Response to Intervention (RtI). States have a variety of multitiered models, but they customarily center on three interwoven elements: high-quality curriculum, instructional interventions, and formative assessments that capture students' strengths and foundational learning needs. Multitiered models were designed to determine whether lower achievement was due to the need for more

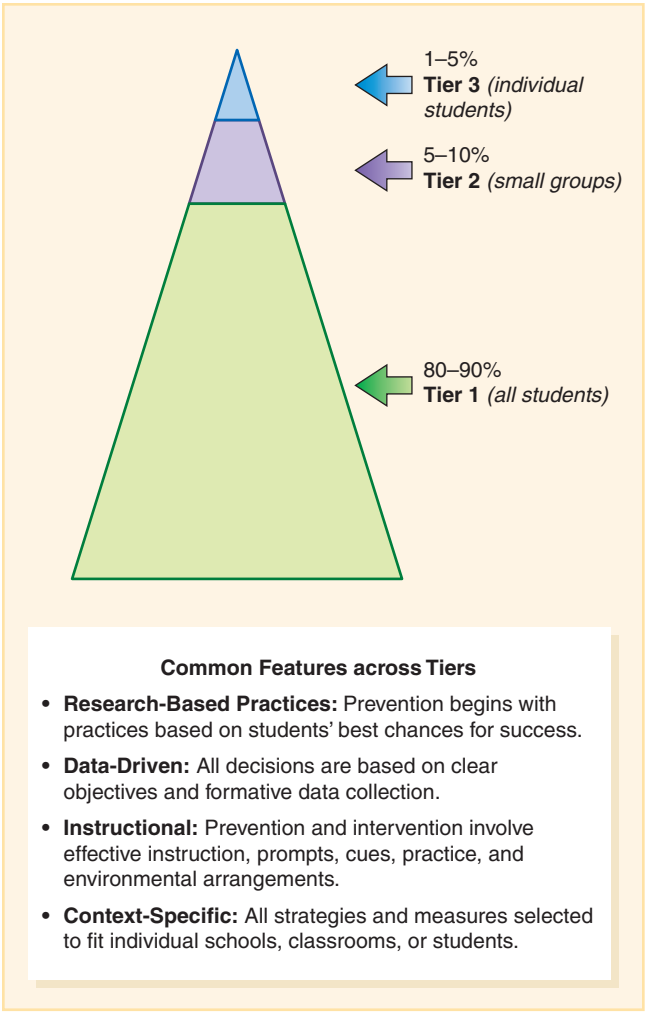

1–5%
Tier 3 (individual students)
5–10%
Tier 2 (small groups)
80–90%
Tier 1 (all students)
Common Features across Tiers
• Research-Based Practices: Prevention begins with practices based on students' best chances for success.
• Data-Driven: All decisions are based on clear objectives and formative data collection.
• Instructional: Prevention and intervention involve effective instruction, prompts, cues, practice, and environmental arrangements.
• Context-Specific: All strategies and measures selected to fit individual schools, classrooms, or students.

FIGURE 6.2 Multitiered systems of support

Source: Based on T. Scott, & H. Lane, (2001). *Multi-Tiered Interventions in Academic and Social Contexts.* Unpublished manuscript, University of Florida, Gainesville.

high-quality mathematics instruction or to a student's actual learning challenges. MTSS can also determine instructional options for students who may need advanced education.

MTSS is a preventive student support system that is often represented by tiers representing a level of instruction in a triangular format (see Figure 6.2). As you move up the tiers, the number of students involved decreases, the teacher–student ratio decreases, and the intensity of the intervention increases.

Tier 1 is the foundational and largest portion of the triangle, representing core instruction presented to all students based on a high-quality mathematics grade-level curriculum, research-based instructional practices, and ongoing monitoring of students' progress. Tier 2 represents students who did not reach the level of achievement expected during Tier 1 instruction or those who showed that they were performing beyond grade level and required advanced interventions. If further assessment after Tier 2 interventions reveals that students who were struggling made favorable progress, interventions are frequently faded and discontinued. But if difficulties and struggles remain for students, or students continue to show performance well above grade level, interventions can be adjusted in intensity, and in some cases, they can be referred to Tier 3 for more intensive instruction. Students should receive all three tiers of instruction and intervention before assignment to special education.

An important part of MTSS is the guiding of students' movement within tiers, which includes the examination of their growth and development through *progress monitoring*. Some schools use a universal screener, and all students are considered for movement in and out of interventions to build strengths. Another way to collect evidence of student knowledge of concepts and skills is by using the diagnostic interviews described in Chapter 5 and shared throughout this book in formative assessment notes.

Supporting Students Who Struggle and Those with Special Needs in Math

Learning Outcome 6.3 **Identify successful interventions for students who struggle and students with special needs in math.**

To begin this section, we want to state clearly that it is aligned with efforts to address deficit perspectives. We are not fixing students; we are fixing structures. Students with special needs in math do not have deficits; they have cognitive differences and, like all students, must be positioned as capable. Approximately 14% (7.1 million) of students receive special education services (NCES, 2020). These students have individualized education programs (IEPs) that guarantee their access to grade-level mathematics content—preferably in a general education classroom.

It is from this background about repositioning from deficits to assets that we suggest the use of respectful and inclusive language. We have purposefully moved away from the word *disability* to the phrase *special needs in math*. By doing, so we honor the reality that students often do not have special needs in all content areas. We also recognize that the language of disabilities can limit, and in its place, we prefer to expand the idea that all students can learn at high levels by the way we speak about children.

MTSS emphasizes ways for students who struggle to get immediate assistance and support rather than waiting for them to fail on a high-stakes test before they receive formalized help. Students moved to Tier 2 should receive supplemental targeted instruction, using interventions that incorporate systematic teaching of foundational skills and concepts, more frequent instructional opportunities, and more supportive and precise prompts (Torgesen, 2002).

Strategies for interventions in the three tiers are outlined in Table 6.3.

TABLE 6.3 INTERVENTIONS FOR TEACHING MATHEMATICS

Tiers	Intervention
Tier 1	**A highly qualified regular classroom teacher:** • Incorporates high-quality curriculum based on grade-level content standards • Develops the Mathematical Practices in learners • Uses multiple representations such as manipulatives, visual models, and symbols • Monitors progress to identify students who struggle and students who excel • Uses flexible student grouping • Fosters active student engagement in doing math • Communicates high expectations
Tier 2	**A highly qualified regular classroom teacher, with collaboration from other highly qualified educators (i.e., special education teacher, multilingual learner specialist, speech pathologist):** • Presents interventions to students in small groups in supplemental sessions to the core instruction that often revisits foundational concepts from prior grades • Conducts diagnostic interviews to target a student's strengths and needs • Creates lessons that emphasize big ideas or themes • Incorporates CSA (concrete, semi-concrete, abstract) approach • Shares *think-alouds* to demonstrate how to make problem-solving decisions • Incorporates systematic strategy instruction (i.e., connects new learning to prior learning and summarizes key points or concepts) • Models specific tools and strategies, such as how to use measuring tools or number lines • Uses prompts or self-questions written on cards or posters to help students follow, for example, the stages solving word problems or summarizing the properties of the operations • Uses peer-assisted learning, in which another student can provide assistance • Supplies families with additional instructional support materials to use at home • Provides opportunities for retrieval of important information
Tier 3	**A highly qualified special education teacher, general education teacher, or mathematics interventionist:** • Works one-on-one with students • Uses tailored instruction based on specific areas of need • Modifies instructional methods, motivates students, and adapts curricula • Uses explicit contextualization of skills-based instruction

Implementing Interventions

The updated *What Works Clearing House (WWCH) Practice Guide: Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades* (Fuchs et al., 2021) suggests a set of effective, research-based recommendations and strategies for teaching the subset of students for whom the initial core instruction (Tier 1) was not effective. These six recommendations include using systematic instruction, precise mathematical language, carefully selected representations, number lines, deliberate instruction on word problems, and fluency-building activities. We will examine several of the WWCH recommendations here with others, such as number lines, word problems, and fluency activities integrated throughout the book in Part II.

Systematic Instruction. Systematic instruction is characterized by highly structured, teacher-led instruction on specific strategies and skills. The approach centers on an orchestrated mix of previously taught content and new concepts to help students connect prior learning to new learning and is considered a high-leverage practice in special education (McLeskey et al., 2017). When engaging in systematic instruction, you do not merely demonstrate the strategy and have students practice it; instead, you try to illuminate the decision making along the way. For example, use think-alouds where you verbally share what's in your head to describe your decision-making process and actions you are taking as you work through a problem. Students can do think-alouds, too. Another option is to examine students' (anonymous or current)

worked examples—a process that may be initially challenging for some learners without a level of support from teachers or peers. Systematic instruction focuses on building meaning by consistently pointing out critical connections between new knowledge and concepts they already know. Let's look at a classroom teacher using systematic instruction (Figure 6.3). Notice how the teacher guides the instruction and provides structured opportunities for practice.

As you enter Mr. Logan's classroom, you observe a small group of students seated at a table listening to the teacher's questions and looking at visuals that represent equivalent fraction concepts. The students are using manipulatives, as suggested by Mr. Logan, and moving through carefully selected and sequenced experiences. He tells the students to take out the red one-fourth pieces and asks them to explore how many one-fourths will exactly cover the blue one-half piece. Mr. Logan asks, "Is equivalent a word you know?" Then, to make sure students don't allow for any gaps or overlaps in the pieces, he asks them to talk about their reasoning process by asking, "What are some things you need to keep in mind as you cover the half with fourths?" Mr. Logan writes their responses on the adjacent whiteboard as $\frac{1}{2} = \frac{2}{4}$ and as "one-half is the same as two-fourths" to connect the ideas that they are looking for how many fourths cover the one-half piece. Then he asks students to compare the brown eighths and the yellow sixths to the piece representing one-half and records their responses. The students take turns answering these questions aloud. During the lesson, Mr. Logan frequently stops, asks for points of clarification, and directly highlights components of the task. For example, he asks, "Are you surprised that it takes more eighths to cover the half than fourths?" He uses a think-aloud to verbalize how he is making sense of that relationship. Vocabulary words, such as *whole, fourth, half*, and *equivalent*, are written on the math wall nearby, and the definitions of these terms are written or a visual is shown to reinforce this mathematical language throughout the lesson. At the completion of the lesson, students try some similar examples, and then they come back together for a whole-class discussion. A number of aspects of the systematic instruction can be seen in Mr. Logan's approach to teaching fractions. He employs a teacher-directed instructional format, carefully describes the use of manipulatives as related to the problem, and incorporates a model-prompt-practice sequence. This sequence starts with verbal discussions and demonstrations with concrete models, followed by prompting, questioning, and then independent practice. The students derive mathematical knowledge from the teacher's oral, written, and visual clues.

FIGURE 6.3 Systematic instruction during an intervention

Manipulatives support explicit strategy instruction. For example, a teacher demonstrating a multiplication array with one-inch squares might say, "Watch me. Now make a rectangle with the squares that looks just like mine. Let's point out the groups—one group of five, two groups of five, three groups of five, and four groups of five" to show 4×5. In contrast, a teacher with a more inquiry-oriented approach might say, "Using these squares, how can you show me a representation for four groups of five?" Although initially more structured, the use of concrete models in this fashion provides students with special needs in math greater access to abstract concepts.

The use of self-instructive prompts, or self-questions, can help students structure the entire learning process from beginning to end. Unlike more inquiry-based instruction, the teacher may initially model these steps and explain components, using terminology that is easily understood by students who did not discover these ideas independently. Consider a problem in which fourth-grade students are asked to determine how much paint is needed to cover the walls of their classroom. Rather than merely demonstrating, for example, how to use a ruler to measure the distance across a wall, the think-aloud strategy involves talking through the process and identifying the reasons for each action while measuring the space. As you place a tick mark on the wall to indicate where the ruler ended in the first measurement, you might state, "I used this line to mark off where the ruler ends. How should I use this line as I measure the next section of the wall? I know I have to move the ruler, but should I repeat what I did the first time?" All of this dialogue occurs prior to placing the ruler for a second measurement. When you use this metacognitive strategy of talking about and modeling approaches (and the reasons behind these approaches), you make your invisible thinking processes visible to students.

Using strategy discussion (for example, in the *after* phase of a lesson) makes more explicit the covert thinking that others use in mathematical problem solving, but some students with special needs in math may not be able to keep up with the rapid pace of the sharing. Without extra time to reprocess the conversation, some students with special needs in math may not get access to these helpful strategies. Recall that equitable instruction requires facilitating discourse that supports the mathematics identities and agency of all learners. For example, support students with a structure for providing explanations or describing their thinking, such as using sentence starters, verbal prompts, or guiding questions. Guiding questions can support connections such as "Tell me how the drawing you made shows how you would divide the 15 apples among the 3 friends."

Effective systematic instruction is scaffolded, meaning it moves from a highly structured, focused strategy approach toward multiple models, including examples and non-examples. The key is not to over-scaffold; you want to move students to independence. Over-scaffolding involves taking thinking away from students rather than supporting their thinking. Examples include when asking low-level or leading questions, telling students how to solve a problem, or presenting a shortcut or trick that is not connected to a student's understanding. Instead of altering our expectations, we want students to discover ideas through doing math and thereby have space to develop their inquiry-oriented strategies. To be effective, make mathematical relationships explicit so that students, rather than only learning how to do that day's mathematics, make connections to other mathematical ideas.

For example, when teaching students to subtract two-digit numbers, systematic instruction might include a series of problems like 87 – 9, 56 – 12, and 94 – 25 with discussion of strategies for thinking through these problems. Then, explore problems like 87 – 79, 56 – 48, and 94 – 89, asking students to look first at how these three problems are different from the first set (what is being subtracted is larger) and how that might affect how it is solved. Notice that the first set is efficiently solved by taking away the subtrahend, whereas the second set is more efficiently solved by determining how far apart the two numbers are (comparing). The teacher has systematically sequenced tasks (aka, scaffolded) to help students notice an important relationship that informs how they can solve subtraction problems, without taking away their thinking.

Representations. The use of multiple representations as described in the concrete, semi-concrete, abstract (CSA) approach has been used in mathematics education for many years (Dobbins, Gagnon, & Ulrich, 2014; Griffin, Jossi, & van Garderen, 2014; Heddens, 1964; Peltier & Vannest, 2018). Based on Bruner and Kennedy's stages of representation (1965), this framework reflects concrete representations that encourage learning through movement or action with manipulative materials; semi-concrete representations using drawings, illustrations, or sketches; and abstract symbols or ideas. Built into this approach is the return to concrete representations and visual models as needed or as students begin to explore new concepts or extensions of concepts previously learned. This is to say that CSA is not a rigid approach where you only move to abstraction in a step-by-step sequence starting from C to S and then A. Instead, it is an integrated model where there is parallel modeling and use of concrete models, visual representations, and corresponding numerals and equations (see Figure 6.4). CSA also includes links of all three representations to mental conversations that you share (think-alouds), students' articulation of their thinking, and written work. Used in a combination with systematic instruction, this approach met with high levels of success for students with special needs in math (Agrawal & Morin, 2016; Flores, Hinton, & Strozier, 2014).

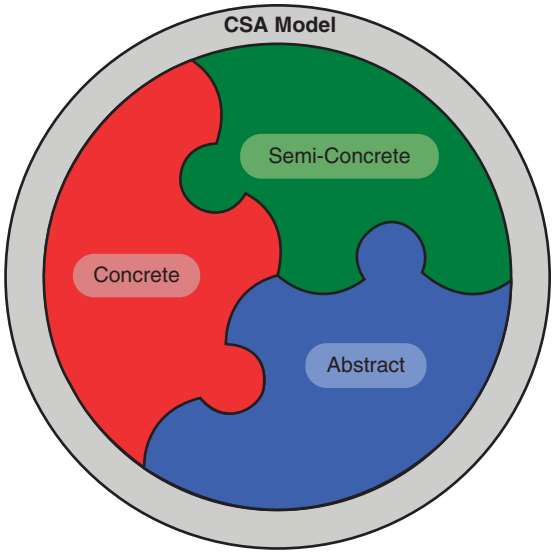


FIGURE 6.4 Integrated CSA model

Mathematics Language. Explicit attention to mathematics language includes correct and consistent use of vocabulary, terminology, phrases, and labels that helps all students, particularly students with special needs in math, learn, discuss, and write about mathematical ideas. When students hear one teacher talk about the "commuter" property that goes back and forth and another year they hear of the "flip-flop" property, instead of the correct terminology of the commutative property, that lack of precision causes long-term confusion. Student–teacher communication is most effective when the conversation is clear through shared language that is used across all the grades in the school. Students can then better

articulate questions that they still need answered or explanations of their thinking. Unlike reading lessons, where pre-teaching vocabulary is a common approach, that route has disadvantages in mathematics instruction. Dixon (2019) refers to pre-teaching mathematics vocabulary as one of six most unproductive Mathematics Teaching Practices. Instead, have students work on developing the idea or concept and at that moment naturally attach the corresponding language when it will be most meaningful in the context. When appropriate, use a math word wall, where you can attach to terms their corresponding meanings and illustrations to provide visual cues.

Movement to higher levels of understanding of content can be likened to the need to move to a higher level on a hill. For some, formal stair steps with support along the way is necessary (systematic strategy instruction); for others, ramps with encouragement at the top of the hill will work (scaffolding). Other students can find a path up the hill on their own with some guidance from multiple representations (CSA approach). All people can relate to the need to have different support during different times of their lives or under different circumstances, and it is no different for students with special needs in math. Yet, students with special needs in math must eventually create a path to new learning on their own, because that is what will be required in the real world after formal education ends. Leaving students only knowing how to climb steps with support and then having them face hills without constant assistance from others will not help them attain their life goals.

Teaching Strategies and Support in the Tier 1 Classroom

Although you will consider all six of the WWCH recommendations as strategies for Tier 2 and 3 interventions, your goal is always to work toward higher levels of student responsibility for learning in the general education classroom for Tier 1. Let's look at strategies and support that can be integrated into the general education classroom in Table 6.4.

TABLE 6.4 SUPPORT FOR COMMON STUMBLING BLOCKS

Stumbling Blocks That Cause Difficulty for Students	What Will I Notice?	What Should I Do to Support Students' Learning?
Forming mental representations of mathematical concepts	• Challenged to interpret a number line • Has difficulty going from a problem about a garden plot (finding area) to graph or dot paper representation	• Explicitly teach the representation—for example, exactly how to draw a diagram (e.g., partition a number line) • Use larger versions of the representation (e.g., number line or coordinate graph) so that students interact with the model
Accessing numerical meanings from symbols or issues with number sense	• Has difficulty with basic facts; for example, doesn't recognize that $3 + 5$ is the same as $5 + 3$ • Cannot describe the meaning of the equal sign • Is overwhelmed by a page of problems.	• Explicitly teach multiple ways of representing a number, showing the variations at the same time • Use a number balance to support understanding of the equal sign • Use multiple representations for a single problem (base-ten blocks, illustrations, and numbers) rather than using multiple problems
Keeping numbers and information in working memory	• Loses counts of objects • Gets confused when other students share multiple strategies during the *after* portion of the lesson • Forgets how to start the problem-solving process	• Use ten-frames to help students organize counts • Model how to skip count • Record in writing the ideas of other students during discussions • Develop mnemonics such as STAR: **S**earch the problem for important information; **T**ranslate words into models, pictures, symbols or actions; **A**nswer the problem; **R**eview your solution for reasonableness (Gagnon & Maccini, 2001). Pick one mnemonic that is used across your entire school.
Organizing information and the ability to self-regulate	• Misses steps in a process • Writes computations in a way that is random and hard to follow	• Use routines as often as possible or provide self-monitoring look-for tools or cues on bookmarks to prompt steps • Use graph paper to record problems or numbers
Overgeneralizing rules	• Applies rules such as "Always subtract the smaller from the larger" too literally, resulting in errors such as $35 - 9 = 34$ • Mechanically applies poorly understood algorithms—for example, incorrectly adds $\frac{7}{8}$ and $\frac{12}{13}$, generating the answer $\frac{19}{21}$.	• Give examples as well as counterexamples to show how and when rules should be used • Tie rules to conceptual understanding; don't emphasize memorizing rote procedures

Each phase of the lesson (before, during, and after) evokes specific planning considerations for students with special needs in math. Some strategies apply throughout a lesson. The following discussion is based on Karp and Howell (2004) and provides some specific suggestions for support while maintaining the challenge.

Structure the Environment.

- *Centralize attention.* Move the student close to the whiteboard or teacher. Face students when you speak to them and use gestures. Remove competing stimuli.
- *Avoid confusion.* Word directions carefully and specifically and ask the student to repeat them. Give one direction at a time. Use the same language for consistency. For example, talk about base-ten materials as ones, tens, and hundreds rather than saying "flats," "rods," and other labels referring to their shape.
- *Create smooth transitions.* Ensure that transitions between activities have clear directions and that there are limited chances for students to get off track.

Identify and Remove Potential Barriers.

- *Use friendly numbers.* Instead of using \$6.13, initially use \$6.00 to emphasize conceptual understanding rather than mixing computation and conceptual goals. Incorporate this technique when computation and operation skills are *not* the lesson objective.
- *Vary task size.* Assign students fewer problems to solve or break the task into component parts. Some students may become frustrated by the enormity of a very large task or a page filled with problems.
- *Adjust the visual display.* The density of words, illustrations, and numbers on a page can overload students. Find ways to put one problem on a page or increase the font size.

Provide Clarity.

- *Repeat the time frame.* Support students in time management by giving them additional reminders about the time left for exploring the materials, completing tasks, or finishing assessments.
- *Ask students to share their thinking.* Use the think-pair-share strategy to provide support and rehearsal for full class discussions.
- *Emphasize connections.* Pose carefully phrased questions to help students connect representations such as visuals, materials, and words. For example, as you fold a strip of paper into fourths, point out the part-whole relationship with gestures as you pose a question about the relationship between $\frac{2}{4}$ and $\frac{1}{2}$.
- *Adapt delivery modes.* Incorporate a variety of visual materials: images, examples, and models. Read problems or assessments or generate audio, using text-to-speech tools. Provide written instructions in addition to oral instructions.
- *Emphasize the relevant points.* Some students may mistakenly focus on the color of a cube instead of the quantity of cubes when filling a prism to measure volume or lose track of the emphasis on counting when using colored bears. Keep using materials that help students generalize the representations to the concept rather than compete with it (Peltier & Vannest, 2018).
- *Support the organization of written work.* Provide tools and templates so students can focus on the mathematics rather than the organization of a table or chart. Use graphic organizers, picture-based models, paper with columns or grids, or appropriate digital tools (i.e., EquatIO (https://www.texthelp.com/products/equatio)).
- *Provide examples and non-examples.* Show examples of acute angles as well as angles that are not acute. Help students focus on the characteristics that differentiate the examples of a word or concept from those that are not examples.

Consider Alternative Assessments.

- *Propose alternative products.* Provide students with options for how to demonstrate understanding (e.g., a verbal response that is scripted by another person, audio recorded, or modeled with a manipulative). Use voice recognition software or word prediction software that can generate a whole menu of word choices when students type a few letters.
- *Encourage self-monitoring and self-assessment.* Students may need additional experiences in self-reflection. Asking them to review an assignment or assessment to explain what was difficult and what they think they got right can help them be more independent.
- *Consider feedback charts.* Help students monitor their growth by charting progress over time.

Emphasize Practice and Summary.

- *Consolidate ideas.* Create or have students create study guides that summarize key mathematics concepts.
- *Provide extra practice.* Use carefully selected problems and allow the use of familiar models.*

Not all of these strategies will apply to every lesson or every student with special needs in math, but as you think about a particular lesson and individual students in your class, you will find that many of these ideas will allow a student to engage in the task and accomplish the learning goals. Explore Strategies for Making Math Accessible for a handy collection of cards that you can use to think about planning instruction for individual students. The Center for Applied Special Technology (CAST) also has a website that contains resources and tools to support students' learning, especially those with cognitive challenges, through Universal Design for Learning (UDL).

Pearson eText Teacher Resource: Strategies for Making Math Accessible

Adapting Instruction for Students Who Need Extensive Support in Math

Students who need extensive support in math include those who are significantly impacted by autism, sensory disorders, limitations affecting movement, cerebral palsy, and combinations of cognitive delays. They may need more wide-ranging modifications and individualized supports to learn mathematics.

Historically, the mathematics curriculum for students who need more extensive supports to respond to their needs focused on life skills such as managing money, telling time, using a calculator, and measuring. Presently, such a curriculum has broadened to address grade-level expectations. This shift encourages the blending of life skills with grade-level content. For example, when the operation of division is studied, use tasks such as figuring how money can be equally shared. Concepts such as parallel and perpendicular lines and curved and straight sides can be connected to interpreting maps. Note that some students may have limited gross or fine motor skills or need tangible manipulatives, so larger versions of the materials, virtual manipulatives, or assistive technology to direct an adult's use of the manipulative for the students may be needed (Jimenez & Stanger, 2017). The Math Activities for Students who Need Extensive Support teacher resource offers ideas across the curriculum, and Additional

*Adapted list reprinted with permission from Karp, K., & Howell, P. (2004). Building responsibility for learning in students with special needs. *Teaching Children Mathematics*, *11*(3), pp. 118–126. Copyright © 2004 by the National Council of Teachers of Mathematics. All rights reserved

Strategies for Supporting Students who Need Extensive Support offers strategies to modify grade-level instruction.

Pearson eText Teacher Resource: Math Activities for Students who Need Extensive Support

Pearson eText Teacher Resource: Additional Strategies for Supporting Students who Need Extensive Support

Multilingual Learners

Learning Outcome 6.4 **Explain mathematics language demands and strategies to support multilingual learners.**

Multilingual learners comprise a fast-growing population in the United States, having increased in number by 28 percent in the 21st century and currently making up about 10 percent of the total population of students (Office of English Language Acquisition (OELA), 2020). The term *multilingual learners* refers to all children and youth who are or have been consistently exposed to multiple languages. It replaces the term *English learner* and similar labels such as English language learner or ESL students (WIDA, 2020). Multilingual learners bring rich resources to the mathematics classroom community, including different representations they use to learn mathematics concepts, unique language to describe mathematics, and culturally based methods for algorithms that vary from the U.S. standard algorithms.

Multilingual learners are not only learning mathematical language that is also new to their peers, they are often also learning mathematical language previously taught in prior grades as well as everyday language. Something important to understand about multilingual learners is this: *conversational* English develops faster than *academic language* (language specific to a content area such as mathematics) (Cummins, 1994). Academic language is harder to learn because it is not frequently used in a student's everyday world, including in their personal interactions and in television and related media. Why is this important? Because you will have multilingual learners in your classroom chatting away with their friends in English and you must realize that this does *not* mean they also have the academic language they need to engage in your mathematics lesson.

Mathematics Language Considerations

Mathematics is commonly referred to as a universal language, but that is not the case. Conceptual knowledge (e.g., what multiplication is) is universal. But symbols and procedures are culturally determined. The ideas shared here are not complete but illustrate how language-intense the subject of mathematics is (and why attending to language benefits more than your multilingual learners).

Specialized Meanings. There is a significant amount of vocabulary in mathematics. Here we share categories of mathematical terminology.

1. Words that are unique to mathematics. For example, *commutative property*, *quadrilateral*, and *subtract* are words only used in reference to doing mathematics, and there are actions that are unique to mathematics, like rounding up (Chval et al., 2021).
2. Phrases that take on specialized meaning. The most notable example in this group is "If…". Compare these two statements and consider the meaning of the word *if*:

   If you are 18, you can vote.
   If a square has sides of 6 cm, what is the area?

   Notice that the word *if* in everyday language means what follows may or may not be the case. But in mathematics, "if" means "assume," as in, "Assume a square has a side of 6 cm, what is its area?"

3. Words that mean something different in a math context than in an everyday context. For example, a table is a grid for recording data in math class and a place to sit at outside of math class. These words abound. Here are just a few more:

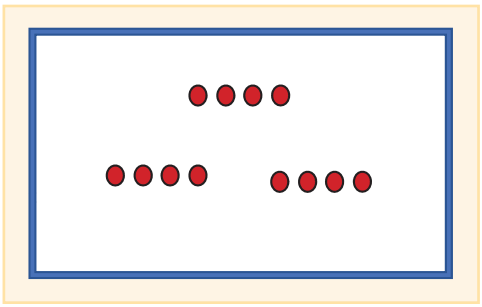
Factor	Difference	Product	Divide	Decompose	Rationale
Variable	Coordinate	Property	Plot	Volume	Negative

Pause & Reflect

For each of these words, consider their specialized meaning within mathematics and everyday meaning outside of mathematics. Then, think of another word not on the list. ●

Students need to use precise and appropriate mathematical language. You support students when you provide language support within a lesson, pointing out when a term is being used in a different way and ensuring that words are developed along with their mathematical meaning.

Symbols. How do you read 3×4? If you thought, "three times four," that is a common way to say this expression. In the United States, this is interpreted conceptually as three groups of four or three rows of four. This is not a universal interpretation of these symbols. In many countries, 3×4 is interpreted as three taken four times, in other words four groups of three. For this whiteboard illustration, can you see three groups of four? Can you see three taken four times? If the latter is harder for you to picture, it illustrates how it can be challenging for a student from another country to follow what is explained in English—it may be more than just a translation issue.



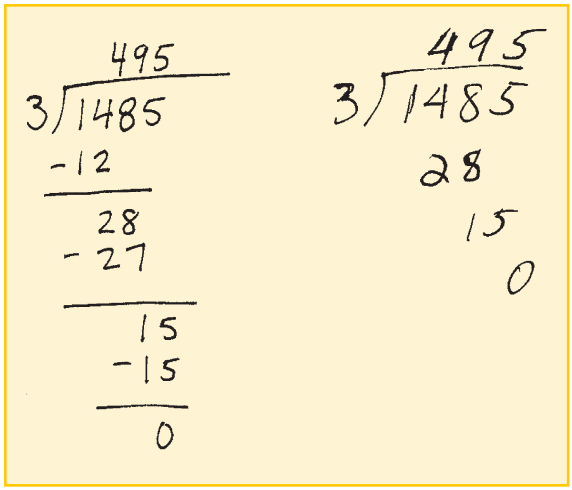
Students thinking of 3×4 as 3 taken four times may use different reasoning strategies (or explain their strategy differently). For example, consider 12×4: This is 12 taken 4 times. That is 12 taken 2 times (24) and then doubled. The answer is 48.

An additional symbol-related challenge (for all students) is that operations can be indicated by more than one symbol. In this case, multiplication can be written in several ways: 3(4) and $3 \cdot 4$. Parentheses may be enclosing a number, as it is here, or indicate what operation should occur first. Compare these expressions: $3(4) + 1$ and $3(4 + 1)$. Here are just some cultural differences related to symbols:

Symbol	Differences in Representation
Numerals	Each country has variations, for example, how 1s, 7s, and 9s are written.
Grouping Symbols	Parentheses (U.S.) versus brackets (many countries).
Billions	10^9 (U.S.) and 10^{12} in Latin countries and UK.
Place Value Groupings	9,456,100 is written as 9.456.100 or 9 456 100. 7.55 (U.S.) may be written as 7,55.
Repeating Decimals	0.555….(U.S.) and 0.5^ (Latin American countries)

In sum, every student needs to understand the meanings (and often multiple meanings) of symbols; multilingual learners are supported when teachers (1) know that differences in symbols exist, (2) help students see the connection between the different symbols, and (3) encourage different ways of thinking through and representing problems.

Procedures. There are many ways a problem like 62 – 47 can be solved, including reasoning strategies and standard algorithms. What may be a standard in one country (or state) is not standard in another. Even when an algorithm follows the same steps, it can vary in the ways students use notations and how much they record or solve mentally. Compare the following two division problems from a fourth-grade classroom (Midobuche, 2001):



Can you understand the thinking of the first student? If you learned division in the United States, that notation is likely easy to follow. But, if you learned division in another country, you may wonder why the first solution has so many numbers recorded. Can you follow the second example? It is, in effect, the same thinking process, but the multiplication and related subtraction are done mentally. The critical equity question, though, is not whether you can follow an alternative approach, but how you will respond when you encounter a student using a different approach:

- Will you require the student to show their steps, disregarding the way they learned it?
- Will you ask the student to elaborate on how they did it?
- Will you have the student show other students their way of thinking?

The latter two responses communicate to students that you are interested in their way of knowing mathematics; thus, you support positive mathematics identities and agency while also strengthening the mathematical knowledge of your classroom community.

Strategies to Support Multilingual Learners

This section describes instructional strategies that provide access for multilingual learners, so that they have the opportunity to learn deep mathematics and develop positive mathematics identities and a sense of mathematical agency (e.g., Celedon-Pattichis & Ramirez, 2012; Chval et al., 2021; Echevarria, Vogt, & Short, 2012; Kersaint, Thompson, & Petkova, 2009).

Honor Use of Native Language. Valuing a student's native language is one of the ways you value their cultural heritage. In a mathematics classroom, students can communicate in

their native language while continuing their English language development (Haas & Gort, 2009; Moschkovich, 2010). For example, a good strategy for students working in small groups is having students discuss the problem in their preferred language. If a student knows enough English, then the presentation in the *after* phase of the lesson can be shared in English. If the student knows little or no English and does not have access to a peer who shares his or her native language, then visuals or pictures, a translator, Web-based mathematics glossary, or self-made dictionary can be components of a support system. Multilingual learners will often *code-switch*—move between two languages. Research indicates that this practice supports mathematical reasoning because the student is selecting the language from which they can best express their ideas (Moschkovich, 2009).

Pearson eText
Video Example 6.2
In this video consider how this teacher speaks both Spanish and English and how she is able to capitalize on her fluency to support some of her multilingual learners.

Certain native languages can support the learning of mathematical words. Because several languages have their roots in Latin, many mathematics words are similar across languages (Celedon-Pattichis, 2009; Gomez, 2010). For example, *aequus* (Latin), *equal* (English), and *igual* (Spanish) are cognates (words in two languages that look and sound similar). See if you can figure out the English mathematics terms for the following Spanish words: *división*, *hexágano*, *ángulo*, *triángulo*, *álgebra*, *circunferencia*, and *cubo*. Students may not make this connection if you do not point it out, so explicitly teaching students to look for cognates is important to supporting language development.

Explicitly Teach Vocabulary. Intentional vocabulary instruction must be part of mathematics instruction for all students. There is strong evidence that teaching academic vocabulary supports the learning of all students, including multilingual learners (Baker et al., 2014; Gibbons, 2015). Vocabulary support can happen throughout a lesson as well as reinforced before or after a lesson. The earlier discussion of mathematical language can help you to think about the language that will need explicit attention within a lesson.

Pearson eText
Video Example 6.3
In this video consider this teacher's strategy for building vocabulary through consistent use across multiple lessons.

Additionally, ongoing reinforcement and practice with language is useful. Here are a few engaging activities:

- Personal math dictionaries that link concepts and terms with drawings or clip art (Kersaint, Thompson, & Petkova, 2009)
- Graphic organizers that look at multiple ways to define a term (see Chapter 3, p. 52)
- Games focused on vocabulary development (e.g., charades; $10,000 Pyramid; Concentration)
- Mathematics word walls, including visuals and translations
- Skits, poems, or songs to address the everyday and the mathematical meanings of words (Seidel & McNamee, 2005)
- Foldables of key words for a topic (e.g., Dinah Zike's *Teaching Mathematics with Foldables* (Zike, n.d.))

Vocabulary support may occur at the beginning of a lesson or at other times in the lesson. However, it can be detrimental to pre-teach words for which students have not yet learned the related concepts (Livers & Bay-Williams, 2014). You may recall this same point made in the section on students with special needs in math. For both populations, this strategy has too often reduced the cognitive demand of the upcoming lesson and reduced the lesson time to engage in that task, both of which hinder a student's opportunity to learn. Importantly, the language support should be *connected* to its mathematical meaning.

Select Culturally Relevant Contexts. Cultural relevancy means that students can relate to the topic and become engaged in the problem because they understand its context and find it interesting (Bay-Williams & Livers, 2009). Certainly, a context must be familiar, or it fails to serve its purpose of building background to help a student make sense of the mathematics. However, students may be familiar with a context, such as vegetables at the

grocery store, but not have an interest in figuring out how many are on a plate. Consider whether the following problem is culturally relevant:

> **Melisa is making braided bracelets. To prepare one, she needs six strands of colored rope, each of them $1\frac{1}{4}$ feet long. She wants to make eight, one for each of her friends who are coming to her party. How much rope does she need to have?**

The answer to whether it is culturally relevant is that it depends. It depends on whether braided bracelets are familiar to the students *and* whether they are interesting for the students. The list of literature in Table 6.1 can be used to explore relevant contexts. One way to select familiar and interesting contexts is to draw on students' community and cultural knowledge (Vomvoridi-Ivanovic, 2012). For example, data provided in textbook tasks can be replaced with data from your students' community, making it much more interesting to students while teaching students important mathematical applications (Bartell et al., 2017; Simic-Muller, 2015; Turner, Sugimoto, Stoehr, & Kurz, 2016). Using everyday, familiar situations can increase student participation, enhance the use of different problem strategies, and help students develop a productive disposition (Tomaz & David, 2015). If you are not sure about your students' community and culture, ask them! For example, one teacher asked her students to put ideas for problems they wanted to explore in a box. From looking through that box of ideas, she was able to develop culturally relevant contexts and involve students.

Use Comprehensible Input. *Comprehensible input* means that the message you are communicating is understandable. Modifications include simplifying sentence structures and limiting the use of nonessential or confusing vocabulary. Note that these modifications do not lower expectations for the lesson. Sometimes unnecessary words and phrases in questions make them less clear to nonnative speakers. Compare the following sets of teachers' directions:

> ***Not Modified:*** **You have an activity page in front of you that I just gave out. For every situation, I want you to determine the total area for the shapes. You will be working with your partner, but each of you needs to record your answers on your own paper and explain how you got your answer. If you get stuck on a problem, raise your hand.**

> ***Modified:*** **Please look at your paper. [Holds paper and points to the first picture.] You will find the area of each shape. What does *area* mean? [Allows wait time.] How can you calculate area? [*Calculate* is more like the Spanish word *calculary,* so it is more accessible to Spanish speakers.] Talk to your partner. [Points to mouth and then to a pair of students as she says this.] Write your answers. [Makes a writing motion over paper.] If you get stuck [shrugs shoulders and looks confused], raise your hand [demonstrates].**

Notice that three things were done: sentences were shortened, confusing words were removed, and related gestures and motions and connections to the assignment on the page were added to the oral directions. Gestures reinforce verbal messages and concepts, emphasize meaning, and provide students access to communicate in nonverbal forms (Takeuchi, 2015). Also, notice the wait time the teacher gives. It is very important to provide extra time after posing a question or

giving instructions to allow multilingual learners time to translate, make sense of the request, translate back, and then participate.

Another way to provide comprehensible input is to use a variety of tools to help students visualize and understand what is verbalized. In the preceding example, the teacher is modeling the instructions. Effective tools to comprehend mathematical ideas include concrete and virtual manipulatives, pictures, real objects, video, demonstrations, and diagrams. For example, if teaching volume of rectangular solids, show a cube and a rectangular container and ask, "How many same-sized cubes will fill this container?" As you ask, physically start to pack cubes into the container to illustrate. Review relevant terms such as *base*, *length*, *width*, and *height* by having students help label a container with sticky notes.

Plan Partnerships to Support Language Development. The use of cooperative groups is a valuable way to support multilingual learners (Baker et al., 2014). Group work provides an opportunity for students to use language, but only if the groups are formed in ways that consider students' language skills (Chval et al., 2021). Placing a multilingual learner with two English-speaking learners may result in the student being left out. On the other hand, grouping all Spanish speakers together prevents the students from having opportunities to participate in the mathematics lesson in English so that they can develop their academic language skills. The key is to help partnerships be productive, and this must be taught and modeled (Chval, Pinnow, Smith, & Rojas Perez, 2018). Chval and colleagues (2018) suggest having students generate partnership norms, such as:

- Give compliments.
- Listen and do not talk over other students.
- Share when you disagree, but be respectful.
- Ask for your partner's ideas.
- Use eye contact.

The key is that each student is able to make contributions to the solving and sharing of the problem. In this way each student is developing a positive identity and sense of mathematical agency.

Engage Students in Discourse That Reflects Language Needs. Discourse, or the use of classroom discussion as a means to make sense of the mathematics, is essential for the learning for all students, but it is particularly important for multilingual learners who need to engage in productive language (writing and speaking) as well as receptive language (listening and reading) (Baker et al., 2014). Listening and speaking emerge first, followed by reading and writing.

There are strategies you can use in classroom discourse that help multilingual learners understand and actively participate in discussions. Practicing an explanation first with a partner can increase participation. Revoicing is an effective strategy. *Revoicing* is when a person (e.g., the teacher) revisits a student's statement in one of these ways: repeating, rephrasing, or expanding (Turner, Dominguez, Maldonado, & Empson, 2013). Revoicing is a way of positioning students as competent! For example, a student might contribute an answer to the question, "What is area?" saying, "It is the whole thing" motioning with hands over the shape. A revoicing might be, "Enrique says area is the whole thing, it is all the tiles inside the shape." Another teacher move is pressing (also sometimes called expansion). *Pressing*, as you might guess, is pressing for detail—asking for more information (Choppin, 2014). Pressing may help a teacher make sense of the concept, but its purpose is to help other students make sense of the idea (Maldonado, Turner, Dominguez, & Empson, 2009). Like revoicing, it assigns competence to the speaker. Calling on another student to add to what the speaker just said can have the reverse impact on the original speaker.

Writing, though potentially more challenging for multilingual learners than for English-speaking learners, is another way to support discourse. If students have already written about their thinking (including words and visuals), they have practiced how to explain their strategy, and they have a written record they can refer to. Writing in and

of itself is important for multilingual learners as they learn mathematics and language (Chval et al., 2021).

Classroom Vignette. Figure 6.5 provides a classroom vignette. The teacher uses several techniques, described earlier, to provide support for multilingual learners.

Ms. Steimer is teaching a third-grade lesson that involves the concepts of estimating length (in inches) and measuring to the nearest half-inch. Students are asked to use estimation to find three objects that are about 6 inches long, three objects that are about 1 foot long, and three objects that are about 2 feet long. Once identified, students are to measure the nine objects to the nearest half-inch and compare the measurements with their estimates.

Ms. Steimer has several multilingual learners in her class, including a student from Korea who knows very little English and a student from Mexico who speaks English well but is new to U.S. schools. These two students are not familiar with the measurement units of feet or inches. She takes time to address the language and the increments on the ruler to the entire class. Because the word *foot* has two meanings, Ms. Steimer decides to address that explicitly before launching into the lesson. She begins by asking students what a foot is. She allows time for students to discuss the word with a partner and then share their answers with the class. She explains that today they are going to be using the measuring unit of a foot (while holding up the foot ruler). She asks students what other units can be used to measure length. In particular, she asks her multilingual learners to share what units they use in their countries of origin, having metric rulers available to show the class. She asks students to study the ruler and compare the centimeter to the inch by posing these questions: "Can you estimate about how many centimeters are in an inch? In 6 inches? In a foot?"

Moving to the lesson objectives, Ms. Steimer uses a ruler on the document camera and enlarges the image so all students can see the demarcations. Below the ruler she has a paper snake whose length ends at the $8\frac{1}{2}$ inch mark. She asks students how long the snake is and writes their reply of $8\frac{1}{2}$ inches. Next, she asks students to tear a paper strip that they estimate is 6 inches long. Students then measure their paper strips with rulers to the nearest half inch. Now she has them ready to begin estimating and measuring.

FIGURE 6.5 A third-grade lesson on length

Pause & Reflect

What specific strategies did Ms. Steimer use to support her students? ●

Ms. Steimer provides support for multilingual learners in these ways: Discussion of the word *foot*, using the think-pair-share technique, recognized the potential language confusion and allowed students the chance to talk about it prior to the task. The efforts to use concrete models (the ruler and the torn paper strip) and to build on students' prior experience (use of the metric system) provided support so that the multilingual learners could succeed in this task. Most important, Ms. Steimer did not diminish the challenge of the task with these strategies. If she had altered the task, for example, by not expecting her students to estimate because they didn't know the inch very well, she would have lowered her expectations. Conversely, if she had simply posed the problem without taking time to have students study the ruler or to provide visuals, she may have kept her expectations high but failed to provide the support that would enable her students to succeed. Finally, by making a connection for all students to the metric system, she showed respect for the students' cultures and broadened the horizons of other students to measurement in other countries.

Incorporating these ideas begins as you lesson plan. As you think about the needs of all your students, you take into account the resources and needs of your multilingual learners.

The Reflective Planning Questions teacher resource provides reflective questions for each lesson phase, with additional considerations for multilingual learners.

Pearson eText Teacher Resource: Reflective Planning Questions for Mathematics Teaching

Promoting Advanced Education and Talent Development

Learning Outcome 6.5 **Apply knowledge of advanced education and talent development.**

Previously in this chapter, we discussed changes in the language we use to describe learners so that the descriptions reflect more equitable and inclusive stances. That emphasis is repeated here as we move away from talking about gifted education and gifted children to embrace the potential of many learners with terms such as *advanced education* or *academic excellence* (Plucker, 2021). Meyers and Plucker (2021) suggest that "learning opportunities need to be rebranded with terms that acknowledge the dynamic and contextual nature of ability, highlight equity-driven practices, and focus on opportunities to excel in talent domains" (p. 4). They say that intellectual gifts are not "bestowed on some and not others" (p. 5).

Advanced education provides challenging, rigorous mathematics learning opportunities to students who aren't getting those experiences via the regular curriculum. As in the MTSS model described earlier, interventions for advanced education fluctuate with academic topic and student needs (Plucker & Peters, 2016). But importantly, additional students need to be identified and nurtured to benefit from expanding opportunities that allow them to grow well beyond mathematical proficiency on grade-level standards.

Right now, approximately nine percent of U.S. fourth graders and 10 percent of eighth graders are performing at advanced levels in mathematics (NCES, 2019), which is undoubtedly not enough. In a large scale study involving hundreds of thousands of students across several states in grades 3–5, Peters et al. (2017) found that 16 percent to 37 percent of students started the school year having already provided evidence that they were learning at least one grade level ahead. In another study, of just fifth graders, two percent showed performance at the 9^{th}-grade level (Peters et al., 2017). In both cases, these studies included low socioeconomic status (SES) schools. These results remind us that we may not fully realize what an exceedingly diverse range of learners exist in every classroom. When we reflect on Vygotsky's Zone of Proximal Development (1978) as discussed in Chapter 2, learning must not be a repetition of already learned content but, instead, be paired with engaging content that stretches the learner's understanding. One reason suggested for students not performing at the advanced level in NAEP or international assessments is that they are significantly under-challenged and have no opportunity to reach those performance levels. The question becomes how to implement research-informed advanced education practices in the diverse ability classroom (Dodillet, 2019; Szabo, 2017).

No one approach can be predicted as best for developing students who are advanced mathematically while taking into consideration environmental influences, so we discuss here a variety of research-based options and strategies for consideration (see Table 6.5). Acceleration research recognizes that your students may already understand the mathematics content that you plan to teach (Assouline et al., 2015; Steenbergen-Hu et al., 2016). Differentiation, highlighted in Chapter 4, represents strategies that support a wide range of learners. Talent development blends the individual learner, the environment of the classroom, and sociocultural influences (Barab & Plucker, 2002). As Plucker and Callahan suggest, "unless advanced learners have their talents fostered and remain challenged in K-12 schools, they will never develop their full potential as creative, real-world problem solvers" (2014, p. 392).

TABLE 6.5 THREE INSTRUCTIONAL APPROACHES FOR ADVANCED EDUCATION

	Acceleration	Differentiation	Talent Development
Student Identification	Administer achievement tests or use teacher recommendations that point to students easily mastering the mathematics curriculum.	Screen students through continuous assessments that unveil strengths and needs.	Include all students, particularly those who are previously untapped for assessing mathematics strengths. Search for students who have an aptitude for mathematics or express an interest or enthusiasm for mathematics.
Instructional approaches	Move the student up a full grade (skipping), put them in pull-out ability groupings, or shift them to a math class at the next higher grade. These approaches are often seen as less socially advantageous or, in the case of ability grouping if used as tracking, actually disadvantageous (Ramberg, 2016). Allow students to increase the pace of their own learning by allowing them to go beyond the topic that the rest of the class is studying. Use curriculum compacting (Reis & Renzulli, 2005) to provide a short overview of the content and then assess students' ability to respond to mathematics tasks that would demonstrate proficiency. This process cuts out repetitive tasks.	Tailor and personalize content in diverse ability classrooms. Consider clustering students into groups based on their needs (Callahan et al., 2020). Use more abstract approaches rather than a step-by-step sequence (Rogers, 2007). Consider pace of presentation and the use of open-ended tasks or products. Provide students options to work independently (Wolfensberger, 2012). Modify the content, process, or products in a variety of ways (Tomlinson et al., 2009).	Assume ability in mathematics can be developed and is not innate. Cultivate development of skills in all learners such as tenacity, a growth mindset, and the ability to see mathematics in a variety of academic and real-life situations. Foster solving challenging nontraditional problems (Nolte & Pamperien, 2017) Provide daily mathematics challenges (Park, Lubinski, & Benbow, 2013) that are engaging and inspire enjoyment (Benölken, 2015). Provide mind-mates or thinking partners who can spur on each other's ideas in a student-to-student ZPD. Facilitate students' use of creative thinking and encourage the use of multiple strategies. Support students' willingness to make mistakes, analyze the outcome, and try again. Encourage students' creativity in culminating performances, such as having them demonstrate knowledge through inventions, experiments, simulations, dramatizations, visual displays, and oral presentations.
The Curriculum	Move students through the existing curriculum and learning standards at a faster pace. You don't really need a different curriculum – just a well-articulated curriculum that every teacher can access and use regardless of the grade at which they're teaching.	Use high quality, pre-differentiated, "prescriptive curricula" that provides lesson plans and resources (Plucker & Callahan, 2014, p. 395). Select or adapt tasks (see chapter 3) to be more challenging and focus on situations where the solution process is not known.	Select and use a challenging curriculum. Implement mathematical tasks that all students find relevant and stimulating, with wide upper boundaries for growth and divergent thinking.
Limitations	Using tests to identify advanced learners must be done carefully to avoid perpetuating inequities (Plucker & Peters, 2016).	Teachers may not have the curriculum materials and related math manipulatives to support students further. Teachers' knowledge of curriculum up to four grades higher may require additional reading or professional development.	Talent development is a long-term undertaking.

Spot and Foster Students' Strengths

Students' mathematical strengths and abilities may become apparent to family members and teachers when the student grasps and articulates concepts at an earlier age than expected. But some students with strengths in mathematics experience boredom (Rubenstein et al. 2012) and instead may daydream or exhibit behaviors that draw negative attention, often leaving them overlooked in the identification process. So you may want to start looking for strengths there instead—who is possibly bored?

Many teachers have a keen ability to spot talent when they note students who have strong number sense or spatial sense (Gavin & Sheffield, 2010). Note that these teachers are not identifying students who are speedy with their basic facts, but those who have the ability to

reason quantitatively, persevere, see patterns, and make sense of mathematics structure—yes, these students are strong in the Mathematical Practices!

Measuring students' performance does not tell you their potential (Plucker, 2021). That premise then leads you to "use different tests" or to "use tests differently" (Matthews & Peters, 2017). By screening everyone, you avoid an overreliance on teacher nominations, which is essential, given that implicit biases may result in low SES and Black, Hispanic, and Indigenous American students being overlooked. Another alternative to identification of strengths is to cultivate your own and develop such skills through challenging tasks and inquiry-based instruction. In advanced education, you grow students by having them use, implement, and extend ideas rather than accumulate a mental collection of facts, rules, and concepts. When only very simple tasks that the whole class can carry out are the basis for designing lessons, students begin to believe that if they cannot solve them quickly they are not able to solve them at all (Nolte & Pamperien, 2017).

Increase the sophistication of a topic by connecting a deeper look at mathematical topics outside of grade-level material to other subject areas. For example, while studying a unit on place value, students in advanced education can deepen their knowledge to study other numeration systems such as Roman, Mayan, Egyptian, Babylonian, Chinese, and Zulu. This approach provides a multicultural view of how our numeration system fits within historical number systems (Mack, 2011). A group might find the beauty of mathematics as it integrates with art (Bush, Karp, Lenz, & Nadler, 2017; Bush, Karp, Nadler, & Gibbons, 2016). In the algebra strand, when studying patterns of numbers, students explore Fibonacci sequences and their appearances in the natural world in spiraling shells and plant life. See the Mathematics Integration Plan to help plan ways to make connections in core content, create independent explorations, or develop research projects. Using integration, students can think about a mathematics topic through another perspective such as economic or cultural ones or through a historic or futuristic viewpoint.

Pearson eText Teacher Resource: Mathematics Integration Plan

Strategies to Avoid. A number of ineffective approaches are not aligned with advanced education, including:

1. *Assigning more of the same work.* This approach is the least appropriate way to respond and the most likely to result in students' hiding their mathematical abilities.
2. *Giving free time to early finishers.* Although students may find this opportunity rewarding, it does not maximize their intellectual growth to go beyond and can lead to students hurrying to finish a task.
3. *Providing general pull-out opportunities.* Unfortunately, generalized advanced programs are often unrelated to the mathematics curriculum (Assouline & Lupkowski-Shoplik, 2011). Disconnected, add-on experiences are not enough to build more complex and sophisticated understandings of mathematical ideas.
4. *Offering independent work as a way for students to develop on their own.* Although excellent enrichment opportunities can be found on the Internet and in apps or through long-term investigations, the practice of having students just take on independent projects does not align with talent development. Students need to be engaged by more knowledgeable others in ways that will enhance conceptual understanding and support their ability to justify their thinking or respond when their ideas are challenged.

Strategies to Consider. Sheffield writes that students in advanced education should be introduced to the "joys and frustrations of thinking deeply about a wide range of original, open-ended, or complex problems that encourage them to respond creatively in ways that are original, fluent, flexible and elegant" (1999, p. 46). Here are options to include:

1. *Provide opportunities to develop the ability to generalize mathematical ideas.* Capable mathematics students are often defined by their ability to think abstractly and generalize. So move away from presenting rules to follow (Karp, Dougherty & Bush, 2021; Szabo, 2017) and have students identify properties that hold for "a large set of mathematical

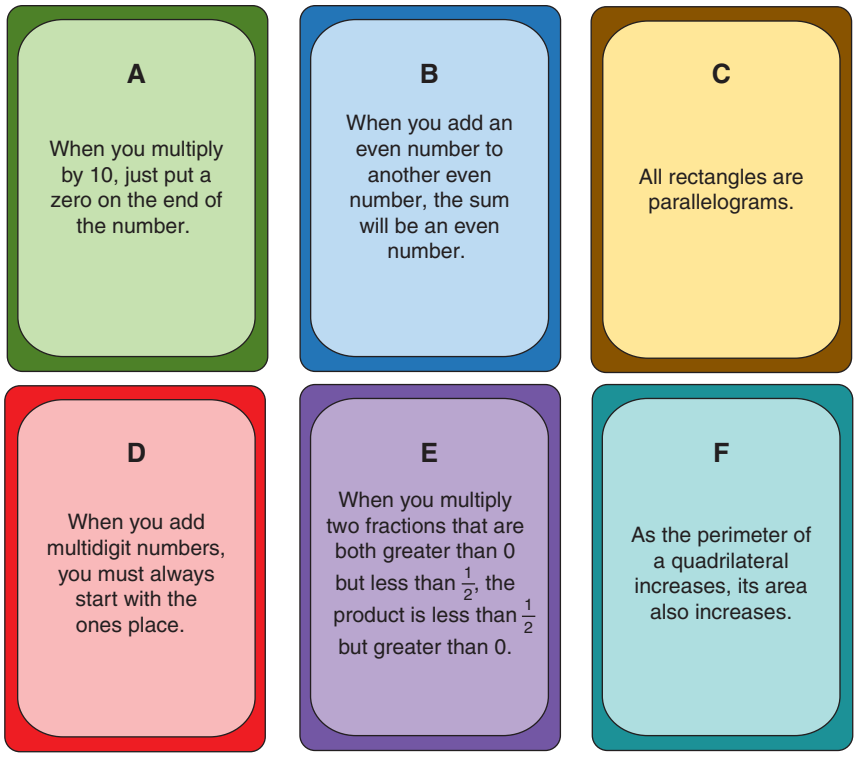

A
When you multiply by 10, just put a zero on the end of the number.
B
When you add an even number to another even number, the sum will be an even number.
C
All rectangles are parallelograms.
D
When you add multidigit numbers, you must always start with the ones place.
E
When you multiply two fractions that are both greater than 0 but less than $\frac{1}{2}$, the product is less than $\frac{1}{2}$ but greater than 0.
F
As the perimeter of a quadrilateral increases, its area also increases.

FIGURE 6.6 Sorting rules and generalizations

Adapted from Karp, Dougherty, & Bush, 2021 and Bush, Karp, & Dougherty, 2021.

Answer: Rules: A, D, F; Generalizations: B, C, E

objects or conditions" or generalize (Carraher et al., 2008 p. 3). Let's reflect on which items in Figure 6.6 are the generalizations that we want students in advanced education to develop independently. Note—some of the rules are not true!

Pearson eText Activity Page: Sorting Rules and Generalizations Cards

2. *Actively combat the excellence gap.* For many years in advanced education, a disproportionality has existed with "persons excluded because of ethnicity or race" (PEERs) (Asai, 2020, p. 754) or students from low SES with an underrepresentation that must be changed (Plucker & Callahan, 2014). Mathematics ability is universal, so there is a consistent set of students with potential for advanced performance who were overlooked or are untapped, and vigorously finding such students is essential (Anderson, 2020). Create opportunities for each and every student to develop and showcase their potential.
3. *Stimulate curiosity in recreational mathematics.* The use of games, spatial puzzlers, brainteasers, mysteries, and escape rooms that can be solved through mathematics, and counterintuitive problems that leave students asking, "How is that possible?" help generate excitement. But the main thrust of the motivation to expand students' learning to higher levels often comes from you as you enthusiastically communicate a passion and curiosity for mathematics.
4. *Help students see mathematics as creative.* Mathematicians see their work as highly creative, and students can experience that too as they look at problems from new vantage points or angles. Engage students in the active risk-taking, methods, communication styles, and tools of mathematicians. This includes the role of accepting failures as part of the process of creative thinking.

RESOURCES FOR CHAPTER 6

WRITING TO LEARN

1. Describe in your own words the central ideas of equitable mathematics instruction.
2. What are two recommendations that the *What Works Clearing House Practice Guide: Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades* (2021) suggests for use in Tier 2 interventions for students with special needs in math?
3. Describe effective and ineffective ways to address mathematical language in teaching.
4. For two of the strategies for multilingual learners, describe how the strategy reflects equitable mathematics instruction.
5. In the context of advanced education, what are two strategies to consider and two strategies to avoid?
6. As you reflect on equitable instruction and the additional suggestions for specific learners, what key ideas emerge as essential to ensuring that every child develops mathematical competence and confidence?

RECOMMENDED READINGS

Articles

Hodges, R., Rose, R., & Hicks, A. (2012). Interviews as RtI tools. *Teaching Children Mathematics*, *19*(1), 30–36.

This article emphasizes ways to identify the strengths and weaknesses of students with special needs in math and the implication for instruction within an RtI framework.

Ruttenberg-Rozen, R. (2020). A wonder-full task leads to a wonder-full intervention. *Mathematics Teacher: Learning & Teaching PL-12*, *113*(6), 474–479.

This mathematical activity supports a sixth grade student with ADHD through a lesson that encourages wonder-filled moments. Video clips that show how the teacher designed the task are included.

Wood, M. B., Sheldon, J., Felton-Koestler, M. D., Oslund, J., Parks, A. N., Crespo, S., & Featherstone, H. (2019). 8 teaching moves supporting equitable participation. *Teaching Children Mathematics*, 25(4), 218–223.

This is a must-read, highlighting specific ideas for enacting equitable instruction from teacher prompts, to ideas for classroom norms. Very practical and important ideas.

Books

Chval, K. B., Smith, E., Trigos-Carrillo, L., & Pinnow, R. J. (2021). *Teaching math to multilingual students, grades K-8: Positioning English learners for success*. Thousand Oaks, CA: Corwin.

Through real vignettes, transcripts, pictures, and videos, this book shows how to support multilingual learners while engaging you in developing your own capacities to do so.

Johnsen, S. K., & Sheffield, L. J. (Eds.). (2013). *Using the common core state standards for mathematics with gifted and advanced learners*. Reston, VA: NCTM.

This book, in collaboration with the National Association for Gifted Children, provides examples and strategies for practitioners. Trajectories for talent development are detailed, as are ideas for encouraging creativity and critical thinking.

CHAPTER 7

Developing Early Number Concepts and Number Sense

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **7.1** Use a developmental approach to teach foundational number concepts.
- **7.2** Demonstrate how to develop children's counting skills, including subitizing.
- **7.3** Develop children's abilities to compare quantities and describe relationships between numbers.
- **7.4** Support children's connections between mathematical ideas and the real world.

Young children enter school with many ideas about numbers, and we build on these ideas by using a variety of experiences to develop new relationships and more advanced mathematics understandings. This chapter emphasizes the development of number concepts up to 20 and number names and counting sequences up to 100 by ones and tens. These foundational ideas extend to content directly affected by how well early number concepts have been developed such as basic facts, place value, and computation as well as content that enhances the development of number (measurement, data, and operations). We begin by introducing critical big ideas about counting and comparing number quantities to nurture and build young children's number concepts and number sense.

BIG IDEAS

- Counting tells how many things are in a set. When counting a set of objects, the last word stated in the counting sequence names the quantity for that set (*cardinality*).
- Numbers relate through comparisons of quantities, including greater-than, less-than, equal-to, and not equal to relationships. These comparisons are made through one-to-one correspondence of objects in sets. For example, the number 7 is 3 more than 4, 2 less than 9, composed of 3 and 4 as well as 2 and 5, 3 less than 10, and can be quickly recognized in several patterned arrangements of dots. These ideas about the number 7 extend to composing and decomposing larger numbers such as 17, 57, and 370.
- Number concepts are closely tied to operations with numbers based on situations in the world around us. Application of number relationships to problem solving marks the beginning of making sense of the world with a mathematical lens.

- *Number sense* means that you can think about different-sized quantities and use numbers and relationships in multiple ways to estimate and solve problems. This sense-making includes understanding the magnitude (or size) of numbers.

Developmentally Appropriate Early Learning

Learning Outcome 7.1 **Use a developmental approach to teach foundational number concepts.**

We cannot overemphasize the importance of early mathematics experiences! By early mathematics, we focus on Pre-K and K experiences. While kindergarten often conjures up ideas of learning to read and socialize, it is early number experiences that must be a central element of this year in a child's life. Here we share some reasons:

- Preschoolers' early mathematical performance establishes a foundation for their future academic success more so than other cognitive measures, including reading ability, memory skills, or spatial sense (Frye et al., 2013; Levine et al., 2010; Watts, Duncan, Siegler, & Davis-Kean, 2014).
- What a five- or six-year-old child knows about mathematics can predict not only their future mathematics achievement, but also their future reading achievement (National Research Council, 2009).
- Kindergarten math has often included too much content children already know. One study found that teachers spent approximately *13 days a month* teaching concepts that the kindergarteners had already learned (Engel et al., 2013). Acknowledging the need for a more challenging experience, early childhood researchers Clements, Fuson, & Sarama (2017) state that the only developmentally *in*appropriate curricula are ones that reteach the mathematics children already know or ones that don't teach enough mathematics.

To ensure developmentally appropriate practices, the National Association for the Education of Young Children (NAEYC), recommends three core ideas for teachers: consider children's development and learning in general (including learning progressions), attend to individual children and their readiness, and know what is culturally important (NAEYC, n.d.). The National Council of Teachers of Mathematics' (NCTM's) position statement on early childhood learning emphasizes that all children need an early foundation of challenging mathematics (2013). That document provides the following research-based recommendations:

1. Enhance children's natural interest in mathematics and assist them in using mathematics to make sense of their world.
2. Build on children's experience and knowledge by using familiar contexts.
3. Base mathematics curriculum and Teaching Practices on a solid understanding of both mathematics and child development.
4. Use formal and informal experiences to strengthen children's problem-solving and reasoning processes.
5. Provide opportunities for children to explain their thinking about mathematical ideas.
6. Assess children's mathematical knowledge, skills, and strategies through a variety of formative assessment approaches.

As both of these national organizations state, early childhood experiences require developmentally appropriate practices, which are based on learning progressions.

Learning Progressions in Early Childhood

Early mathematics learning is based on well-researched learning trajectories (Clements & Sarama, 2021). The standards are also based on learning progressions. Although learning doesn't happen in a precise, linear movement, students do tend to require certain conceptual knowledge before they can grasp more advanced concepts. Concepts or skills early on in the progression are needed for later concepts or skills in the progression. The progressions in this chapter reflect years

of early childhood research and content standards and are reflected in state standards, early learning curriculum materials, and assessments. These progressions provide a reference guide as you work with students to ensure that you are identifying where they are in their learning and moving them forward. For your students who may be struggling, you can see what appears earlier in the progression to provide appropriate interventions on foundational concepts.

Pause & Reflect

Avoiding teaching content that students already know and attending to learning progressions requires significant attention to what each student knows. How might you accomplish this in a kindergarten classroom? ●

Using the aforementioned research-informed recommendations, you will need to give students opportunities to share their thinking and notice how they are working on mathematics activities such as the ones in this chapter. You will also use diagnostic interviews as a formative assessment tool to pinpoint strengths to work from and areas where additional experiences are needed. Then and only then can you use a progression to gauge what your next instructional steps should be. Let's look at a selection of foundational concepts (see Table 7.1) in a progression with other more specific learning progressions developed in greater depth within the chapter.

TABLE 7.1 LEARNING PROGRESSION FOR EARLY NUMERACY

Learning Steps
Compares quantities without assigning number values (discriminates between very large and very small amounts)
Can identify a unit
Knows principles of counting, including subitizing (see Table 7.2 for counting progression)
Reads numbers
Writes numbers and recognizes that written numbers are symbols for number quantities, including zero
Constructs sets of objects, given a number
Classifies and reclassifies groups of objects/sets according to one attribute
Approximates number magnitude—positioning numbers on number line (after working with number lines)
Composes and decomposes numbers
Works with ten as a unit (see progression in Table 7.3)
Counts to solve simple problems, using concrete objects or modeling with role playing

The NRC Committee on Early Childhood Mathematics (2009) identified three foundational areas in mathematics content for early learners: the number core, the relations core, and the operations core. This chapter focuses on the first two core areas; Chapter 8 addresses the third core area by deepening your understanding of how to teach the meaning of the operations.

As you develop children's abilities in counting, the conversations about number relationships begin. Therefore, the activities and concepts in this chapter allow movement among the core ideas and coexist in a rich environment of mathematical experiences where children identify networks of connections between numbers.

The Number Core: Quantity, Counting, and Cardinality

Learning Outcome 7.2 Demonstrate how to develop children's counting skills, including subitizing.

Even two- and three-year-old children count fingers, toys, people at the table, and other small sets of objects. Questions such as "Who has more?" or "Are there enough?" are part of children's daily lives. Evidence indicates that when children have such experiences, they begin

to develop understanding of the concepts of number and counting (Baroody, Li, & Lai, 2008; Clements & Sarama, 2020).

Quantity and the Ability to Subitize

Children explore quantity before they can count when they can identify which cup holds more juice or which of two plates of blueberries has more berries. They, like other animal species, can tell the difference between large and small quantities. Soon they will attach a numerical amount to these quantities to explore them in greater depth. When you look at a number of objects, particularly in a small group, sometimes you can just see how many are there. For example, when you roll a die and immediately know you have five without counting the dots, that ability to just see it is called *subitizing*. More specifically, that instant mental recognition is called *perceptual* subitizing. There are times when you can subitize for even larger amounts when you, for example, decompose two rows of dots shown in a familiar pattern of ten by seeing five in one row and mentally doubling it to get ten. When you recognize these patterns to organize the information, that is called *conceptual* subitizing. Subitizing is a "critical cognitive process" that "builds on innate sensitivity to number" (Clements, Sarama, & MacDonald, 2019 pp. 13, 22). The early childhood researcher Art Baroody calls subitizing "a key foundation for building number sense" (2020, p. 1). Evidence shows that kindergarteners who can subitize five or more items are better able in first grade to build mental units to which they can add two quantities (Wilkins et al., 2021). One way to develop subitizing is through patterned sets (see Figure 7.1).

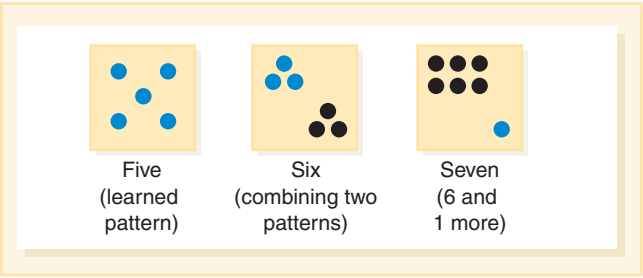


FIGURE 7.1 Recognizing a patterned set

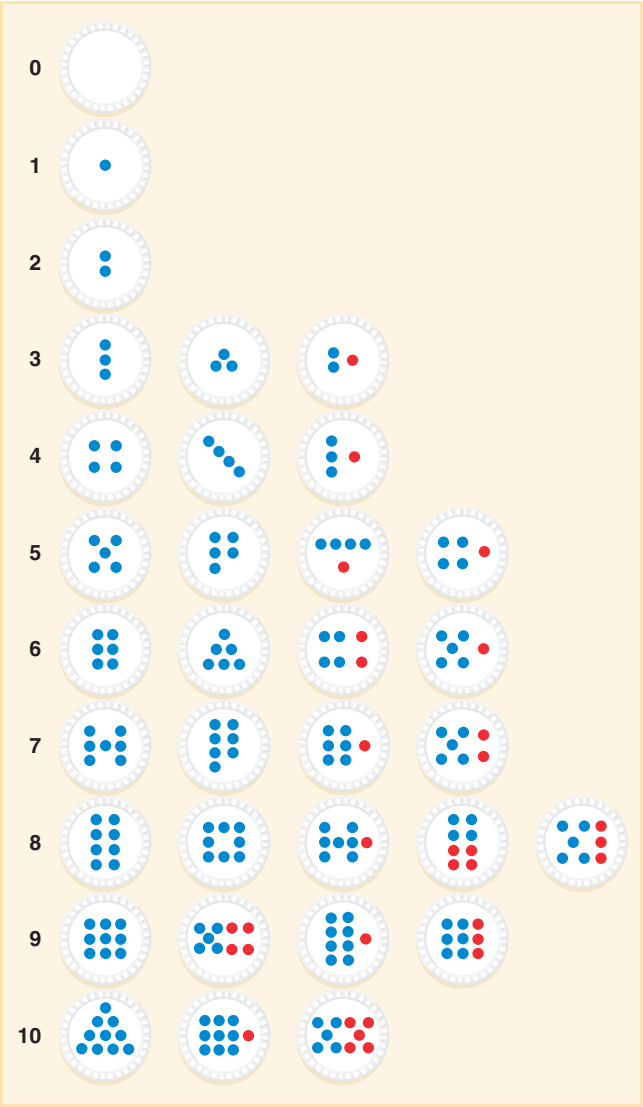


FIGURE 7.2 A collection of patterns for dot plates

Many children easily recognize patterned sets of dots on dice due to the many games they have played. Similar instant recognition (subitizing) can be developed for other patterns. Naming these amounts instantaneously without counting by ones moves students away from an overuse of counting one at a time and toward the use of number patterns. This shift also facilitates counting on (when children count from a known patterned set) or learning combinations of numbers (seeing a patterned set of two known smaller patterns). As described in the learning progression, we want students to move beyond counting by ones to focusing on patterns to determine how many. To support beginning learners in subitizing, use three objects organized in patterns that are symmetric before moving to numbers such as five or more challenging images. Good materials to use in pattern recognition activities include a set of dot plates made with paper plates and the sticky dots available in office supply stores (see Figure 7.2). Note that some patterns are combinations of two smaller patterns or a pattern with one or two additional dots. These should be made in two colors to highlight the patterns. Keep the patterns compact and organized because if the dots are too spread out, the patterns are hard to identify. Seeing a number as composed of a combination of parts is a critical understanding for students because they will need to compose and decompose numbers flexibly as they learn to calculate.

***TECHNOLOGY* Note.** Explore interactive web-based activities such as Subitizing Seeds from toytheater.com (https://toytheater.com/subitizing-seeds). Children find the seed packet that matches the given numeral and drag it in the pot to watch the plants grow. Children can practice subitizing and basic addition, using flashed images of ten frames and base-ten materials. ■

Activity 7.1 Dot Plate Flash

CCSS-M: K.CC.B.4a; K.CC.B.4b

Provide each child with 10 counters and a piece of paper or a paper plate to use as a mat. Hold up a dot plate for about 5 seconds and say, "Make the pattern you saw on my plate using your plate and the counters. How many dots did you see? What did the pattern look like?" Spend time discussing not only the number of dots but the configuration of the pattern. Then show the plate again so children can self-check. Do this activity with a few new patterns each day. To modify for children with special needs in math, you may need to give the child a small selection of premade dot plates. Then instead of creating the pattern with counters, the child finds the matching dot plate. Then move to holding up a dot plate for 3 seconds and say, "How many dots did you see? What did the pattern look like?" When needed, show a plate a second time so children can get another look. Include simple patterns first, and add more dots as children's confidence builds. Children can also use flash dot plates with each other as a learning center activity (see Chapter 4).

Counting

By three years of age, children are commonly engaging in meaningful counting, but by the end of kindergarten, they should be able to count to 100 accurately. For children to have an understanding of counting, they must construct this idea by working through a variety of counting experiences and activities. Only the counting sequence of number words is a rote procedure; the rest requires conceptual understanding. The meaning attached to whole-number counting is the key idea on which all subsequent number concepts are developed.

The Development of Verbal Counting Skills. Research-based learning progressions (also called *learning trajectories)* can help you design instruction that strategically moves children to desired goals by using information about the path on which these ideas typically develop. The counting progression (Clements & Sarama, 2020) identifies how to help a child develop more sophisticated levels of thinking through specific instructional activities. Table 7.2 is based on this research and is a selection of levels and sublevels identified as benchmarks.

TABLE 7.2 LEARNING PROGRESSION FOR COUNTING

Levels of Thinking	Characteristics of the Child
Number Word Sayer	No verbal counting ability. A young child looking at three balls will answer "one" when asked how many. The child does not associate the correct number word with a quantity.
Reciter	Verbally counts using number words, but not always in the right order. Sometimes they say more numbers than they have objects to count, skip objects, or repeat the same number.
Corresponder	Makes a one-to-one correspondence with numbers and objects, stating one number per object. If asked "How many?" at the end of the count, they may have to recount to answer.
Counter	Accurately counts objects in an organized display (in a line, for example) and can answer "How many?" accurately by giving the last number counted (called *cardinality*). They may be able to write the matching numeral and may be able to say the number just after or before a number by counting on from 1.
Producer	Counts out objects to 5 and then moves to 10+. If asked to give you five blocks, they can show you that number of blocks.
Counter and Producer	Counts out objects AND tells how many are in a group. The child can remember which objects are counted and which are not and respond to random arrangements. They begin to separate tens and ones, like 23 is 20 and 3 more.
Counter Backward	Counts backward by removing objects one by one or just verbally, as in a countdown.
Counter from Any Number	Counts on, starting from numbers other than one. They are also able to state immediately the number before and after a given number.
Skip Counter	Skip-counts with understanding by a group of a given number—tens, fives, twos, and so on.

Source: Based on Clements and Sarama (2020).

Standards for Mathematical Practice
MP7. Look for and make use of structure

Verbal counting has at least two separate skills. First, a child must be able to produce the standard string of counting words in order: "One, two, three, four…". Second, a child must be able to connect this sequence by using a one-to-one correspondence with the objects in the set being counted. Each object must get one and only one count. As part of these skills, children should recognize that each counting number identifies a quantity that is one more than the previous number and that the new quantity is embedded in the previous quantity (see Figure 7.3). This knowledge will be helpful later in decomposing numbers (breaking them apart). Go to Counting Games at www.splashlearn.com to view dozens of online games for helping children practice counting.

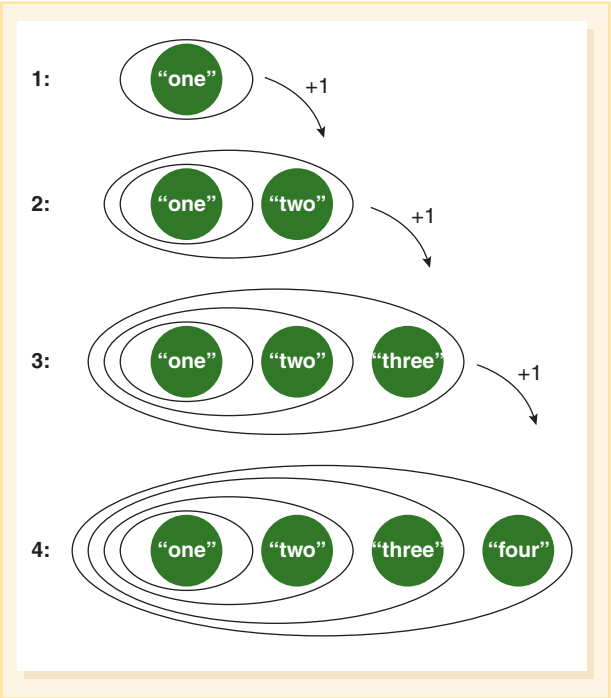


FIGURE 7.3 In counting, each number is one more than the previous number

Source: National Research Council. (2009). *Mathematics learning in early childhood: Paths toward excellence and equity*, p. 27. Reprinted with permission from the National Academy of Sciences, courtesy of the National Academies Press, Washington, DC.

While many children come to kindergarten able to count sets of 10 or more, some children may have less background knowledge and may require additional counting experiences. Sometimes precision will help children grasp the meaning of a number (Frye et al., 2013). For example, try using counterexamples to help clarify a number such as labeling four blocks as "four," stating, "That's four blocks, not six blocks."

For many children, especially children with special needs in math, it is important to have an organized plan for structuring the counting process. To begin, children learning the skills of counting should initially be given sets of blocks or counters. They should count these objects from left to right, pointing and touching them as they orally say each number word in sequence, and move the objects as they count to a space that indicates "already counted." Consistently ask the children, "How many do you have in all?" at the end of each count. The size and arrangement of the set are important factors in students' success in counting. Obviously, longer number strings require more practice to learn. The first 10 counting words involve no pattern or repetition, and many children find there is no consistent pattern in the teen number names (11 – 19) in the English language—often making those numbers a challenge to learn. You may hear children saying "three-teen" and "five-teen" as they attempt to identify a regular sequence.

Activity 7.2 Up and Back Counting

CCSS-M: K.CC.A.1; K.CC.A.2

Counting up to and back from a target number in a rhythmic fashion is an important counting exercise. Line up five children and five chairs. As the whole class counts from 1 to 5, the children sit down one at a time. When the target number, 5, is reached, it is repeated; then the child who sat on 5 now stands, and the count goes back to 1 with the children standing up one at a time, and so on, "1, 2, 3, 4, 5, 5, 4, 3, 2, 1, 1, 2. . . ." Start at numbers other than one. To modify this activity for multilingual learners, give each child who is in front of the class one card from a set of cards with the target numerals (e.g., 1 to 5, 15 to 20, or 55 to 65) and the corresponding number word. These cards will provide a visual to help children connect the written numeral and number word to the number being said.

Activity 7.3 Story Book Counting

CCSS-M: K.CC.A.1; K.CC.A.2

Children's literature with repeated phrases, such as *Goodnight Moon* (Brown, 1947), provides an opportunity for students to count. Students can press the equal sign on a calculator each time the little rabbit says "goodnight" in his bedtime routine. At the completion of the book, they can compare how many "goodnights" were recorded. Follow-up activities include using the same automatic-constant feature on the calculator with different stories or books to skip-count by twos (e.g., pairs of animals or people), fives (e.g., fingers on one hand or people in a car), or tens (e.g., dimes, "ten in a bed," apples in a tree). Students can use tallies instead of calculators, too.

Cardinality

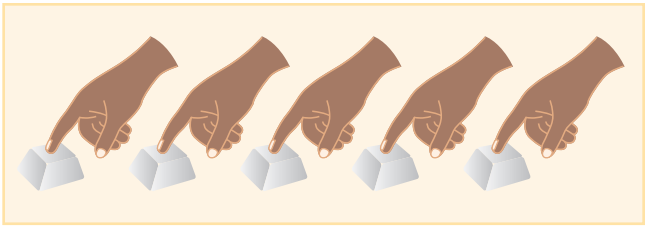


FIGURE 7.4 The child has learned cardinality if, after counting five objects, they answer the question, "How many do you have in all?" with "Five."

As described in the learning progression, children will learn *how* to count (matching counting words with objects) before they understand that the last count word stated in a count indicates the amount of the set (how many you have in all) or the set's *cardinality*, as shown in Figure 7.4. Children who make this connection are said to have the cardinality principle, which is a refinement of their early ideas about quantity and is an expectation for kindergartners (NGA Center & CCSSO, 2010) and a predictor of future mathematics achievement (Chu, vanMarle, & Geary, 2015). Fosnot and Dolk (2001) state that an understanding of cardinality and its connection to counting is not a simple task for four-year-olds, but most children by age four and a half make this connection. Remember, for children to be successful with cardinality, they will tax their working memory (ability to store and mentally manipulate information at the same time) with thinking about: knowledge of the counting words, touching each item once in the count, what number is next in the count, and how many do I have in all (labeling the quantity in the set). As such, this skill is an important milestone.

FORMATIVE ASSESSMENT Notes. Children who count successfully orally may still not attach meaning to their counts. Here is a diagnostic interview to assess a child's understanding of cardinality and whether they are using counting as a tool. Show a card with five to nine large dots in a row so that they can be easily counted. Ask the child to count the dots. If the count is accurate, ask, "How many dots are on the card?" Early on, children may need to count again, but a child who is beginning to grasp the cardinality principle will state the quantity. Now say, "Please give me the same number of counters as there are dots on the card." Here is a sequence of indicators to watch for, listed in order from a child who is not yet attaching full meaning to the count to one who is using counting as a tool:

The child does not count but instead takes out counters and makes a similar pattern.

The child recounts the dots on the card.

The child takes out counters in a one-to-one correspondence with the dots.

The child remembers the number of dots and retrieves the correct number of counters.

The child shows that there is the same number of counters as dots.

As the child shows competence with patterned sets in an easy to recognize organization, move to random dot patterns. ■

To build children's full understanding of the cardinality principle, one of the best approaches is having them subitize small collections and verbally label them with the correct number word (Paliwal & Baroody, 2020) as described in the previous section. Also, engage children in games or activities that involve counts and comparisons, such as Activity 7.4.

Activity 7.4 Fill the Tower

CCSS-M: K.CC.B.5

Children take turns rolling a die and collecting the indicated number of counters. They place these counters on one of the towers on their Fill the Tower Game Board. The object is to fill all towers with counters. As an option, require that the towers be filled exactly so that a roll of 5 cannot be used to fill four empty squares. To modify this activity for children with special needs in math, use a die with only 2 or 3 (dots or numerals) on the sides. You can increase the number choices on the die when you have evidence that the child is counting accurately. For a challenge, have children use a die with higher numbers and a game board with larger towers.

Pearson eText Activity Page: Fill the Tower Game Board

Playing Fill the Tower provides opportunities for you to talk with children about numbers and assess their thinking. Observe how they count the dots on the die. Ask, "How do you know you have the right number of counters?" and "How many counters did you put in the tower? How many more do you need to fill the tower?"

Regular classroom routines, such as counting how many snacks for each table, what materials are needed for an activity, how many children plan to eat the school lunch, or even simply taking attendance, are additional opportunities for engaging children in purposeful counting. Have students count groups of children, as in the "Mingle and Count" video, found at Teaching Channel (www.teachingchannel.org). Look for ways to make classroom situations into real mathematical problems requiring counting and discuss children's strategies. Explore The Find Activity Page for another opportunity to build children's counting strategies.

Pearson eText Activity Page: The Find

Activity 7.5 Counting with the Count

CCSS-M: K.CC.A.1; K.CC.A.2

Engage families in counting as they gather data about their home or community. Each week, a different student takes home the Count (the Sesame Street character or a stuffed animal called the Count), and the family selects a numeracy project (e.g., How many glasses of milk does my family drink in a day (or a week)?) (Muir, 2016). They determine how the data will be gathered, collect it, and a week later share their findings (count) and pass the Count to the next student.

Thinking about Zero

Children need to discover the number zero (Clements & Sarama, 2020), particularly as understanding its value is a required standard of kindergarteners. Surprisingly, it is not a concept that is easily grasped without intentionally building understanding. Three- and four-year-old children can begin to use the word *zero* and the numeral 0 to symbolize that there are no objects in the set. With the dot plates discussed previously (see Figure 7.2), use the zero plate to discuss what it means when there is no dot on the plate. We find that because early counting often involves touching an object, zero is sometimes not included in the count. Zero is one of the most important digits in the base-ten system, and purposeful conversations about it and its position on the number line are essential. Activities 7.1, 7.2, 7.8, and 7.14 are useful in exploring the number zero.

Numeral Writing and Recognition

In many states, kindergarteners are expected to write numbers from 0 to 20. Helping children read and write the 10 single-digit numerals is similar to teaching them to read and write letters of the alphabet. Neither has anything to do with number concepts. In other words, this skill sits outside of conceptual understanding. Students may initially struggle with number writing or recognition, but it should not stop them from gaining more mathematics experiences. Numeral writing can be engaging when you ask children to trace over pages of numerals, make numerals from clay, trace them in shaving cream on their desks, press the numeral on a calculator, write them on the interactive whiteboard or in the air, and so on.

Activity 7.6 Number Necklaces

CCSS-M: K.CC.A.3

Place a necklace (a card with a yarn string) that has a numeral (use numeral cards) on each (you can also show the corresponding number with objects) on the child. Initially, display a number and ask children who are wearing that number to step forward. Now is a good time to show numbers written backward to highlight how those images do not match anyone in the room. Call out different numbers or show different amounts, and children should either step forward or find a peer with the same amount. This game can progress to calling out a number, and then two children who add to that number pair up or having children separate into groups of odd or even numbers.

Pearson eText Activity Page: Number Cards

Activity 7.7 Number Tubs

CCSS-M: K.CC.B.4; K.CC.B.5

Give each child four to six covered margarine tubs, each containing a different number of pennies or counters. Ask the child to find a tub with a particular number of counters. First, they can estimate. Then after the child looks inside and counts to find the correct tub, add a twist. Allow the child to label the tubs with sticky notes to show the amount inside. At first, the child may make four dots to represent four counters, but with encouragement, the numeral can be written. Discuss the value of writing the numbers in a form that all can understand and how that helps avoid recounting.

Activity 7.8 Line Them Up!

CCSS-M K.CC.B.4c; K.CC.A.2

Use painters' tape on the floor or a clothesline as a number line and print the number cards or use the sequence of numbers you want the children to investigate (such as 0 through 20). Mix the cards and place them face down in a pile. Ask a child to take the top card and place/attach the card on the number line in the appropriate position. Have a second child take the next card and so on. As children place their number cards, ask, "How did you know where to place your number?" "Is your number before or after…?" and "Does your number go on the left or the right of…?" Continue placing the cards until all the cards are correctly positioned on the number line in the conventional sequence. Have the children read the sequence forward as well as backward and discuss the need for equal units on the number line. Modify this activity by using smaller sets of easier numbers, longer sequences of larger numbers, placing the numbers out of sequence or improperly spaced or by using number sequences that start at a number other than zero or one. Also, try the Line 'Em Up! Activity Page to order any ten numbers of your choice.

Pearson eText Activity Page: Number Cards

Pearson eText Activity Page: Line 'Em Up!

Common pre-K and kindergarten exercises have children match sets of objects with numerals. These tasks are called set-to-number and number-to-set. Children are given pictured sets (e.g., frogs) and asked to write or match the number that tells how many. Alternatively, they may be given a number and told to make a set with that many objects. The NCTM Illuminations website offers activities in the "Let's Count to 5" unit, such as Focus on Two, Three in a Set, and Finding Four, where children make sets (0–5 objects) and connect number words or numerals to the sets. Illuminations also has Okta's Rescue (http://illuminations.nctm.org), which matches sets to a numeral. Songs, rhymes, and activities that appeal to visual, auditory, and kinesthetic learners are included. Also, try the Button Counts Activity Page, where children can roll a die and place buttons on a shirt. These tasks tap different understandings and should both be used to inform assessment of preschoolers' math abilities (Mou, Zhang, Piazza, & Hyde, 2021). When children are successful with these matching-numeral-to-sets activities, it is time to move on to more advanced concepts, such as counting on and counting back.

Pearson eText Activity Page: Button Counts

Counting On and Counting Back

Counting on is the ability to start counting from a given number other than one, and it is considered a landmark skill on a child's path to number sense (Fosnot & Dolk, 2001). For example, if a child is given a group of five cubes and then given two more and asked how many in all, if the child does not recount the five cubes but just uses the counting sequence and states "six, seven," they are counting on. Counting on from a particular number and counting back can be difficult skills. In particular, for multilingual learners, counting back is more difficult. (Try counting back in a second language you have learned.) One strategy is to encourage students to bob their head as they count to keep track of several things: the counting sequence, the starting number, and how many they need to count beyond that number (Betts, 2015).

Because this double-counting process (Voutsina, 2016) is not easily kept in working memory, teachers can model the actions and use a think-aloud to share their thinking orally as they pull these multiple components together.

Eventually, first-graders should be able to start from any number less than 120 and count on from there. Later, children will make the connection that counting on is adding and counting back is subtracting. Frequent use of Activities 7.8, 7.9, and 7.10 is recommended.

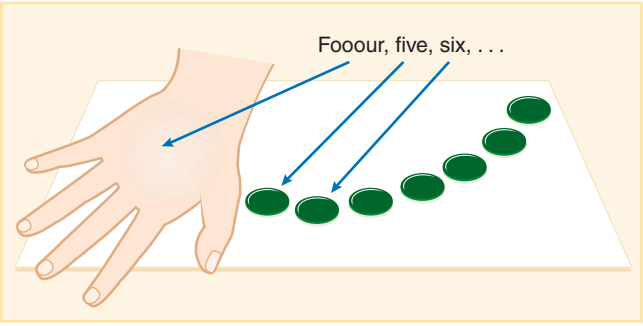


FIGURE 7.5 Counting on: "Hide four. Count, starting from the number of hidden counters."

Activity 7.9 Counting On with Counters

CCSS-M: K.CC.A.2; K.CC.B.5

Have children line up a set 10 or 12 counters from left to right on their desk. Tell them to count four counters and push them under their left hands or place them out of sight in a cup (see Figure 7.5). Then, pointing to the hidden counters in the hand or cup, ask "How many are there?" (Four.) "So, let's count like this: fooour . . . (pronouncing the first number slowly and pointing to their hand), five, six . . . ." Repeat with other numbers of hidden counters.

Activity 7.10 Real Counting On

CCSS-M: K.OA.A.1; 1.OA.B.5

This game for two requires a deck of cards (numbers 1 to 7), a die, a paper cup, and counters. The first player turns over a number card and places the indicated number of counters in the cup. Place the card next to the cup as a reminder of how many are inside. The second player rolls the die and places that many counters next to the cup (see Figure 7.6). Together, they decide how many counters in all. Give children the Real Counting on Recording Sheet with columns for "In the Cup," "On the Side," and "In All" to support their organization. Increase to higher numbers when children are proficient with the smaller numbers. For children with special needs in math, at first keep the number of counters in the cup constant (such as 5) and have them count on from that number. Then move on to the full game.

Pearson eText Activity Page: Real Counting on Recording Sheet

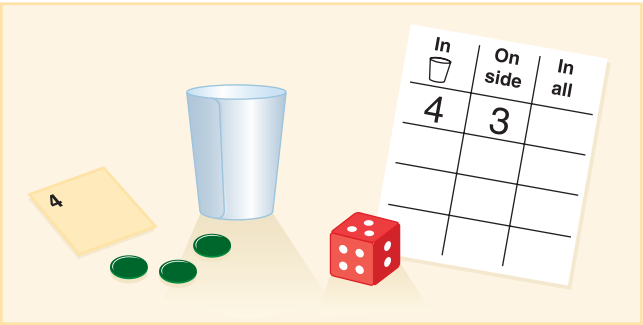


FIGURE 7.6 How many in all? How do children count to tell the total?

Observe how children determine the total amounts in Activity 7.10. Children who are not yet counting on may empty the counters from the cup or will count up from one without emptying the counters. As children continue to play, they will eventually use counting on as that strategy becomes meaningful and useful.

Board games have been shown to support students from low SES in expanding their ability to sense the size or magnitude of numbers, which expands their early numerical development (Siegler, 2010).

Activity 7.11 Counting Moose Tracks

CCSS-M: K.CC.B.4.A

Before this activity, read *Moose Tracks!* (Wilson & Davis, 2006) (or other book with animal tracks), a delightful story about a moose that thinks someone has been in his house because of all the footprints he finds. In the end, he is the one creating all the footprints! Give pairs a Moose Tracks game board, one die, and two game pieces, such as differently colored centimeter cubes. Children take turns rolling the die and moving their game piece along the game board. Play continues until both children have traveled the length of their tracks. Because the board has separate spaces for each moose track, children work on one-to-one correspondence as they move their game piece along the path on the board.

An alternative to moving one game piece along the board is to have children place the number of centimeter cubes along their track that corresponds to the number on the die. For example, if a child rolls a 4, they place 4 cubes along the track in 4 separate spaces on their game board. For the next turn, they place 3 cubes for a 3 on the die. By using several of the cubes, as opposed to moving just one cube along the board, there is a visual record of the one-to-one correspondence related to numbers in the counting sequence.

Pearson eText Activity Page: Moose Tracks Game Board

The Relations Core: More Than, Less Than, and Equal To

Learning Outcome 7.3 **Develop children's abilities to compare quantities and describe relationships between numbers.**

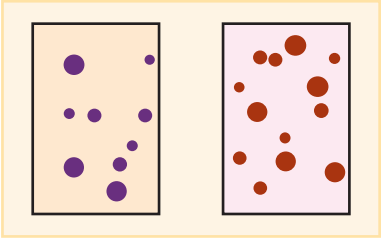


FIGURE 7.7 Which has more?

The concepts of "more," "less," and "same" are basic relationships contributing to children's overall understanding of number. Most kindergarteners can choose the set that is *more* if presented with two sets that are obviously different in number (see Figure 7.7). In fact, Baroody (1987) states, "A child unable to use 'more' in this intuitive manner is at considerable educational risk" (p. 29). Classroom activities should help children build on and refine this basic notion.

Although the concept of less is logically related to the concept of more (selecting the set with more is the same as *not* selecting the set with less), the concept of *less* proves to be more difficult for children than that of *more*. A possible explanation is that young children have many opportunities to use the word *more* but may have limited exposure to the word *less*. To help children, frequently pair the idea of *less* with *more* and make a conscious effort to ask "Which is less?" questions as well as "Which is more?" questions. Teacher language for comparisons is precise:

Countable objects: Ask, "Which has fewer?" or "Which has more?" Example: Comparing 5 and 8 cubes. When we state the full comparison, we say 5 is less than 8 or 8 is greater than 5.

Non-countable objects: Ask, "Which is less?" or "Which has more?" Example: Comparing two piles of dirt on the playground.

For all three relationships (more/greater than, less/less than, and same/equal to), children should construct sets, using counters, as well as make comparisons or choices between two given sets. Conduct the following activities in a spirit of inquiry with requests for children's explanations, such as "Can you show me how you know this group has fewer?"

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

FORMATIVE ASSESSMENT Notes. Observe children during Activity 7.14. Children whose number ideas are completely tied to counting and nothing more will select cards at random and count each dot. Others may begin by estimating and selecting a card that appears to have about the same number of dots. This performance demonstrates a significantly higher level of understanding. Also, observe how the dots are counted. Are the counts made accurately? Is each dot counted only once? Does the child touch the dot or just use a visual scan? A significant milestone (subitizing) occurs when children recognize small patterned sets without counting. ■

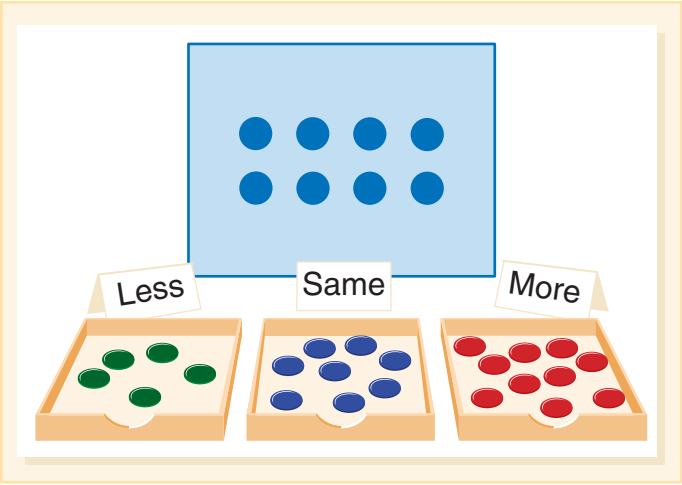


FIGURE 7.8 Making sets that are more, less, and the same

Activity 7.12 Make Sets of More/Less/Same

CCSS-M: K.CC.C.6

Provide about eight cards each with pictures of sets of anywhere from 4 to 12 objects (or use dot cards), a set of counters, word cards labeled *More, Less*, and *Same* (see the More-Less-Same Activity Page), and paper plates or low boxes that can define the work space, to support children with special needs in math. Next to each picture card, have children make three collections of counters: a set that is more than the amount in the picture, one that is less, and one that is the same (see Figure 7.8). Have children who struggle begin by creating a collection of items that matches the picture. For children who are ready to use symbols $>, <, =$ between the quantities, use the Relationship Cards Activity Page.

Pearson eText Activity Page: Dot Cards
Pearson eText Activity Page: More-Less-Same
Pearson eText Activity Page: Relationship Cards

Activity 7.13 Find the Same Amount

CCSS-M: K.CC.A.3; K.CC.C.6

Give children a collection of cards with pictures of sets on them, such as Dot Cards. Have children pick any card in the collection and then find another card with the same amount to form a pair. Continue finding other pairs. This activity can be altered to have children find dot cards that are "less" or "more." Some children with special needs in math may need a set of counters with a blank ten-frame to help them make a match instead of finding a pair.

Pearson eText Activity Page: Dot Cards

FIGURE 7.9 Materials to play More, Less, or the Same

Activity 7.14 More, Less, or the Same

CCSS-M: K.CC.A.3; K.CC.C.6; K.O.A.A.1

This activity is for partners or a small group. Make four to five sets of the cards from the more-or-less cards and two sets of number cards (see Figure 7.9). To play, one child draws a number card, places it face-up, and puts that number of counters into a cup. Next, another child draws one of the more-or-less cards and places it next to the number card. For the more cards, the designated number of counters is added to the cup; for the less cards, counters are removed; and, for zero cards, no change is made. Children then predict how many counters are in the cup. The counters are emptied and counted and then the game is repeated by drawing new cards. Eventually, the words *more* and *less* can be paired or substituted with *add* and *subtract* to connect these ideas with the arithmetic operations.

Pearson eText Activity Page: More-or-Less Cards
Pearson eText Activity Page: Number Cards

Developing Number Sense by Building Number Relationships

Number sense is a "good intuition about numbers and their relationships. It develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms" (Howden, 1989, p. 11). Number sense has been a priority in mathematics learning for decades. The ability to think about numbers flexibly and represent numbers with concrete materials and symbols is an indication of number sense.

Pearson eText
Video Example 7.1
In this video the teacher engages students in an activity on that supports their understanding of one more than and two more than a number.

Pause & Reflect

You already have seen some of the early foundational ideas about number. Stop now and make a list of all the important ideas that you think children should *know about* the number 8. Put your list aside, and we will revisit your ideas later. ●

The discussion of number sense begins as we look at the relationships and connections children should make about numbers up to 20. But good intuition about numbers does not end with these smaller whole numbers. Children continue to develop number sense competencies as they use numbers in operations, build place-value understanding, and devise flexible methods of computing and making estimates involving large numbers.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Relationships between Numbers 1 through 10

Once children acquire cardinality and can meaningfully use their counting skills, they need to expand their collection of ideas about number sense to develop the knowledge needed to work meaningfully with operations. We want children to move away from counting by ones to being able to use number sense to add and subtract more efficiently. This requires development of number relationships, in particular the three identified in Figure 7.10 and explained in the section that follows.

The principal tool young children will use as they construct these relationships is the one number tool they possess: counting. Initially, you will notice a lot of counting by ones, and you may wonder whether you are making progress. Have patience! As children construct new relationships and begin to use more powerful ideas, counting by ones will become less and less necessary.

One and Two More, One and Two Less. When children count, they don't automatically think about the way one number is related to another. Their goal is only to match number words with objects until they reach the end of the count. To learn that 6 and 8 are related by the corresponding relationships of "two more than" and "two less than" requires reflection on these ideas. Children should know that 7, for example, is 1 more than 6 and 2 less than 9. Counting on (or back) one or two counts is a useful tool in constructing these ideas and will soon link to Add 1 (or 2) or Subtract 1 (or 2). Begin with Activity 7.15, which focuses on the two-more-than relationship.

In activities where children find a set or make a set, they can also select the matching number cards that identify the quantity in the set. Children can be encouraged to take turns reading the associated number sentence to their partner. If, for example, a set has been made that is two more than a set

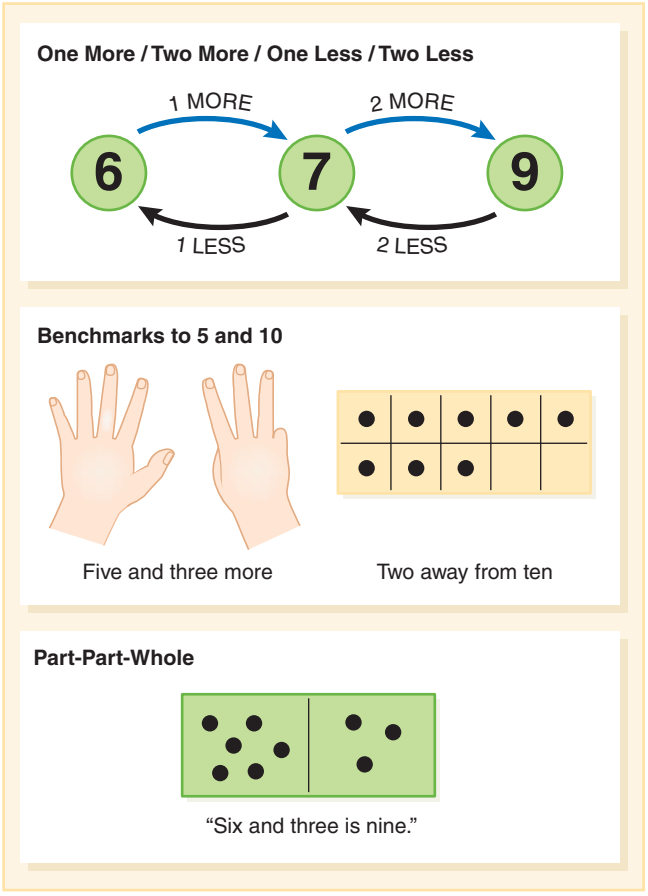


FIGURE 7.10 Three number relationships for children to develop

of four, the child can say, "Two more than four equals six" or "Six is the same as two more than four." Activity 7.16 combines these relationships with a symbolic representation.

Pearson eText Activity Page: Number Cards

Activity 7.15 Make a Two-More-Than Set

CCSS-M: K.OA.A.2; 1.OA.C.5

Provide children with six dot cards. For each card, children should display a set of counters that is two more than the set shown on the card. Similarly, spread out eight to ten dot cards, and for each card, find a card that is two less than it (omit the 1 and 2 cards for two less than). Also, see the Expanded Lesson Two More Than/Two Less Than and the Two More Than and Two Less Than Activity Pages.

Pearson eText Activity Page: Dot Cards

Pearson eText Activity Page: Two More Than and Two Less Than

Pearson eText Expanded Lesson: Two More Than/Two Less Than

Activity 7.16 A Two-More-Than Machine

CCSS-M: 1.OA.C.5

Teach children how to make a two-more-than machine. Press 0 [+] 2 [=] on a calculator. Now have them press any number—for example, 5. Children hold their finger over the [=] key and predict the number that is two more than 5. Then they press [=] to confirm. If they do not press any of the operation keys ([+], [−], [×], [÷]), the two-more-than machine will continue to perform in this way.

Using Benchmark Numbers 5 and 10. Because 10 plays such a large role in our numeration system and because two fives equals 10, we want to help children develop relationships for the numbers 1 to 10 connected to the benchmarks of 5 and 10. For example, consider how the knowledge of 8 as "5 and 3 more" and as "2 away from 10" can play a role in how a child thinks about $5 + 3$, $8 + 6$, $8 - 2$, $8 - 3$, $8 - 4$, and $13 - 8$. For example, $8 + 6$ may be thought of as $8 + 2 + 4$ ("Making 10" strategy). Later, similar relationships can be used in the development of mental computation skills with larger numbers, such as $68 + 6$.

The most common models for exploring the *benchmark* numbers 5 and 10 (and effective templates for structuring counting of manipulatives) are five-frames and ten-frames. The *five-frame* is a 1×5 array, and the *ten-frame* is a 2×5 array, both positioned horizontally, in which counters or dots are placed to illustrate numbers. See the Math Learning Center for free apps at https://www.mathlearningcenter.org/apps/ with frames for 5, 10, 20, and 100.

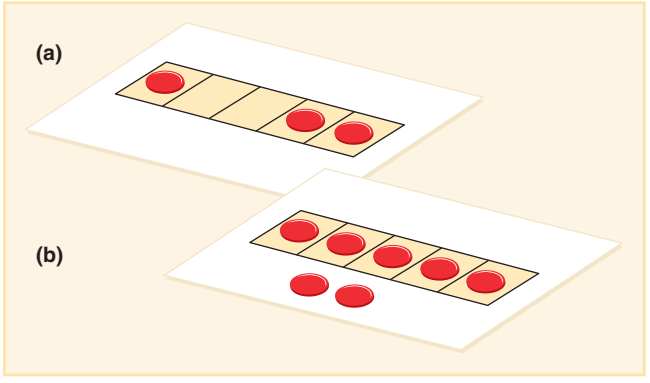


FIGURE 7.11 A five-frame focuses on five as the benchmark.

Pearson eText Blackline Master: Five-Frames

Pearson eText Blackline Master: Ten-Frames

For pre-K or kindergarten children, it is a good idea to begin with a five-frame (see Figure 7.11). Provide about 10 counters that will fit in the five-frame sections and try Activity 7.17.

Activity 7.17 Five-Frame Tell-About

CCSS-M: K.CC.C.7; K.OA.A.5

Explain that only one counter is permitted in each section of the five-frame. Have children show three on their five-frame, as seen in Figure 7.11(a). "What can you tell us about 3 from looking at your five-frame?" After hearing from several children, try other numbers from 0 to 5. Children may initially place their counters on the five-frame in any manner. For example, with four counters, a child may place two on each end and say, "It has a space in the middle" or "It's two and two." There are no wrong answers with the initial placements. Focus attention on how many more counters are needed to make five. Next, try numbers between 5 and 10. As shown in Figure 7.11(b), numbers greater than 5 are shown with a full five-frame plus additional counters on the mat but not in the frame. In discussion, focus attention on these larger numbers as 5 and some more: "Seven is the same as five and two more."

Pearson eText Blackline Master: Five-Frame

When students have a strong foundation in 5 as a benchmark (from the five-frames), introduce ten-frames. Ten frames are two rows of five and help students notice number relations for both 5 and 10. Play a ten-frame version of a Five-Frame Tell-About. When the top row is full, place counters in the bottom row, also from left to right. This structure encourages seeing a full row in the ten-frame as five without counting, as in Figure 7.12. At first, children may continue to count every counter by ones. Then ask questions such as, "What are you looking at in the ten-frame to help you find how many?" and "How does knowing you have a full row help you find how many you have?"

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

Pearson eText Blackline Master: Ten-Frame

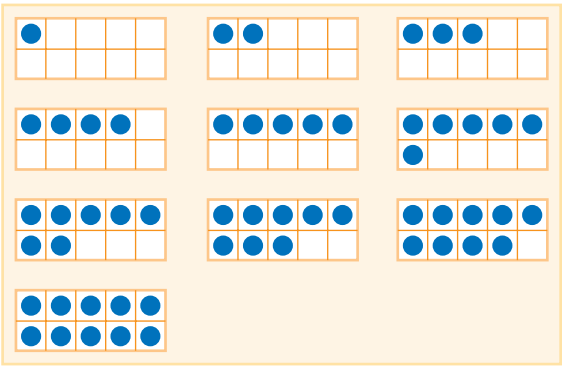


FIGURE 7.12 Ten-frames

Move children to a problem-solving situation by having them create different numbers, as suggested in Activity 7.18.

Activity 7.18 Number Medley

CCSS-M: K.CC.C.6; K.CC.C.7

Begin by having all children make the same number on their ten-frame. Then call out or hold up random numbers between 0 and 10. After each number, the children change their ten-frames to show the new number. Ask, "How do you decide to change your ten-frame?" If working with multilingual learners, consider saying the number in their native language or writing the number. Children can play this game independently by using a prepared list of about 15 random numbers. One child plays teacher, and the rest use the ten-frames.

Pearson eText Blackline Master: Ten-Frame

FORMATIVE ASSESSMENT Notes. How children use the ten-frame can provide you with insights into children's number concept development; therefore, Activity 7.18 can be used as a diagnostic interview. Observe that some children will take all counters off and begin each new number from a blank frame. Others will have learned what each number looks like, and others will soon learn to adjust numbers by adding on or taking off only what is required, often subitizing a row of five. These strategies of moving from one number to another number more efficiently need to come from the children. Look for opportunities to draw attention to how other children are making these changes by having them share strategies. To add another

dimension to the assessment, have the child tell, *before* changing their ten-frames, how many more counters need to be added (addition) or removed (subtraction). For example, if the frame showed 6, and the teacher said "4," the child would respond, "Subtract two!" and then change their ten-frames accordingly. A discussion of how they know what to do is valuable. ■

Small ten-frame cards are an important variation of ten-frames and can be made from cardstock. Use the Little Ten-Frames Activity Page, with a set of 20 cards that consists of a 0 card, a 10 card, and two of each of the numbers 1 to 9. The cards allow for simple practice activities to reinforce the benchmarks of 5 and 10, as in Activity 7.19.

Pearson eText Blackline Master: Little Ten-Frames

Activity 7.19 Little Ten-Frame Flash

CCSS-M: K.CC.B.5

Flash little ten-frame cards to the whole class or a small group and see how quickly the children can tell how many dots are shown. This activity is fast-paced, takes only a few minutes, can be done at any time, and is highly engaging. For multilingual learners, producing the English word for the number may take more time, so either make pairs with similar language skills or encourage children to use their preferred language.

Pearson eText Activity Page: Little Ten-Frame Cards

Standards for Mathematical Practice
MP7. Look for and make use of structure

Important variations of Little Ten-Frame Flash include:

- Saying the number of empty spaces on the card instead of the number of dots
- Saying one more than the number of dots (or two more, one fewer, or two fewer)
- Saying the "10 fact"—for example, flash six dots, and they say, "Six and four equals ten"
- Saying a sum by adding the flashed card to a card they have at their desk

Ten-frame tasks are surprisingly challenging for children, because there is a lot to keep in their working memory. Children must reflect on the two rows of five, the empty spaces, how a particular number is more or less than 5 and how far away it is from 10.

Pearson eText
Video Example 7.2
In this video researchers explain executive function and how young children keep ideas in their working memory.
https://www.youtube.com/watch?v=efCq_vHUMqs

How well children can respond to Little Ten-Frame Flash is a quick diagnostic assessment of their current number concept level. Because the distance to 10 is so important, another assessment is to point to a numeral less than 10 and ask, "If this many dots were on a ten-frame, how many empty spaces would there be?" Or, you can also simply ask, "If I have seven, how many more do I need to equal ten?" There are virtual activities with five-frames and ten-frames on the NCTM Illuminations webpage, with free games that develop counting and basic computation skills. These activities can be used with individual learners, because a voice-over option will ask children questions about the tasks.

Pause & Reflect

Before reading on, gather eight counters. Count out the set of counters in front of you as if you were a 4- or 5-year-old. ●

Part-Part-Whole Relationships. Any child who has learned how to count meaningfully can count out eight objects as you just did. What is significant about the experience is what it did *not* cause you to think about. Nothing in counting a set of eight objects will cause a child to focus on the fact that it could be made of two parts. Of the three number relationships (Figure 7.10), *part-whole* ideas are easily the most important, and the ability to think about numbers in terms of the *part-part-whole* relationship is a major milestone in the development of number sense. To conceptualize a number as being composed of two or more parts is the

most important relationship that can be developed about numbers. For example, separate the counters you just counted into two piles and reflect on the combination. It might be 2 and 6, 7 and 1, or 4 and 4. Make a change in your two piles of counters and say the new combination to yourself. Focusing on a quantity in terms of its parts is *decomposing numbers*.

Many times, part-part-whole activities focus on a single number for the entire experience. For example, a pair of children might work on breaking apart or building the number 7 throughout the activity. They can either build (*compose*) the designated quantity in two or more parts, or they can start with the full amount and separate it into two or more parts (*decompose*). This latter situation is known as a "both addends unknown" problem ($7 = ? + ?$), which should be developed in kindergarten, possibly starting by working on the numbers 4 or 5. As concepts develop, children extend to numbers 6 to 12 and beyond.

When children do these activities, have them say or read the parts aloud and then write them down on some form of recording sheet. Reading and writing the combinations serve as a means of encouraging reflective thought focused on the part-part-whole relationship and help in the identification of patterns. Writing can be in the form of sketches, numbers written in blanks (a group of _____ cubes and a group of _____ cubes), or addition equations ($3 + 5 = 8$ or $8 = 2 + 6$). There is an important connection between part-part-whole concepts and addition and subtraction ideas.

Activity 7.20 and its variations may be considered the classic part-part-whole task.

Activity 7.20 Build It in Parts

CCSS-M: K.OA.A.3; 1.OA.C.6

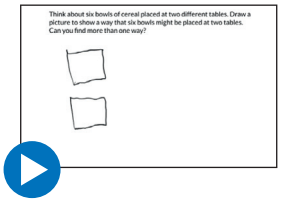
Provide children with one type of material in different colors, such as connecting cubes or squares of colored paper. Ask children, "How many different combinations for a particular number can you make using two parts?" (If you wish, you can allow for more than two parts.) Then students would take, for the number 7, 3 yellow cubes and 7 blue cubes. You can use a familiar, everyday context or consider a situation from a piece of children's literature. For example, ask how many different combinations of six hats the peddler in the picture book *Caps for Sale* (Slobodkina, 1938) can wear, limiting the color choices to two and keeping the colors clustered together (i.e., 3 red and 3 blue caps, 4 red and 2 blue, etc.). (Note that the book is also available in Spanish for multilingual learners.) Each different combination can be displayed on a small mat. Here are just a few ideas, each of which is illustrated in Figure 7.13.

- Use two counters, such as lima beans spray-painted on one side (available commercially in plastic).
- Make two different-colored bars of connecting cubes. Keep the same colors together on the bar.
- Make combinations of two Dot Strips.
- Make combinations of Two-Column Cards. All pieces except the single squares are cut from two columns of the cardstock.
- Make combinations of two Cuisenaire rods connected as a train to match a given amount.
- Use beads on a pipe cleaner or use an arithmetic rack that has 10 beads in two rows. See the number rack app at the Math Learning Center website.

Pearson eText Blackline Master: Dot Strips

Pearson eText Blackline Master: Two-Column Cards

As you observe children working on the Build It in Parts activity, ask them to read to their partner a number sentence that matches the representation. Two or three children working together may have quite a large number of combinations, including repeats. Encourage them to look for patterns by asking, "Do you have all possible combinations? How do you know?"


Think about six bowls of cereal placed at two different tables. Draw a picture to show a way that six bowls might be placed at two tables. Can you find more than one way?

Pearson eText
Video Example 7.3
In this video consider how the student solves the Bowls on Two Tables problem (a both addends unknown problem), illustrating different combinations of the number 6.

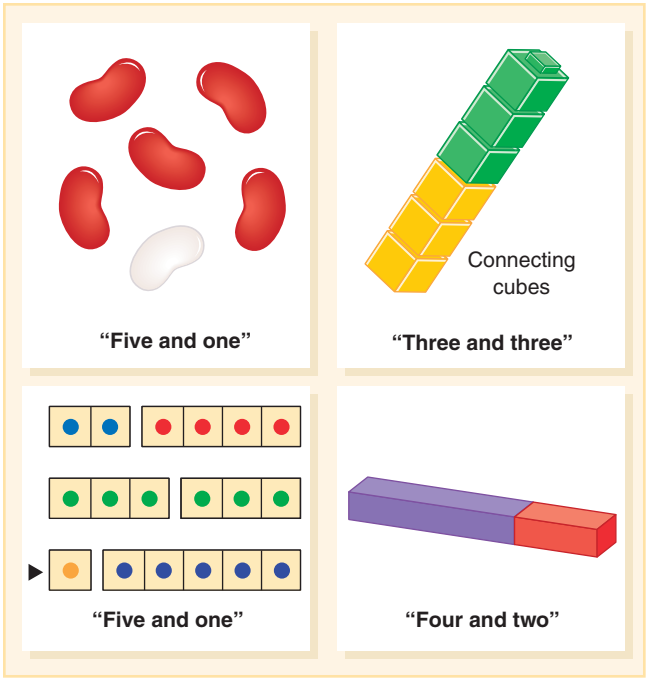


FIGURE 7.13 Assorted materials for building parts of six

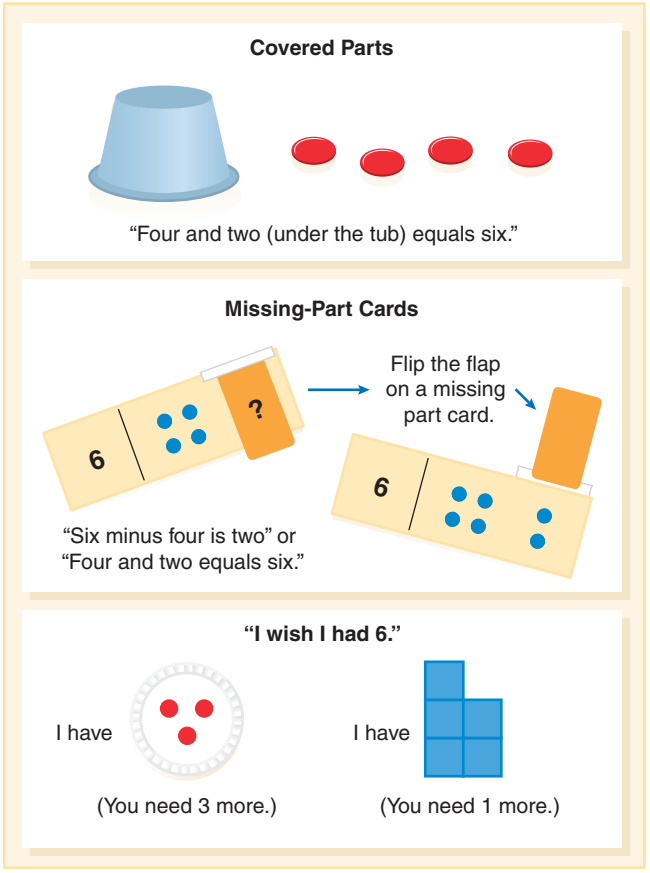


FIGURE 7.14 Missing part activities

Activity 7.21 Ten Hot Dogs

CCSS-M: K.CC.C.6; K.OA. A.3; 1.OA.A.1

Read the book *10 Little Hot Dogs* by Himmelman (2010) or any other book with a set of 8–10 animals. This predictable-progression counting book highlights 10 dachshund puppies climbing on and off a chair. Children can create their own stories, using a mat illustrated with a chair and move counters representing the puppies on or off. Two children can compare the numbers of dogs on their chairs. Who has more puppies? How many more? How many different ways can you put 6 puppies on the two chairs? Students can also record the equations as both-addends-unknown problems.

Pearson eText Activity Page: Chair Mat

Missing-Part Activities. An important variation of part-part-whole activities is referred to as *missing-part* activities. In a missing-part activity, children are given the whole amount and part of the whole. They then use their already developed knowledge of the part-part-whole relationship to tell what the covered or hidden part is. If they are unsure, they simply can uncover the unknown part and say the full combination. Missing-part activities increase the difficulty level because they involve an unknown quantity that is not visible and that to solve, children need to reflect on the combinations for a number. They serve as the forerunner to subtraction. With a whole amount of 8 but with only 3 showing, the child can later learn to say and write "$8 - 3 = 5$." Splat! activities are excellent ways to engage students in missing activities. (See Chapter 4 and check out the Splat! website: https://stevewyborney.com/2017/02/splat/.)

The next three activities illustrate variations of this important idea of a missing part. For any of these activities, you can use a context from familiar classroom events or from a children's book, such as the disappearing treats in *Pete the Cat and the Missing Cupcakes* (Dean & Dean, 2016).

Activity 7.22 Covered Parts

CCSS-M: K.OA.A.2

A set of counters equal to a target amount is counted out, and the rest are put aside. One child places these counters under a margarine tub or piece of cardstock. The child then pulls some counters out into view. (This amount could be none, all, or any amount in between.) For example, if 6 is the whole and 4 are pulled out and showing, the other child says, "Four and *two* equals six." Then they look under the tub to confirm. If there is hesitation or if the hidden part is unknown, the hidden part is immediately shown (see Figure 7.14).

Activity 7.23 Missing-Part Cards

CCSS-M: K.OA.A.4

Use the missing-part cards, which have a number from 4 to 10 (see Figure 7.14). Each card has a numeral for the whole and two dot sets with one set covered by a flap. For the number 8, you need to have nine cards with the visible part ranging from 0 to 8 dots. For a card showing 6, four dots showing and two dots hiding under the flap, children use the cards, as in "Covered Parts," saying, "Four and two equals six."

Pearson eText Activity Page: Missing-Part Cards

Activity 7.24 I Wish I Had

CCSS-M: K.OA.A.1; K.OA.A.2

Hold out a bar of connecting cubes, a dot strip, a dot plate or a two-column card showing six or less and say, "I wish I had six." The children respond with the part that is needed to equal 6. Counting on can be used to check. The game can focus on a single number (especially as a starting point for children with special needs in math), or the "I wish I had" number can change each time (see Figure 7.14). Consider adding a familiar context, like "I wish I had six books to read." See the corresponding Expanded Lesson: I Wish I Had.

Pearson eText Blackline Master: Dot Strip

Pearson eText Blackline Master: Two-Column Cards

Pearson eText Expanded Lesson: I Wish I Had

***TECHNOLOGY* Note.** There are many virtual manipulatives that are useful when teaching early number concepts. Here are a few:

Connecting Cubes	https://www.didax.com/apps/unifix/	Multicolored cubes for counters, making 10, etc.
Ten Frames	https://www.didax.com/apps/ten-frame/	Basic frame for use as demonstration or for students to fill
Double Ten Frames	https://www.mathlearningcenter.org/apps	Double ten frames (or more) for numbers greater than 10.
Rekenrek (bead frame)	https://www.didax.com/apps/rekenrek/	20-bead frame to use for counting or part-part-whole tasks

Pause & Reflect

Remember the list you made about what children should know about the number 8? Before reading on, refer back to it and see if you would add to it or revise it based on what you have read to this point.

Here is a possible list of the kinds of things that children should learn about the number 8 (or any number up to about 20) while they are in pre-K and kindergarten:

- Count to 8 (know the number words and their sequence).
- Count 8 objects and know that the last number word tells how many.
- Recognize, read, and write the numeral 8 and pair it with an accurate number of objects or units.
- State more and less than 8 by 1 and 2—8 is one more than 7, one less than 9, two more than 6, and two less than 10.

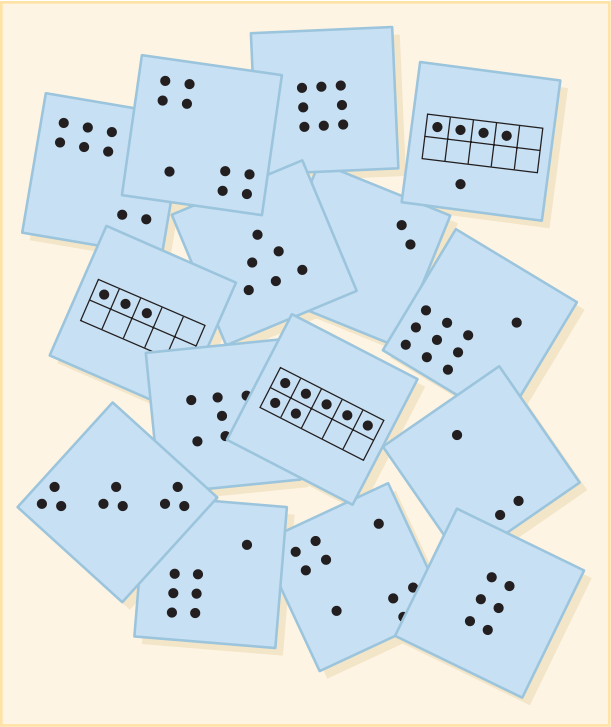
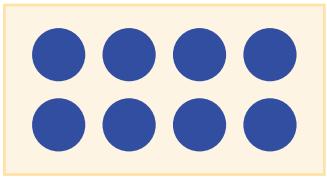


FIGURE 7.15 Dot cards

- Recognize *patterned sets* for 8 such as:



- Relate 8 to the *benchmark numbers* of 5 and 10: 8 is 3 more than 5 and 2 away from 10.
- State part-part-whole relationships: 8 is the same as 5 and 3, 2 and 6, 7 and 1, and so on (including knowing the missing part of 8 when some of the quantity is hidden).
- Identify doubles: double 4 equals 8.
- State relationships to the real world: my brother is 8 years old; my reading book is 8 inches wide.

Then extend these ideas to numbers beyond ten and further refine children's understanding of number.

Moving toward More Challenging Relationships. As children learn about ten-frames, patterned sets, and other relationships with an expanded collection of numbers beyond ten, the dot cards provide a wealth of activities (see Figure 7.15) and help them think flexibly about numbers. The full set of cards contains dot patterns, patterns that require counting, combinations of two and three simple patterns, and ten-frames with standard as well as unusual dot placements. These dot cards add another dimension to many of the previous activities and can be used effectively in Activities 7.25, 7.26, and 7.27.

Activity 7.25 Double War

CCSS-M: K.CC.C.6

The game of Double War (Kamii, 1985) is played like the classic War card game, but on each play, both players turn up two dot cards instead of one. The winner is the player with the larger total. Children can and should use reasoning and many number relationships to determine the winner without actually finding the total number of dots. A modification of this activity for children with special needs in math would have the teacher (or another child) do a think-aloud and describe their thinking about the dots, using relationships. This modeling is critical for children who struggle.

Pearson eText Activity Page: Dot Cards

Activity 7.26 Find the Missing Dot Card

CCSS-M: K.OA.A.2

Place a number card on the table (for example, 8 if you are working on combinations that equal 8). Use dot cards that are less than eight. Lay the cards out like in a large array, with all cards face-up. Partner 1 pulls one card from the set and gives it to their partner. The partner finds the other card (part) to equal 8. Then a challenge could be to have the cards face-down and look for the two parts that equal 8 (or whatever number is selected). Differentiate by selecting different totals.

Pearson eText Activity Page: Dot Cards

Activity 7.27 Number Sandwiches

CCSS-M: K.OA.A.2; 1.OA.C.6

There are several engaging ways to make number sandwiches and build understanding of part-part-whole (and addition families):

1. Select a number between 5 and 12 and post it. A child finds a combination of two dot cards that total that number and makes a sandwich with the two cards by placing them back to back with the dot sides facing out. The child shows one side of their sandwich to a partner (or the class) and the partner names the quantity on the other side. The sandwich builder flips the sandwich over to see if the missing part was identified correctly.
2. Distribute dot cards to the students and have them create their own mystery sandwich. The two dot cards are still on the outside, back to back, but inside the sandwich is a dot card with the whole amount. Use paper clips or binder clips to keep the sandwich together. Children can swap sandwiches with their peers and figure out what is inside the sandwich, open the sandwich for the reveal, and then trade sandwiches again.
3. Begin as in version 1, but rather than show-and-find, each child tries to make as many sandwiches as they can from their set of cards. This can be a wonderful Math Center activity.

Pearson eText Activity Page: Dot Cards

FORMATIVE ASSESSMENT Notes. To assess part-part-whole relationships, use a missing-part diagnostic interview that begins with a number you believe the child can name all the combinations of, say, 5. Have the children count out that many counters into your open hand. Close your hand around the counters and confirm that the child knows how many are hidden. Then remove some and show them in the open palm of your other hand (see Figure 7.16). Ask the child, "How many are hidden now?" "How do you know?" Repeat with different amounts removed, trying three or four missing parts for each number. If the child responds quickly and correctly and is clearly not counting in any way, call that an understood number and check it off on the child's assessment record. Repeat the process with the next higher number. Continue until the child begins to struggle. In early kindergarten, you will find a range of learned numbers from 4 to 8. By the end of kindergarten, children should know all number combinations through 10. ■

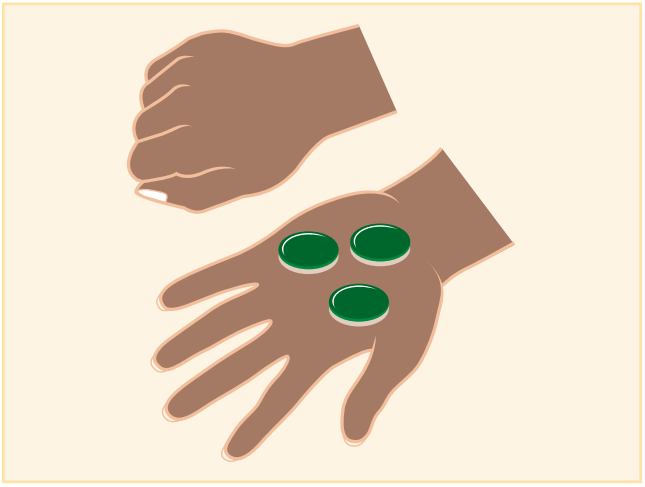


FIGURE 7.16 A missing-part assessment. "There are eight in all. How many are hidden?"

Relationships for Numbers 10 through 20 and Beyond

Although pre-K, kindergarten, and first-grade children experience numbers up to 20 and beyond daily, it should not be assumed that they will automatically extend the set of relationships they developed with smaller numbers to numbers beyond 10. And yet these larger numbers play a big part in many counting activities, in basic facts, and in much of what we do with mental computation. In fact, several researchers suggest the movement to number instruction beyond 10 and even beyond 20 as soon as possible sets the stage for developing initial place-value knowledge (Wright, Stranger, Stafford, and Martland, 2006).

Pre–Place-Value Concepts. Recognizing a set of ten plays a major role in children's early understanding of numbers between 10 and 20. When children see a set of six together with a set of ten, they should know without counting that the total is 16. This work with composing and decomposing numbers from 11 through 19 in kindergarten is seen as an essential foundation for place value.

Pause & Reflect

Say to yourself, "One ten." Now think about that idea from the perspective of a child just learning to count to 20! What could "one ten" possibly mean when ten tells me how many fingers I have and is the number that comes after nine? How can it be one of something? ●

Wright and colleagues (2006) outlined a three-level learning progression (Table 7.3) of children's understanding of ten:

TABLE 7.3 LEARNING PROGRESSION FOR UNDERSTANDING TEN

Levels of Thinking	Characteristics of the Child
Initial Concept of Ten	Understands ten as ten ones and does not see the ten as a unit. When children at this level work on a task involving tens, they will count by ones.
Intermediate Concept of Ten	Understands ten as a unit composed of ten ones but relies on materials or representations to help complete tasks involving tens. They cannot yet carry out addition and subtraction with tens and ones presented in number sentences.
Facile Concept of Ten	Solves addition and subtraction situations involving tens and ones without using materials or representations. Children at this level mentally think about two-digit numbers as groups of tens and ones.

First-graders should know that "10 can be thought of as a bundle of ten ones—called a 'ten'" (NGA Center & CCSSO, 2010, p. 15). Fuson, Clements, and Sarama (2021) describe working with the numbers from 11 to 19 as "one of the most important kindergarten standards" (p. 2) with its connection to place value. The difficulty in children discussing "one ten and six ones" (what's a one?) does not mean that a set of ten should not figure prominently in the discussion of the teen numbers. Initially, children do not see a numeric pattern in the numbers between 10 and 20. Rather, these number names are simply ten additional words in the number sequence. In some languages such as Chinese, the teen number names are actually formed from their words for 10 and 1, 10 and 2, 10 and 3, and so on. But because this logical pattern is not the case in English, for many children, the teens provide a significant challenge.

Activity 7.28 Ten and Some More Stories

CCSS-M: K.NBT.A.1

Use a simple two-part mat and a story situation that links to whatever number of counters you are using. Then if using 15, for example, have children count out ten coffee stirrers onto the left side of the mat. Next, have them put five stirrers on the other side. Together, count all of the stirrers by ones. Chorus the combination: "Ten and five equals fifteen." Rotate the mat: "Five and ten equals fifteen." Repeat with other teen numbers in a random order, but then always keep 10 on the left side of the mat. After playing the game for a while, bundle the 10 stirrers with a rubber band.

Following this activity, explore numbers through 20 in a more open-ended manner. Provide each child with a double ten-frame mat. In random order, have children show numbers up to 20 on the frames and have them discuss how counters can be arranged on the mat so that it is easy to see how many are there. At first, not every child will create a full set of 10, but as this idea becomes more popular, they will develop the notion that teens are 10 and some more. Then challenge children to find ways to show 26 counters or even higher numbers.

Pearson eText Blackline Master: Double Ten-Frame

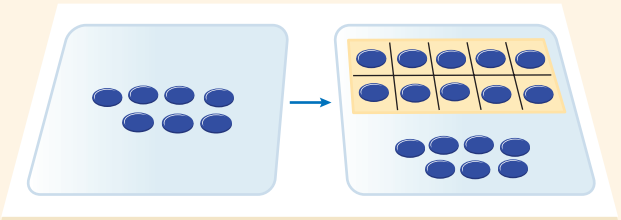
Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

Extending More-Than and Less-Than Relationships. The relationships of one more than, two more than, one less than, and two less than are important for all numbers and are built from children's knowledge of the same concepts for numbers less than 10. The fact that 17 is one less than 18 is connected to the idea that 7 is one less than 8. Children may need explicit support in making this connection.

Activity 7.29 More and Less Extended

CCSS-M: K.CC.C.6; K.NBT.A.1

Project an image of seven counters and ask what is two more, one less, and so on. Now add a filled ten-frame to the display (or 10 in any pattern) and repeat the questions. Pair up questions by covering and uncovering the ten-frame, as in Figure 7.17.



How many?
What is one more?
Two less?

How many?
What is one more?
Two less?

FIGURE 7.17 Extending relationships to the teens

Numbers to 100: Early Introductions. By the end of the school year, kindergarteners are expected to be able to count to 100 (NGA Center & CCSSO, 2010). Therefore, early exposure to numbers to 100 is important. Although it is unlikely that children in kindergarten or first grade will initially have a facile understanding of tens and ones related to place value, they should learn much about the sequence of numbers and the counting patterns to 100, if not beyond.

The classic hundred chart (Figure 7.18) is an essential tool for every K–2 classroom. A useful version of the chart is made of transparent pockets into which each of the 100 numeral cards can be inserted. You can hide a number by inserting a blank card in front of a number in the pocket. You can insert colored pieces of paper in the slots to highlight various number patterns. And you can remove some or all of the number cards and have children replace them in their correct location (see Chapter 10 for more ideas and activities). A digital version of a hundred chart can be found at Math Playground (https://www.mathplayground.com/interactive_hundreds_chart.html). A 0–99 chart is available by visiting https://toytheater.com/number-chart-0-to-99.

Pearson eText Blackline Master: Hundred Chart

Another version of the hundred chart is a bottom-up version (Bay-Williams & Fletcher, 2017). This version organizes the numbers from a 1 in the lower left-hand corner to a 100 in the upper right (see Figure 7.19). A bottom-up format is based on number sense—as a number

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

FIGURE 7.18 A hundred chart

91	92	93	94	95	96	97	98	99	100
81	82	83	84	85	86	87	88	89	90
71	72	73	74	75	76	77	78	79	80
61	62	63	64	65	66	67	68	69	70
51	52	53	54	55	56	57	58	59	60
41	42	43	44	45	46	47	48	49	50
31	32	33	34	35	36	37	38	39	40
21	22	23	24	25	26	27	28	29	30
11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10

FIGURE 7.19 Bottom-Up Hundred Chart

increases in quantity, it also goes up the chart. For example, you can stack rows of ten connecting cubes to show counting by tens. Many of the same activities for the classic hundred chart can be modified for use with the flipped chart. Brainingcamp.com has a hundred chart adaptable to top-down or bottom-up.

Pearson eText Blackline Master: Bottom-Up Hundred Chart

FORMATIVE ASSESSMENT Notes. Replacing the number cards from a blank hundred chart is a good learning center activity for two children. By listening to how they go about finding the correct locations for numbers, you can learn a lot about how well they have constructed an understanding of the 1-to-100 sequence. ■

Pearson eText Blackline Master: Blank Hundred Chart
Pearson eText Blackline Master: Bottom-Up Hundred Chart

Number Sense in Their World

Learning Outcome 7.4 **Support children's connections between mathematical ideas and the real world.**

Young students need to examine relationships of numbers to real-world quantities and measures to enhance their number sense. The use of numbers in simple estimations can help children develop flexible, intuitive ideas about numbers. Jointly published by the National Association for the Education of Young Children (NAEYC) and NCTM (2010) and summarized by Clements, Fuson, and Sarama (2017, p. 182), here are two developmentally appropriate Teaching Practices:

- Teachers support children's ability to make meaning and mathematize the real world.
- Teachers provide settings and bring young learners' attention to the connections between mathematics language and symbols and quantities and actions in the world.

The home becomes a fertile area for real-world mathematics activities. The Erickson Institute is a collaborative that focuses on early math (https://earlymath.erikson.edu/grade-level/1-pre-k/), and there you can encourage home projects such as math around the house, putting away groceries, math walks, and math in outdoor spaces.

Additionally, young learners can begin to explore the world through integrated STEM experiences to see how mathematics links with science, technology, and engineering ideas (Sarama et al., 2018). These opportunities to make connections often emerge in activities in measurement, geometry (which will be developed in a later chapter), and data.

Estimation and Measurement

One of the best ways for children to think of real quantities is to associate numbers with measures of things. Measures of length, weight, and time are good places to begin. Just measuring and recording results will not be very effective unless there is a reason for children to be interested in or think about the result. To help children think about what number might tell how long the desk is or how heavy the book is, it is important if they can first write down or tell you an estimate.

To produce an estimate is, however, a very difficult task for young children. They do not easily grasp the concept of "estimate" or "about." For example, suppose that you have cut out a set of very large footprints, each about 18 inches long. You would ask, "About how many of the giant's footprints will it take to measure across the rug in our reading corner?" The key word here is *about*, and it is one that you will need to spend a lot of time helping your children understand.

To this end, the request of an estimate can be made in ways that help with the concept of "about." For example, the following questions can be used with early estimation activities:

- *More or less than* _____? Will the rug be more or less than 10 giant footprints long? Will the apple weigh more or less than 20 blocks? Are there more or less than 15 connecting cubes in this long bar?
- *Closer to* _____ *or to* _____? Will the rug be closer to 5 giant footprints or closer to 20 giant footprints long? Will the apple weigh closer to 10 blocks or closer to 30 blocks? Does this bar have closer to 10 cubes or closer to 50 cubes?
- *About* _____? (You can even suggest possible numbers as options.) About how many giant footprints long is the rug? About how many blocks will the apple weigh? About how many cubes are in this bar?

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Asking for estimates by using these formats helps children learn what you mean by *about*. Every child can make a close estimate with some supportive questions and examples. However, rewarding children for the closest estimate in a competitive fashion will often result in their attempt to seek precision and not actually estimate. Instead, discuss all answers that fall into a reasonable range. One of the best approaches is to give children ranges as their possible answers: "Does your estimate fall between 10 and 30? Between 50 and 70? Or 100 and 130?" Of course, you can make the choices more divergent until they grasp the idea.

Here are some activities that can help children connect numbers to real situations.

Activity 7.30 Add a Unit to Your Number

CCSS-M: K.MD.A.1

Write a number on the board. Now suggest some units to go with it and ask the children what they can think of that fits. For example, suppose the number is 9. "What do you think of when I say 9 *dollars*? 9 *hours*? 9 *cars*? 9 *kids*? 9 *meters*? 9 *o'clock*? 9 *hand spans*? 9 *gallons*?" Spend time discussing and exploring each unit. Let children suggest other appropriate units. Children from different cultures and multilingual learners may be able to share different units used in their culture.

Activity 7.31 Is It Reasonable?

CCSS-M: K.MD.A.1

Select a number and a unit—for example, 15 feet. Could the teacher be 15 feet tall? Could a house be 15 feet wide? Can a man jump 15 feet high? Could three children together stretch their arms 15 feet? Pick any number, large or small, and a unit with which children are familiar. Then make up a series of these questions. Also ask, "How can we find out if it is reasonable? Who has an idea about what we can do?" For students who would like a challenge, have them select the number and unit and create the questions.

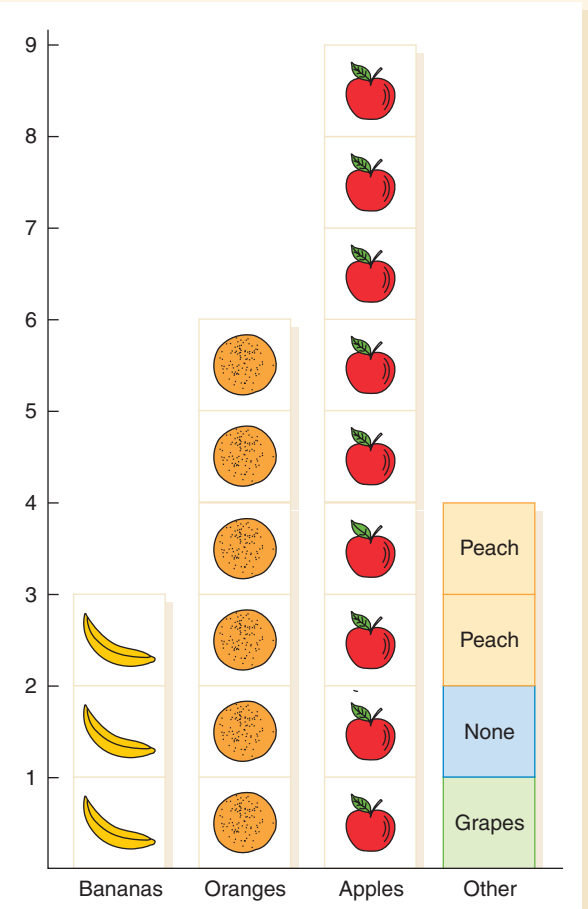
These activities are problem-based in the truest sense. Not only are there no clear answers, but children can easily begin to pose their own questions and explore the numbers and units most interesting to them.

Pearson eText
Video Example 7.4
In this video the teacher engages students in a graphing activity where students represent and interpret data.

Represent and Interpret Data

Graphing activities are good ways to connect young children's worlds with number and relationships. Graphs can be quickly made from any data gathered, such as favorites (ice cream flavors, sports teams, pets), number of sisters and brothers, and transportation modes to school. Once children classify and count the number of objects in each category (a kindergarten standard in the CCSS-M), graphs can be used to share the information. Here are some other options for graphs that are linked to children's literature:

- *Chrysanthemum* (Henkes, 2008): Create a graph of the length of children's first names (i.e., use categories 3, 4, and 5 letters and so on).



Class graph showing fruit brought for snack. Paper cutouts for bananas, oranges, apples, and cards for "others."

- Which snack (or refer to what the graph represents) is most, least?
- Which are more (less) than 7 (or some other number)?
- Which is one less (more) than this snack (or use fruit name)?
- How much more is _____ than _____ ? (Follow this question immediately by reversing the order and asking how much less.)
- How much less is _____ than _____ ? (Reverse this question after receiving an answer.)
- How much difference is there between _____ and _____ ?
- Which two bars together are the same as _____ ?

FIGURE 7.20 Relationships and number sense in a bar graph

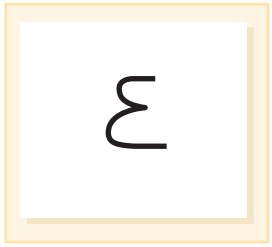
- *This Is the Way We Go to School* (Baer, 1992): Make a graph of the way children come to school.
- *We're Going on a Leaf Hunt* (Metzger, 2008): Have children collect a leaf from around their home or around the school, create a graph of different types of leaves.
- *3 Little Firefighters* (Murphy, 2003) or *When I Grow Up* (Ryant, 2019): Use children's career aspirations to make a graph (i.e., sort jobs into public service, entertainment, education, etc.).

In the early stages of number development, the use of graphs is primarily for developing number relationships and for connecting numbers to real quantities in the children's environment. The graphs focus attention on tallies and counts of realistic things. Once a simple bar graph is made, it is very important to take the time to ask questions (e.g., "What do you notice about our class and our ice cream choices?"). Equally important, graphs clearly exhibit comparisons between numbers that are rarely made when only one number or quantity is considered at a time. See Figure 7.20 for an example of a graph and corresponding questions. At first, children may be challenged with questions involving differences, but these comparison concepts add considerably to children's understanding of number.

Common Challenges Related to Early Number Concepts from Research

Starting here and throughout Part II of this book, you will find that each chapter contains a table of common error patterns identified from research that you can look for in your students' work or listen for in their thinking (see Table 7.4). Students may acquire these beginning conceptual understandings, or bugs, in how they perform a procedure in a variety of ways. Using a constructivist approach, we identify these systematic patterns as an anticipatory framework to support your knowledge of the potential ideas students might exhibit in your classroom. Then we offer ways to develop next instructional steps to advance the continuous development of students' prior knowledge to new knowledge through discussion and additional experiences.

TABLE 7.4 COMMON CHALLENGES RELATED TO EARLY NUMBER CONCEPTS

Common Error Pattern	What It Looks Like	How to Help
1. The child does not yet know the counting sequence.	The child might skip numbers or go out of order, saying, "one, two, four, nine, three, five." Child may be able to count orally to a certain point (e.g., from 1 to 10) but then uses student-generated number words (e.g., ten, one teen, two-teen, three-teen,...).	• Use a puppet that makes a variety of counting errors, having students correct the mistakes made. • Practice counting together out loud (forward and backward). • Read counting books. • Prepare cards with numerals and have children place them in order on a number line (see Activity 7.8). • Match the written numeral with the written number word.
2. The child counts without using a one-to-one correspondence.	Child does not attach one number word to each object. There is either a double count, saying two number words for one object, or touching more than one object as the number word is stated. For example, they may touch two objects as they say two syllable counting words such as "se-ven" or "thir-teen."	• As the child counts objects, have them place one object in each space of an egg carton or ten-frame. • Make a plan for counting: Arrange objects in a row, count objects from left to right, touch one object and say each number word out loud, move each object as it is counted across a line on a work mat or placed into a bag or box. • Have children use a pointer to touch each object as they count.
3. The child does not yet count on.	Child counts out one set and then, when given more objects, recounts the first set, starting back at one. The child, when counting on 3 more from 7, counts 1, 2, 3... rather than 8, 9, 10.	• After child counts out one set and states how many, cover the collection with a sheet of paper or put the collection in a cup. The idea is to remove the objects from sight, forcing the child to create a mental image. Let the child peek at the hidden collection if needed, but encourage the child to think about how many before peeking. Use Activities 7.9 or 7.10. • Use quick images to work on child's subitizing skills such as in Activities 7.1, 7.18 or 7.19.
4. The child is distracted by perceptual cues such as spacing or size of counters.	Child uses a visual cue and thinks that a smaller set of larger counters is greater than a larger set of smaller counters. Given two rows of counters with the same amount, the child thinks the top row has more circles because the row is longer. ● ● ● ● ● ○ ○ ○ ○ ○	• Students lack the ability to conserve, which develops through more experiences and activities with counting. Pose situations that ask students to use one-to-one correspondence to prove that the two amounts are equal. • Use matching to compare sets. For example, stack counters on top of images to match the sets.
5. Child does not yet grasp the cardinality principle.	When asked "How many?" the child recounts all of the objects or points to the last object counted but does not connect the last number to the number of objects.	• Play board games that have a linear path and instead of moving one marker along the path, leave one counter in each space. For example, if the child rolls a 4 on the die, they place 4 counters, one in each space on the board. Ask, "How many spaces did you travel?" • Provide lots of counting opportunities, followed by asking "How many?" • When counting collections together, say, "1, 2, 3, 4, 5. We have 5 pencils."
6. Child has difficulty counting the teen numbers or decade numbers.	The child may say one-teen, two-teen and so on The child may say twenty-ten or forty-ten	• Play numerous games with counting or count as children line up to leave the room. Consistent practice will provide familiarity with the teen numbers. • Play a game that crosses the decade, where one child in a pair has the numbers ending in 9 up to 100 such as 29, and the other has the decade number such as 30. One student holds up a card the other must find in their set of cards, either the number that comes before or after.
7. Child writes a numeral backward or reverses the digits in the teen numbers.	The child hears nineteen and writes 91. Writes the following for 3. 	• Use a vertical number line to show the pattern in writing the teen numbers • Give children a sheet of numerals and ask them to circle the numerals that are not written correctly. It is important for students to see counterexamples of numerals written backwards to sort between the visual appearance of correctly written numerals and incorrectly written ones.

Common Error Pattern	What It Looks Like	How to Help
8. Child is not yet sure of the magnitude of numbers from 1–20.	Student consistently is unsure or incorrect when asked to show which number is greater than or less than another.	• Use a walk-on number line to have the students count the number of units (placing down cardstock pieces of the length as the unit) from the start (or zero) of the number line. Show how the lengths of the two rows of units compare.

RESOURCES FOR CHAPTER 7

WRITING TO LEARN

Assess your understanding and application of chapter content by responding in writing to the following:

1. What must a child be able to do to accurately count a set of objects?
2. What are three types of relationships for numbers 1 through 10? Explain briefly what each means, and suggest an activity for each.
3. How can you assess part-whole number relationships?
4. For numbers between 10 and 20, describe how to develop each of these ideas:
   a. The idea of the teens as a set of ten plus some more
   b. Extension of the one-more/one-fewer concept to the teens and beyond

LITERATURE CONNECTIONS

Children's literature abounds with wonderful counting books and visually stimulating number-related picture books. Research demonstrates that using picture books to teach mathematics helps children develop conceptual understanding and flexibility (Hassinger-Das et al., 2015). Have children talk about the mathematics situations they find in the story. Begin by talking about the book's birthday (copyright date) and how old the book is. Here are a few ideas for making literature connections to number concepts for early learners.

One Fox: A Counting Book Thriller

Read (2019)

This award-winning counting book starts with 1 famished fox and ends with a surprise. With wonderful illustrations and humor, children will count their way through the pages.

The Ultimate Animal Counting Book

Cossins (2020)

Go on an adventure through accurately illustrated pages of sets of animals with numbers extending to 100. Facts about the different species will be learned along the way.

Pete the Cat and His Four Groovy Buttons

Litwin (2012)

Pete is one cool cat who never gets flustered. He starts with four groovy buttons on his shirt, but one by one, they pop off and he ends up with zero—or does he? As he counts down, the numeral is shown, along with the written word and the recording of a related subtraction equation. The story is a perfect lead-in to counting objects and counting backward. Using a five-frame with buttons can help children model Pete's situation.

RECOMMENDED READINGS

Articles

Dyson, N., Jordan, N., & Hassinger-Das, B. (2015). The story of Kyle. *Teaching Children Mathematics*, *21*(6), 354–361.

This useful article presents evidence-based interventions that support children's number sense. Suggestions are shared for interventions developing counting and cardinality as well as for using contexts for thinking about operations.

Books

Karp, K., & Dougherty, B. (2019). E*ssential understanding of number and numeration in practice, grades K-2.* Reston, VA: NCTM.

This book describes the big mathematical ideas a teacher needs to know about number and how to teach and assess this pivotal topic in a child's development of number sense.

Richardson, K. (2003). *Assessing math concepts: The hiding assessment.* Bellingham, WA: Mathematical Perspectives.

This book is part of a series of nine assessment books with diagnostic interviews covering number topics from counting through two-digit numbers. Extensive explanations and examples are provided.

CHAPTER

8

Developing Meanings for the Operations

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **8.1** Develop students' skills in generalizing the problem structures with additive situations involving joining, separating, part-part-whole, and comparison where the unknown can be in any position.
- **8.2** Explain how to teach the meaning of the operations of addition and subtraction.
- **8.3** Develop students' skills in generalizing problem structures with multiplicative situations involving equal groups, comparison, area, and arrays where the unknown can be in any position.
- **8.4** Explain how to teach the meaning of the operations of multiplication and division.
- **8.5** Develop methods for helping students solve word problems.

Helping students learn the different meanings, interpretations, and relationships of the four operations of addition, subtraction, multiplication, and division can assist them in accurately and fluently applying these operations in real-world settings. This is the goal of this chapter, in which students learn to see mathematical situations in their day-to-day lives or in story problems and begin to make models of these situations in mental images, words, pictures, models, and/or numbers (e.g., equations). This reasoning develops *operation sense*.

The operations core builds and expands on the National Research Council's (NRC, 2009) number core and relations core, discussed in Chapter 7, and will be extended in the discussion of place value in Chapter 10. As students learn to connect the big ideas that follow, they can and should simultaneously be developing more sophisticated ideas about number, recognizing ways to think about basic number combinations and accurately and fluently applying these operations in real-world situations.

BIG IDEAS

- Addition and subtraction are connected. Addition names the whole in terms of the parts, and subtraction names a missing part.
- Addition and subtraction have an inverse relationship. Subtraction names a missing part in terms of the known part and the whole.

- Multiplication involves counting groups of equal size and determining how many are in all (multiplicative thinking) or using a reference set as a unit in a multiplicative comparison.
- Multiplication and division have an inverse relationship. Division names a missing factor in terms of the known factor and the product.
- Models can be used to solve contextual problems for all operations and to figure out what operation is involved in a problem regardless of the size of the numbers. Models also can be used to give meaning to number sentences. Representing contextual situations with equations is at the heart of algebraic thinking.

Developing Addition and Subtraction Problem Structures

Learning Outcome 8.1 **Develop students' skills in generalizing the problem structures with additive situations involving joining, separating, part-part-whole, and comparison where the unknown can be in any position.**

Contextual problems are the primary teaching tool to help students activate problem-solving strategies (Jong & Magruder, 2014; Schwartz, 2013) and gain a rich understanding of the operations. Although these are often word-only problems, there is evidence that problems that focus on strong visual supports result in higher student performance (Fuchs et al., 2015; Hoogland et al., 2018).

Table 8.1 provides a learning progression based on Clements et al. (2020, p. 639) that emphasizes addition and subtraction.

What might an effective lesson built around contextual problems look like? The answer comes more easily if you think about students not just solving the problems but also using multiple representations to explain how they solved the problem, justifying why they think they are correct, and describing how their solution pathway makes sense within the context of the situation. If they are recording their ideas on paper, whether a written explanation or a drawing they used to help them solve the problem, it should explain what they did in enough detail to allow someone else to understand it. With the emphasis on students explaining their ideas, strategies, and reasoning, lessons should focus on two or three problems and the related writing of representations and thinking and discussions of those strategies. A useful children's book to read as you embark on a problem-solving emphasis is *How to Solve a Problem: the Rise (and Falls) of a Rock-Climbing Champion* (Shiraishi, 2020). Here they will learn of a girl who sees rocks as problems and understands there are both mistakes and multiple pathways to success. An important connection to what it really means to do mathematics.

TABLE 8.1 ADDITION AND SUBTRACTION LEARNING PROGRESSION

Levels	Description of Problem Structures
Small Number	Sums to 5 with result-unknown problems
Find Result +/−	Sums to 10 with the addition of part-part-whole with whole unknown
Find Change +/−	Addition of change-unknown problems in add-to, take-from, part-part-whole part-unknown, or compare difference unknown
Counting Strategies +/−	All previous using counting strategies
Part-Whole +/−	All previous with the addition of start-unknown problems
Number-in-Numbers +/−	Numbers less than 21 start-unknown problems
Deriver +/−	All types of single-digit problems, using derived strategies (See Chapter 9)
Problem Solver +/−	All types of problem structures for single-digit and multi-digit numbers, using strategies and known combinations.

From an adult's perspective, the ideas of addition and subtraction seem quite simple. For example, think about how you would solve the following problem.

Zoe had 7 beads. After she bought some more beads, she had 15 beads. How many beads did Zoe buy?

Most adults and older students solve this problem by subtracting 15 – 7. But students may not initially view this situation as subtraction because 15 – 7 is the opposite action implied in the problem. Instead, they will model the action in the problem, solving it by adding on or counting up from 7 to 15, which makes sense because it mirrors the situation in the problem.

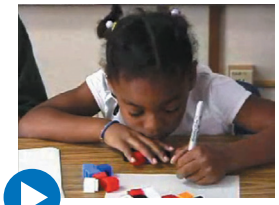
Eventually, after having experience making sense of story problems, working with different combinations of numbers, and examining the results of doing addition and subtraction, students will begin to draw generalizations that you get the same result by subtracting and will eventually use subtraction for these problems. This example illustrates how children's initial conceptions differ from adults—even with something we consider so basic as addition and subtraction. But, these instinctive conceptions from students' thinking about a situation are the best foundation on which to build future learning. Therefore, the perspective on addition and subtraction we take in this chapter is based on what has been learned from numerous research studies about students' understanding of the operations. Through this research, we know students can solve contextual or story problems by imagining the situations and reasoning through the relationships in the problems.

We also know that different problems have different structures that can affect the difficulty level of the problem. When teachers are familiar with these structures, they are better able to plan and differentiate instruction. These structures support the learning of additive situations and don't expire or run out of usefulness, because they are not a reliable or lasting approach (Karp, Bush, & Dougherty, 2014). These same structures support students' thinking about addition and subtraction with multi-digit whole numbers as well as with fractions and decimals.

Researchers have separated addition and subtraction problems into structures based on the kinds of relationships involved (Verschaffel, Greer, & De Corte, 2007). These include *change* problems (*join* and *separate*), *part-part-whole* problems, and *compare* problems (Carpenter, et al., 2017). The structure for each of these three categories of problems is illustrated in Table 8.2.

Standards for Mathematical Practice

MP1. Make sense of problems and persevere in solving them

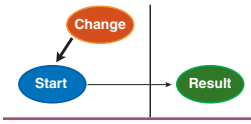
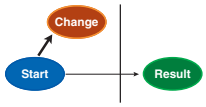


Pearson eText

Video Example 8.1

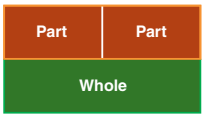
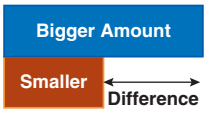
In this video Arriel solves the following problem: Suppose Tricia has 6 marbles. How many more marbles does she need to buy to have 13 all together. Consider the type of problem this is and how Arriel solves the problem.

TABLE 8.2 STRUCTURES FOR ADDITION AND SUBTRACTION STORY PROBLEMS

Problem Type and Structure with Physical Action Involved: Change Problems	Result Unknown	Change Unknown	Start Unknown
Join (add to) **(a)** 	Sandra had 8 pennies. George gave her 4 more. How many pennies does Sandra have altogether? 8 + 4 = ☐	Sandra had 8 pennies. George gave her some more. Now Sandra has 12 pennies. How many did George give her? 8 + ☐ = 12	Sandra had some pennies. George gave her 4 more. Now Sandra has 12 pennies. How many pennies did Sandra have to begin with? ☐ + 4 = 12
Separate (take from) **(b)** 	Sandra had 12 pennies. She gave 4 pennies to George. How many pennies does Sandra have now? 12 – 4 = ☐	Sandra had 12 pennies. She gave some to George. Now she has 8 pennies. How many did she give to George? 12 – ☐ = 8	Sandra had some pennies. She gave 4 to George. Now Sandra has 8 pennies left. How many pennies did Sandra have to begin with? ☐ – 4 = 8

(*Continued*)

TABLE 8.2 ***(Continued)***

Problem Type and Structure with No Physical Action Involved: Part-Part-Whole and Compare Problems			
	Whole Unknown	**One Part Unknown**	**Both Parts Unknown**
Part-Part-Whole **(c)** 	George has 4 pennies and 8 nickels. How many coins does he have? $4 + 8 =$	George has 12 coins. Eight of his coins are pennies, and the rest are nickels. How many nickels does George have? $12 = 4 + \square$ or $12 - 4 = \square$	George has 12 coins. Some are pennies and some are nickels. How many of each coin could he have? $12 = \square + \square$
	Difference Unknown	**Larger Quantity Unknown**	**Smaller Quantity Unknown**
Compare **(d)** 	Situations of How many more?		
	George has 12 pennies, and Sandra has 8 pennies. How many more pennies does George have than Sandra? $8 + \square = 12$	George has 4 more pennies than Sandra. Sandra has 8 pennies. How many pennies does George have? $8 + 4 = \square$	George has 4 more pennies than Sandra. George has 12 pennies. How many pennies does Sandra have? $\square + 4 = 12$
	Situations of How many fewer?		
	George has 12 pennies. Sandra has 8 pennies. How many fewer pennies does Sandra have than George? $12 - 8 = \square$	Sandra has 4 fewer pennies than George. Sandra has 8 pennies. How many pennies does George have? $\square - 4 = 8$	Sandra has 4 fewer pennies than George. George has 12 pennies. How many pennies does Sandra have? $12 - 4 = \square$

Standards for Mathematical Practice
MP7. Look for and make use of structure

Each of the problem structures is illustrated with the story problems that follow using the number family 4, 8, and 12. The story context determines the structure. Depending on which of the three quantities is unknown, a different equation results. Note that the problems are described in terms of their structure and interpretation and not based on what operation you will use to solve for the unknown. Contrary to what you may have thought, a joining action does not always involve addition, nor does separate or remove always use subtraction.

Pearson eText
Video Example 8.2
In this video Jocelyn solves the following problem: Suppose Eva has 6 marbles. How many more marbles does she need to buy to have 13 all together. Consider the type of problem this is and how Jocelyn solves the problem.

Change Problems—Join and Add To. For the action of joining, three quantities are involved: an initial or *start amount*, a *change amount* (the part being added or joined), and the *resulting amount* (the total amount after a change takes place). In Table 8.2(a), this action is illustrated by the change being "added to" the start amount. Provide students with a Join Story Graphic Organizer where they work with counters and model the problem's story situation on the graphic organizer.

Pearson eText Activity Page: Join Story Graphic Organizer

Change Problems—Separate and Take From. *Separate* problems are commonly thought of *take away* or *take from* problems in which part of a quantity is physically being removed from the start amount (see Table 8.1[b]). Notice that in *separate* problems, the start amount is the whole or the largest amount, whereas in the *join* problems, the result is the largest amount (whole). Have students use the Separate Story Graphic Organizer to help them interpret the story.

Pearson eText Activity Page: Separate Story Graphic Organizer

Part-Part-Whole Problems. *Part-part-whole* problems, also known as *put together and take apart* problems, involve two parts that are conceptually or mentally combined into one collection or whole, as in Table 8.1(c). These problems are different from change problems in that there is no action of physically joining or separating the quantities. In these situations, either the missing whole (total unknown), one of the missing parts (one addend unknown), or both parts (two addends unknown) must be determined. There is no meaningful distinction

between the two parts in a part-part-whole situation, so there is no need to have a different problem for each part as the unknown. The third situation in which the whole or total is known and the two parts are unknown creates opportunities to think about all the possible decompositions of the whole and as such instead of having one answer, this situation usually produces a set of correct answers (Caldwell, Kobett, & Karp, 2014; Champagne, Schoen, & Riddell, 2014). This structure links directly to the idea that numbers are embedded in other numbers. For example, students can break apart 7 into 5 and 2, where each of the addends (or parts) is embedded in the 7 (whole). A Part-Part-Whole Story Graphic Organizer can help students represent this type of story (see Activity 8.2).

Pearson eText Activity Page: Part-Part-Whole Story Graphic Organizer

Compare Problems. As the name implies, *compare* problems involve the comparison of two quantities. The third amount in these problems does not actually exist but is the *difference* between the two quantities (see Table 8.1[d]). Looking for a difference and not a resulting quantity makes compare stories challenging for students (Fuson, 2018). Like part-part-whole problems, comparison problems do not typically involve a physical action. The corresponding Compare Story Graphic Organizer can help students model the situation. The unknown quantity in compare problems can be one of three quantities: the smaller amount, the larger amount, or the difference. For each of these situations, two examples are provided: one problem in which the difference is stated in terms of "how many more?" and another in terms of "how many less?" (or "how many fewer?"). Note that initially the language of "more," "less," and "fewer" may require that you present students with additional opportunities to interpret the relationships between the quantities and understand the meaning of these phrases.

Pearson eText Activity Page: Compare Story Graphic Organizer

Pause & Reflect

Revisit the story problems in Table 8.2 and match the numbers in the problems with the components of the structures. Print off the corresponding graphic organizer (or sketch it). Use a set of counters to model (solve) the problem as you think students might do. Write addition and/or subtraction equations that represent the problem as you carried it out with counters. Finally, compare your equation(s) to the ones in Table 8.2. ●

As you look back at the equations for each of the problems, did you notice you wrote some equations where the unknown quantity is not isolated on the right side of the equal sign? For example, the equation for the join problem with start part unknown is $\square + 4 = 12$. This equation is referred to as the *semantic* equation for the problem because the numbers are written in the order that follows the meaning of the story problem (see Table 8.3). When an equation does not isolate the unknown on one side of the equal sign, an equivalent equation can be written for the same problem; in this case, for $\square + 4 = 12$, we can write $12 - 4 = \square$. This equation is referred to as the *computational* form of the equation, which isolates the unknown quantity and would be used if you were to solve this equation with a calculator.

Students typically begin to think about and model the situation based on the sequence played out in the problem, which is modeled more appropriately by the semantic equation.

TABLE 8.3 THE SEMANTIC AND COMPUTATIONAL EQUATIONS FOR THE SIX JOIN AND SEPARATE PROBLEMS

Quantity Unknown	Join Problems	Separate Problems
Result Unknown	$8 + 4 = \square$	$12 - 4 = \square$
Change Unknown	$8 + \square = 12$ (semantic) $12 - 8 = \square$ (computational)	$12 - \square = 8$ (semantic) $12 - 8 = \square$ (computational)
Start Unknown	$\square + 4 = 12$ (semantic) $12 - 4 = \square$ (computational)	$\square - 4 = 8$ (semantic) $4 + 8 = \square$ (computational)

Standards for Mathematical Practice
MP4. Model with mathematics

Help students see how these equations are equivalent and that there are several ways to represent a given situation in an equation. A discussion of how different equations can be written for the same situation is an important opportunity to connect the operations of addition and subtraction and strengthen students' relational understanding.

Presenting the variety of problem types in Table 8.1 supports students in developing schema to structure their thinking and understand the meaning of the operations. Researchers suggest that students (particularly those students with special needs) should be explicitly taught these underlying structures so that they can identify important characteristics of the situations and determine when to add or subtract (Fagnant & Vlassis, 2013; Fuchs, et al., 2021; Yip et al., 2020). Over time, "concreteness fading" (Goldstone & Son, 2005) occurs when students incrementally move from the need to model the physical situation to a more abstract mental representation. When students are exposed to new problems, familiar characteristics will assist them in generalizing from the similar problems when they practiced the actual actions.

Beyond Whole Numbers. These structures for addition and subtraction story problem types apply to *all* addition and subtraction situations. The stories have focused on single-digit examples to focus on the situations (rather than the computation), but the ideas apply to all types of numbers. When you are teaching decimal addition, for example, you will begin with stories, and those stories need to include situations that are join, separate, part-part-whole, and compare, and the missing value in the stories should be in various places (not just the result). The graphic organizers described in this chapter can be adapted to all number types; rather than using counters, students simply write the numbers in the locations they fit.

Problem Difficulty. The structure of some problem types is more difficult than others. For example, problems in which a physical action is taking place, as in join and separate problems with result-unknown or both-parts unknown problems, are easier because students can directly model the physical action or act out the situation (Clements et al., 2020). That is why these types start being presented in kindergarten. However, the join or separate problems in which the start is unknown (e.g., Sandra had some pennies) are often the most difficult (Clements et al., 2020), probably because students attempting to model the problems directly do not know how many counters to put down to begin. Problems in which the change amounts are unknown can also be difficult.

Pearson eText
Video Example 8.3
In this video consider how the teacher engages a student in a diagnostic interview. Note the student has difficulty with the problems. What surprised you in this interview? What might be some next steps in this student's instruction?

Part-part-whole problems may be initially challenging for students for two reasons: (1) there is no action to model because the situation describes a conceptual bringing together of quantities, and (2) it is a challenge for students to grasp that a quantity can represent two things at once. For example, if the problem describes three cars and four trucks in a parking lot and asks how many vehicles are in the lot, students have to understand that the cars and trucks are also part of the larger category of vehicles.

Students often have more experience with the relationships of more and greater than, so you need to ensure that they have opportunities to think about relationships described using the language *fewer* and *less than*. Even so, some students misinterpret the language of "how many more?" as meaning "to add" instead of "find the difference." Note that when the larger amount is unknown, stating the problem by using the term *more* is easier for students because the relationships between the quantities and the operation more readily correspond to each other. In the smaller unknown situation, stating the problem by using the term *fewer* is easier for students for the same reason. Similar to the part-part-whole problems, the lack of a physical action in these situations makes it difficult for students to model or act out these kinds of problems. As a result, many students will solve compare problems as part-part-whole problems without making separate sets of counters for the two amounts. The whole is used as the large amount, one part for the small amount and the second part for the difference.

Teaching Addition and Subtraction

Learning Outcome 8.2 **Explain how to teach the meaning of the operations of addition and subtraction.**

As in other places in the content chapters of the book, we will refer to the approach of considering CSA (concrete, semi-concrete, abstract, discussed in Chapter 6) as we think about instruction. That means starting with concrete experiences. Concrete experiences include stories and

manipulatives. Semi-concrete representations include mathematical representations like bar diagrams and number lines, and abstract representations include equations and mental images.

Connecting the use of representations is important in helping students construct a deep understanding of addition and subtraction. Building the understanding of these operations now will support these operations with larger numbers, fractions, and decimals later on. Note that addition and subtraction are taught at the same time to reinforce their inverse relationship.

Contextual Problems

Beyond varying the structures of stories, there is more to think about when selecting or creating contextual problems. The structures provide the groundwork for more complex problems that incorporate meaningful contexts (Verschaffel et al. 2020). Meyer and colleagues (2001, p. 526–527) suggest several critical attributes of meaningful problem contexts. As you consider evaluating ideas, ask yourself:

Checklist for Evaluating a Problem Context

- Does the **context** focus attention on an important mathematical idea?
- Does the **context** support engagement with the concept(s)?
- Is the **context** one that students can act out or imagine?
- Is the **context** varied but still sensitive to multilingual learners' need for language consistency?
- Does the **context** motivate the students?
- Is the solution realistic for the **context** rather than contrived?
- Is the choice of **context** culturally responsive?
- Does the **context** invite participation from every student?

Pearson eText Teacher Resource: Checklist for Evaluating a Problem Context

Fosnot and Dolk (2001) point out that in story problems, students tend to focus on getting the answer. "Context problems, on the other hand, are connected as closely as possible to students' lives, rather than to 'school mathematics.' They are designed to anticipate and to develop students' mathematical modeling of the real world" (p. 24). They have potential to provide "memorable imagery that can act as a touchstone for teacher and learners in building and discussing abstract concepts" (Gerofsky, 2009, p. 36). Contextual problems might derive from recent experiences in the classroom; a field trip; a discussion in art, science, or social studies; or from children's literature.

Because contextual problems connect to life experiences, they are important for multilingual learners too, even though it may seem that the language complicates the process. Some instructional approaches to support comprehension of contextual problems include using a noun–verb word order, replacing words such as "his/her" and "it" with a name, and removing unnecessary vocabulary words. A visual aid, or actual students modeling the story, would also be effective methods for multilingual learners and students with special needs in math.

Yesterday, we measured how long you were, using cubes. Dion and Rosa asked how many cubes long they are when they lie down head to foot. Rosa was $49\frac{1}{2}$ cubes long, and Dion was 59 cubes long. How long are Rosa and Dion when lying head to foot? (See Figure 8.1.)

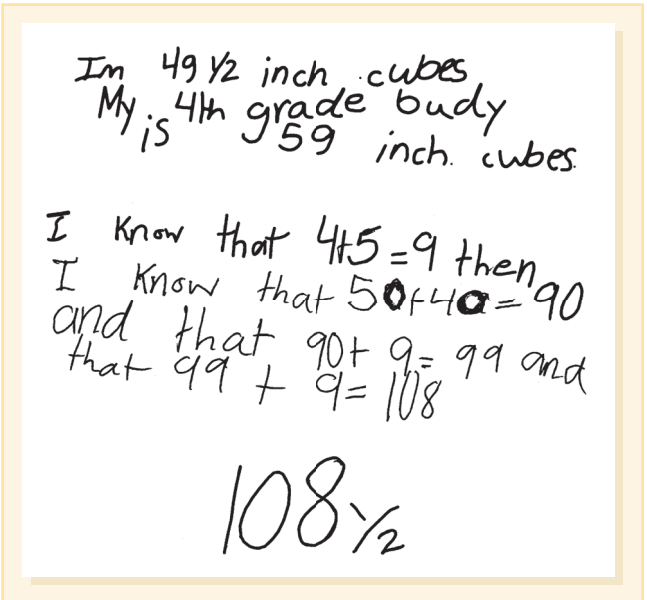


FIGURE 8.1 Student work sample showing the thinking about the combined measurement of Rosa's and Dion's heights

For example, here is a contextual problem written to support comprehension:

What might a before-during-after lesson built around word problems look like? The answer comes more naturally if you think about students not just solving the problems but also how they will use words, pictures, and numbers to explain how they solved the problems and justify why they are correct. In a single lesson, focus on a few problems with in-depth discussions rather than multiple problems with little elaboration. In the before phase, the context is discussed in an authentic way, making sense of the situation (but not discussing how to solve the problem). In the during phase, students select and use their choice of physical materials, role play, or drawings to help them solve the problem. Whatever they record should explain what they did well enough to allow someone else to understand their thinking. In the after phase, students have the opportunity to learn from their peers by listening to and seeing how other students solved the problem(s).

Choosing Numbers for Word Problems. The structure of the problem you use will change the challenge, but you can also vary the problem's complexity by the numbers you use. In general, the numbers in the problems should align with the students' number development. If at first a student struggles with a problem, use smaller numbers to see if it is the size of the numbers causing the obstacle. You can also intensify the challenge by increasing the size of the numbers. One option is to provide number choice for students (Land et al., 2014):

> Devon made _____ cookies on Saturday. He needs _____ cookies for a school party. How many more does he need to make?
>
> (12, 20) (35, 50) (28, 80) (36, 75)

For more ideas on using Choice Tasks, see p. 76 in Chapter 4.

Also use word problems as an opportunity for students to learn about number and computation simultaneously. For example, a problem involving the combination of 30 and 42 has the potential to help first- and second-graders focus on sets of 10. As they decompose 42 into 40 and 2, it is not at all unreasonable to think that they will add 30 and 40 and then add 2 more. The structure of a word problem can strongly influence the type of strategy a student uses to solve multi-digit problems. This thinking is especially true for students who have not been taught the standard algorithms for addition and subtraction, which are not usually required until third or fourth grade.

Introducing Symbols. Preschoolers listen and respond orally to addition and subtraction situations, with no need for the symbols, however, by kindergarten and first grade the symbolic conventions of + and – are expected. Therefore, whenever students are engaged in solving story problems, introduce symbols as a way to record what they did as they share their thinking. Say, "You had the whole number of 12 in your problem, and the number 8 was one of the parts of 12. You found out that the unknown part was 4. Here is a way we write that: $12 - 8 = 4$." The minus sign should be read as "minus" or "subtract" but *not* as "take away." The addition sign is easier because it is typically a substitute for "and." Some students may describe a counting-up strategy to find the solution of 4, so also record the equivalent equation as $8 + 4 = 12$.

Some care should be taken with the big idea of the equal sign, because it is a relational symbol meaning "is the same as" and is not an operations symbol. However, many students think of it as a symbol that signals "give me the answer" or that the "answer is coming up next," which is an obstacle to understanding equations (Byrd, McNeil, Chesney, & Matthews, 2015).

An equation such as $4 + 8 = 3 + 9$ has no answer and is still true because both sides represent the same quantity. As you record and read the equal sign in equations, use the phrase "is the same as" in place of or in conjunction with "equals." Avoid reading the equation as $2 + 2$ "makes" 4. Using a variety of *nonstandard equations* such as $9 = 5 + 4$, $6 = 6$, and $3 + 3 = 2 + 4$ supports the relational thinking that can help students understand the meaning of the equal sign (Knuth & Stephens, 2016; Powell, Kearns, & Driver, 2016) and develop their algebraic thinking. This knowledge is expected in first grade. In fact, students' knowledge of the equal sign in the second grade predicts their equation-solving and algebraic-thinking skills in the fourth grade more than IQ, attentiveness, SES, or whether they were suggested for tutoring (Matthews & Fuchs, 2020) do. In plain language, learning the meaning of the equal sign is essential.

Another approach is to think of the equal sign as a balance; whatever is on one side of the equation "balances" or equals the other side (see Chapter 13 for additional details).

This concept of the equal sign will support algebraic thinking in future grades if developed early (Knuth, Stephens, McNeil, & Alibali, 2006; Powell, Kerns, & Driver, 2016). However, it is important for students in all grades beyond first grade to be assessed for their knowledge of the meaning of the equal sign; teachers have been shown to overestimate their students' understanding of this symbol (Miller-Bennett, 2017).

FORMATIVE ASSESSMENT Notes. Observing how students solve story problems will provide information about their understanding of number, strategies they may be using to answer basic facts, and methods they are using for multi-digit computation. Look beyond the answers they get. For example, a student who uses counters and counts out each addend and then recounts the entire set (this approach is called *count all*) needs to develop more sophisticated strategies. With more practice, they will count on from the first set then later count on from the larger set; that is, for $4 + 7$, the student will begin with 7 and count on. Eventually, students use facts retrieved from memory, and they use counters only when necessary. Observations of students solving problems can inform which numbers to use in problems and what questions to ask to focus students' attention on more efficient strategies. ■

Representing Addition and Subtraction: Concrete Representations

Concrete materials are important to support students' thinking about problem situations. When the parts of a set are known, addition is used to name the whole in terms of the parts. This simple definition of addition serves both *action situations* (join and separate) and static or *no-action situations* (part-part-whole). Subtraction should be learned at the same time as addition and it is the case when the whole is known, one of the parts is known, but one of the parts is unknown. Addition also includes naming both parts (addends) when the sum or whole is known. Activity 8.1 explores the both-addends-unknown problem situation using manipulatives to work out the options.

Activity 8.1 Guinea Pigs in Cages

CCSS-M: K.OA.A.3; 1.OA.A.1; 1.OA.B.4

Use the *Guinea Pigs Add Up* (Cuyler, 2010) book to think about a changing population of class pets—guinea pigs! There are many options for using this story, such as starting with two addends unknown: "How many ways can you place 10 guinea pigs in two cages?" Use the Activity Page Both Addends Unknown. Another idea is to use the book as a context for missing-part thinking for subtraction. Use the Guinea Pig Problems Activity Page to pose questions to the students about the numbers of pigs in the cage, numbers adopted, and how many are left. Students can use the cage mat and the little guinea pig counters to model the problem. Encourage students to pose their own questions and record the corresponding number sentences.

Pearson eText Activity Page: Both Addends Unknown
Pearson eText Activity Page: Guinea Pig Problems
Pearson eText Activity Page: Cage Mat
Pearson eText Activity Page: Guinea Pig Counters

Each part-part-whole model shown in Figure 8.2 represents $5 + 3 = 8$. Some of these are the result of a definite put-together or joining action, and some are not. Notice that in every example, both parts are distinct, even after the parts are combined. For students to see the relationship between the two parts and the whole, the image of the five and three are initially retained as two separate and distinct sets. Therefore, the two parts should be in different piles, on different sections of a mat, or (in the case of counters) in different colors. This structure helps students reflect on the action after it has occurred. "These red chips are the ones I started with. Then I added these three blue chips, and now I have eight chips altogether."

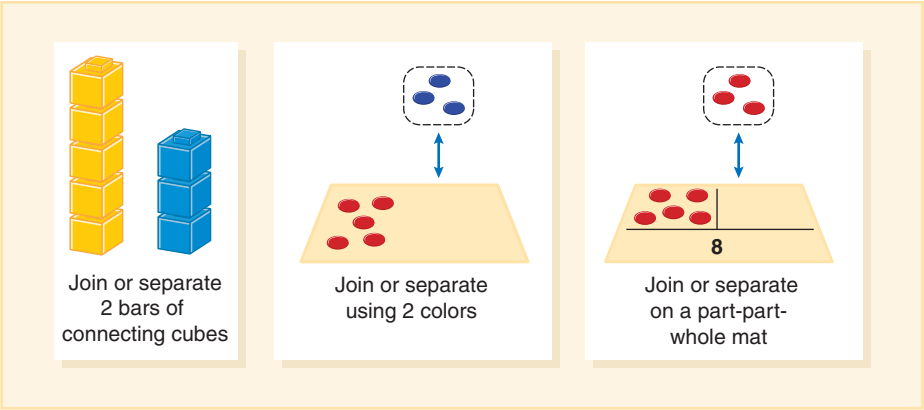


FIGURE 8.2 Part-part-whole models for $5 + 3 = 8$ and $8 - 3 = 5$

As stated earlier, in a part-part-whole model, when the whole and one of the parts are known, subtraction can be used to name the missing part. If you start with a whole set of 8 and remove a set of 3, the 2 sets that you know are the sets of 8 and 3. The expression $8 - 3$, read "eight minus three," names the set of 5 that remains. (Note that we didn't say "take away," because not all subtraction problems are a take-away situation.) Notice that the addition models in Figure 8.2 are for subtraction (except for the action). Helping students see that they are using the same models or pictures connects the two operations through their inverse relationship. Unfortunately, many times this relationship is presented solely through the use of fact families, and often, students just observe that they are using the same numbers and do not see the central structure of part + part = whole and whole − part = part (Ding, 2016). There is also a need to point explicitly to the usefulness of using the inverse operation to check answers, as in $61 - 43 = 18$, because $43 + 18 = 61$ (Ding & Auxter, 2017). The next Activity engages students in thinking about missing parts.

Pearson eText
Video Example 8.4
This video explores the concept of equality and common challenges students face in interpreting the equal sign.
https://www.youtube.com/watch?v=eO3OQI9Jwts

Activity 8.2 Missing-Part Subtraction

CCSS-M: K.OA.A.2; 1.OA.A.1; 1.OA.A.2; 2.OA.A.1

Use a context or story about hiding, as in the lift-the-flap book *Who's Hiding on the Farm?* (McEwen, 2019), where animals are concealed in various locations. Have students model the animals by using a fixed number of tiles placed on a mat. One student separates the tiles into two parts while another covers their eyes. The first student covers one of the two parts with a sheet of paper (see Figure 8.3[b]) or uses the part-part-whole graphic organizer. The second student says the subtraction sentence, for example, "Nine minus four [the visible part] is five [the hidden part]." The hidden part can be revealed for the student to self-check. Have students record both the subtraction equation and the addition equation. While multilingual learners will benefit from prompts such as "____ minus ____ equals ____," all students may find it useful to structure their response with a sentence frame.

Pearson eText Activity Page: Part-Part-Whole Story Graphic Organizer

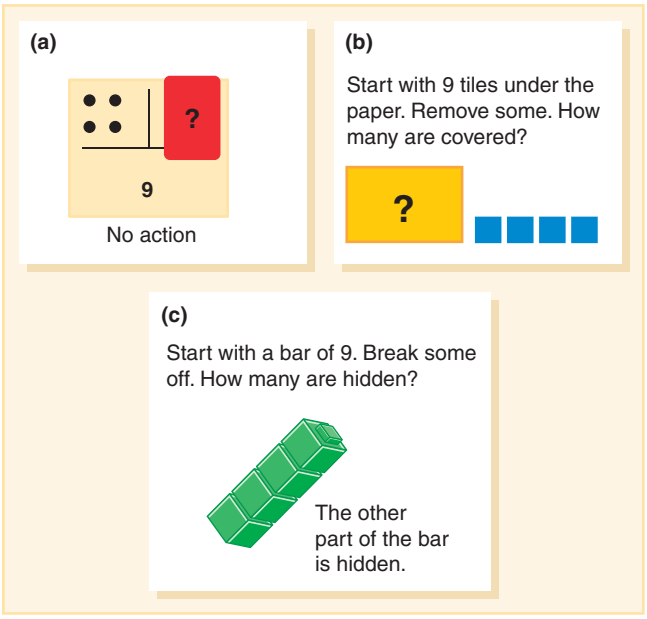

(a)
?
9
No action
(b)
Start with 9 tiles under the paper. Remove some. How many are covered?
?
(c)
Start with a bar of 9. Break some off. How many are hidden?
The other part of the bar is hidden.

FIGURE 8.3 Concrete models for $9 - 4$ as a missing-part problem

Comparison situations involve two distinct sets or quantities and the difference between them. Several ways of modeling the difference relationship are shown in Figure 8.4. The same model can be used whether the difference or one of the two quantities is unknown.

Have students make two amounts (use two bars of connecting cubes) to show how many pencils are in their backpacks. Discuss the difference between the two bars to generate the third number (or the difference as shown in Figure 8.4). For example, if the students make a bar of 8 and a bar of 5, ask, "How many more do we need to match the bar of ?" The unmatched cubes in the larger bar show that the difference is 3. "What equations can we make with these three numbers?" Have students create other story problems that involve the two amounts of 8 and 5. Discuss which equations go with the problems that are created.

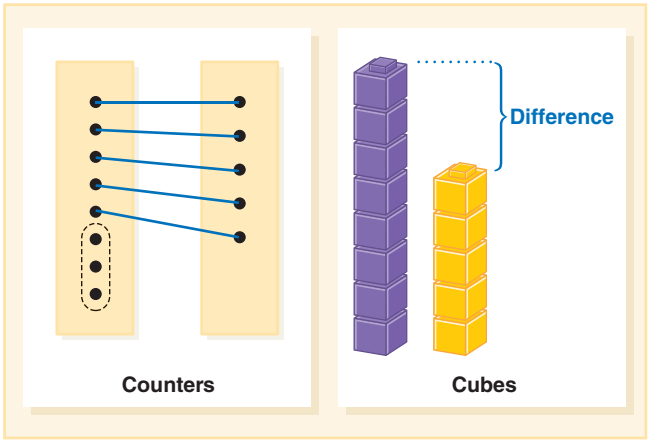


FIGURE 8.4 Concrete models for the difference between 8 and 5

Representing Addition and Subtraction: Semi-Concrete Representations

There are a variety of semi-concrete representations that are very useful in helping students represent their thinking, in particular bar diagrams and number lines.

Bar Diagrams. The use of *bar diagrams* (also called *strip* or *tape diagrams*) as semi-concrete visual representations is a central fixture in many curricula. These diagrams support students' mathematical thinking by generating "meaning-making space" (Murata, 2008, p. 399) and significantly enhance students' performance (Osman et al., 2018). Murata states, "Tape diagrams are designed to bring forward the relational meanings of the quantities in a problem by showing the connections in context" (p. 396). Here is an example problem to consider. Notice how the corresponding bar diagram (see Figure 8.5) visually connects to the part-part-whole representation students have been using since kindergarten. See Table 8.2c, 8.2d, and Figures 8.2 and 8.3 for the part-part-whole mats with concrete materials that connect to the use of this semi-concrete bar diagram.

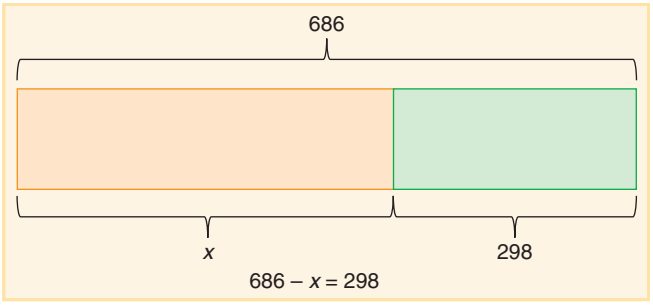


FIGURE 8.5 A bar diagram that supports students' thinking about the problem

Peyton had 686 boxes of Girl Scout cookies to sell. She sold some of them. If 298 boxes were left over, how many boxes of cookies did she sell?

You can find online bar diagram tools called Thinking Blocks at Math Playground's website and as a free app in the iTunes Store. Thinking Blocks can be used to help students model and solve contextual problems involving operations with whole numbers and fractions.

Number Lines. A *number line* is an essential model (Fuchs et al., 2021). Experiences with using number lines are essential because they will be used with third graders to locate fractions and add and subtract time intervals, fourth graders will locate decimals and use them for measurement, and fifth graders will use perpendicular number lines in coordinate grids. At third grade and beyond, students use vertical number lines to read liquid measures and temperatures.

Initially, the number line may present conceptual difficulties for some students and students with special needs in math (National Research Council Committee, 2009). This beginning challenge is partially due to a difficulty in seeing the unit in a continuous line when they are accustomed to counting individual items and even moving the items aside as they count them. This shift from counting a number of individual objects in a collection to counting continuous-length units needs to be explicitly connected.

A *number path* (Figure 8.6) is the first step in a progression of number lines across the grades to make sense of the representation (blending the idea of countable units to a length interpretation). Here the numbers are written on the unit so students are focused on precisely what is being counted:

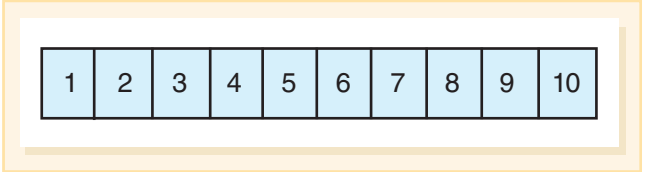


FIGURE 8.6 Number path

Activity 8.3 Bears' Picnic

CCSS-M: K.CC.A.1; K.CC.A.2; 1.OA.A.1

This is a small group activity that can be used for join and separate story situations as bears are coming and going from a picnic. The picnic stories in the Bears' Picnic Activity Page focus on one more and one less, but it can be adapted to any values. Distribute the Activity Page and teddy bear counters (or other counters). Tell students to use either the ten-frame (picnic table) or the number path to solve each problem. Invite students to tell or write their own bear picnic stories.

Pearson eText Activity Page: Bears' Picnic

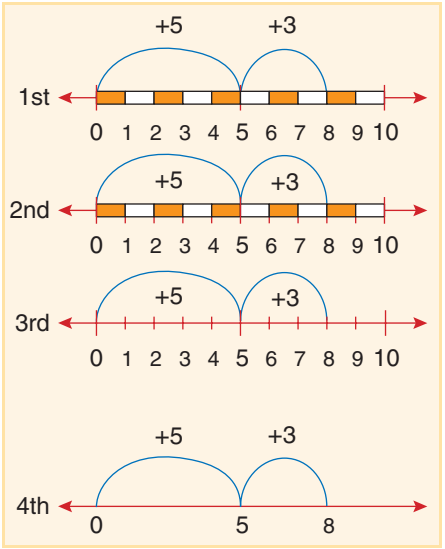


FIGURE 8.7 Sequence of number lines

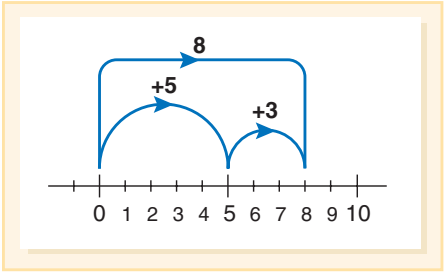


FIGURE 8.8 Two lengths, one of 5, the other of 3, combine for a length of 8

A number line differs from a number path in that the numbers represent a length or distance marked at the end with a unit. Figure 8.7 shows ways to support learners as you introduce and model number lines after students have experience with the number path.

Here you see the number line (Figure 8.8), and like other representations, this model develops from more concrete experiences in kindergarten, when students walked on the number path to this semi-concrete version.

A number line measures distances from zero the same way a ruler does. If you don't emphasize the unit (length), students may focus on the tick marks or numerals instead of the spaces (a misunderstanding that becomes apparent when their answers are consistently off by one). At first, students can build number paths by using a given length, such as a set of Cuisenaire rods of the same color. This path will show that each length unit is "one unit," and that same unit is repeated over and over (iterated) to form the number line (Karp & Dougherty, 2019). Also, playing board games with number paths helps students develop a better concept of number magnitude and estimate more accurately on a number line (Siegler & Ramani, 2009). In each version of the number line sequence, arrows (lengths) are drawn for each number in an expression to illustrate the unit length concept more clearly. To model the part-part-whole concept of $5 + 3$, start by drawing an arrow from 0 to 5, indicating "This much is five." Do *not* point to the tick mark for 5, saying, "This is five." Then show the three lengths and count "six, seven, eight" (not "one, two, three") to demonstrate the counting-on model and reinforce the mental process. Eventually, the use of a ruler or a scale in a bar graph or coordinate grid will build upon this model.

There's an interactive number line at the Math Learning Center https://apps.mathlearningcenter.org/number-line/ that helps students visualize number sequences and model strategies for addition, subtraction, multiplication, and division. It can be used to represent sequences of numbers, including whole numbers and multiples of a variety of numbers.

When looking at the comparison situation from a semi-concrete perspective on the number line, note that it is often not immediately clear to

Activity 8.4 Up and Down the Number Line

CCSS-M: K.OA.A.2; 1.OA.A.1; 1.OA.A.2; 2.OA.A.1

Create a large number line on the floor of your classroom, or display one in the front of the room. (Make sure your line starts with a tick mark at zero and has arrows at each end of the line.) Use a stuffed animal for moving units or, if the number line is large enough, ask a student to walk the number line. Pose a variety of problem situations and talk about the movement required for each. (Use Table 8.2 to help you vary your story types.) Start with a context that requires moving a distance, such as the frog moving away from the lily pad, to emphasize the spaces (units of length) on the number line. This activity supports students' development of a mental image for thinking about the meaning of addition and subtraction.

students how to associate either addition or subtraction with a comparison situation (see Figure 8.9). From an adult vantage point, you can see that if you match part of the larger amount with the smaller amount, the large set can now be thought of as the whole in a part-part-whole model. In fact, many students do model compare problems in just this manner. But you need to provide experiences for students to discuss the idea of identifying how many more or finding the difference so they can begin to construct this structure themselves.

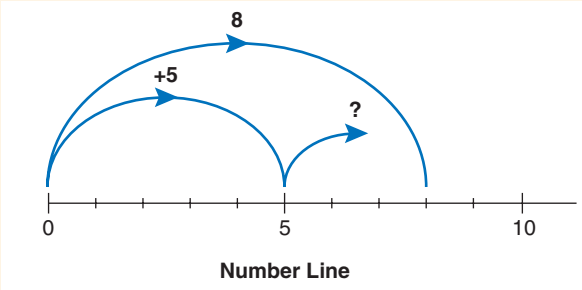


FIGURE 8.9 Compare situation represented on the number line

Representing Addition and Subtraction: Abstract Representations

Students should be translating the equations they see into concrete and semi-concrete representations while also writing equations to match the situations they are modeling. Then they will engage in thinking about problems with mental images and reasoning while responding with more symbolic approaches, as in Activity 8.5.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Activity 8.5 More Than Two Addends

CCSS-M: 1.OA.B.2

Give students six sums to find involving three or four addends, as in this More Than Two Addends Activity Page. Within each problem, include at least one pair with a sum of 10 or perhaps a double: $4 + 7 + 6$, $5 + 9 + 9$, or $3 + 4 + 3 + 7$. Students should show how they added the numbers (see Figure 8.10). For students with special needs in math, you may need to support them initially in their decision making, suggesting that they look for a 10 or a double and have them underline or circle those numbers as a starting point.

Pearson eText Activity Page: More Than Two Addends

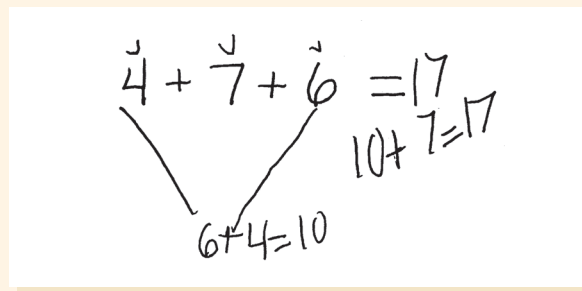


FIGURE 8.10 A student using check marks to keep track of numbers added

As students share their solutions, some will have added using different pairs and in a different order but got the same result. From this discussion, you can help them conclude that they can add numbers in any order by using both the associative property and the commutative property. (Please see a full exploration of the properties of all of the whole-number operations in Chapter 13.) Learning to adjust strategies to fit the numbers puts learners on the road to computational fluency.

Developing Multiplication and Division Problem Structures

Learning Outcome 8.3 **Develop students' skills in generalizing problem structures with multiplicative situations involving equal groups, comparison, area, and arrays where the unknown can be in any position.**

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

A key component of the third-grade curriculum is to help students develop *operation sense* with respect to multiplication and division. This sense-making means facilitating students connecting the different meanings of multiplication and division to each other, as well as connecting

to addition and subtraction. Operation sense supports students' effective application of these operations in real-world settings.

As with addition and subtraction, there are problem structures that will help you in formulating and discussing multiplication and division tasks. These structures will also benefit your students in generalizing as a way to identify familiar situations. And, as with addition and subtraction, these structures apply to all numbers, multi-digit whole numbers, fractions, and decimals. The examples in this section use single-digit numbers simply to focus on the structures.

Most researchers identify four classes of multiplicative structures: equal groups, comparison, area (some standards include arrays with area), and combinations (see Table 8.4). Arrays begin to be considered in second grade with addition situations with equal groups, and equal-group and area problems (as a component of measurement) are introduced in the third grade. Multiplicative comparison models are a focus in the fourth grade, and problems with combinations are found in seventh grade when students explore probability and identify all of the possible outcomes in situations.

Equal-Group Problems

Equal-group problems involve three quantities: the number of groups (sets or parts of equal size), the size of each group (set or part), and the total of all the groups (whole or product). The parts and wholes terminology is useful in making the connection to addition, and is why equal groups is the way multiplication and division are introduced. When the number of groups and size of groups are known, the problem is a *multiplication* situation with an unknown product (How many in all?). When either the group size is unknown (How many in each group?) or the number of groups is unknown (How many groups?), then the problem is a *division* situation. But note that these division situations are not alike. Problems in which the size of the group is unknown are called fair-sharing or *partition division* problems. The whole is shared or distributed among a known number of groups to determine the size of each group. If the number of groups is unknown but the size of the equal group is known, the problems are called *measurement division* or sometimes *repeated-subtraction* problems. The whole is measured off in groups of the given size. Use the illustration in Table 8.4(a) as a reference.

Pearson eText
Video Example 8.5
In this video Andrew solves the following problem: Suppose 20 children are driven to the park. If each car only had 4 seatbelts, how many cars will be needed to drive all 20 children to Coronado Park. Consider the type of division problem and how Andrew solves the problem.

If you choose to create your own problems, keep in mind this suggestion from Jong and Magruder (2014): provide clarity so students know exactly who is involved in the sharing. The problem should plainly answer this question: Who is sharing the items? Is the person doing the sharing also a part of the share or not?

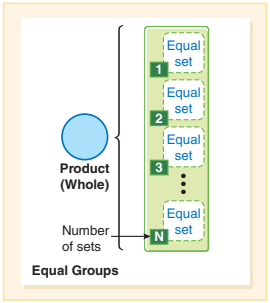
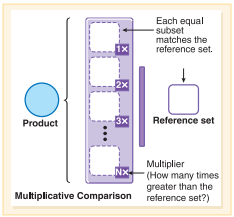
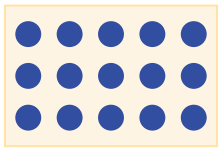
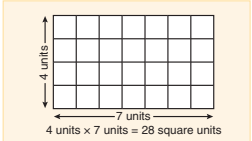
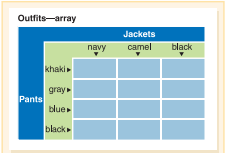
Sometimes equal-group problems are called *repeated-addition* problems, because the equal group is being added over and over. And in fact, multiplication is an efficient way to carry out a repeated-addition situation. This link may be an important initial connection for learners to make, because the multiplication and the repeated addition produce the same results for positive whole numbers. But by the time students are multiplying fractions, this notion falls apart. Additionally, repeated addition is not an efficient way to carry out multiplication—it is just the opposite (try 23×57), so move students away from additive thinking to thinking of a multiplier and equal sets as soon as possible.

Comparison Problems

In multiplicative comparison problems, there are really two sets or groups, as there were with comparison situations for addition and subtraction. In additive situations, the comparison is an amount or quantity difference between the two groups. In multiplicative situations, the comparison is based on one group being a particular multiple of the other (a reference set). With multiplication comparison, there are three possibilities for the unknown: the product, the group size, and the number of groups (see Table 8.4[b]).

When students have experiences considering these possibilities for the unknown, you can discuss how the situations are different, how each is related to multiplication, and how each

TABLE 8.4 STRUCTURES FOR MULTIPLICATION AND DIVISION STORY PROBLEMS

	Unknown Product (multiplication)	Group Size Unknown (partition division)	Number of Groups Unknown (measurement division)
Equal Groups (second set of problems are rate problems)			
(a) Equal Groups 	Mark had 4 bags of apples. There are 6 apples in each bag. How many apples does Mark have altogether? $4 \times 6 = \square$	Mark has 24 apples. He wants to give equal shares of them to his 4 friends. How many apples will each friend receive? $24 \div 4 = \square$	Mark has 24 apples. He put them into bags containing 6 apples each. How many bags did Mark use? $24 \div 6 = \square$
	If apples cost 7 cents each, how much did Jill have to pay for 5 apples? (*rate*) $7 \times 5 = \square$	Jill paid 35 cents for 5 apples. What was the cost of 1 apple? (*rate*) $35 \div 5 = \square$	Jill bought apples at 7 cents each. The total cost of her apples was 35 cents. How many apples did Jill buy? (*rate*) $35 \div 7 = \square$
	Peter walked for 3 hours at a speed of 4 miles per hour. How far did he walk? (*rate*) $3 \times 4 = \square$	Peter walked 12 miles in 3 hours. How many miles per hour (speed) did he walk? (*rate*) $12 \div 3 = \square$	Peter walked 12 miles at a speed of 4 miles per hour. How many hours did it take Peter to walk the 12 miles? (*rate*) $12 \div 4 = \square$
Comparison Problems			
	Product Unknown (multiplication)	Group Size Unknown (partition division)	Multiplier Unknown (measurement division)
(b) 	Jill picked 6 apples. Mark picked 4 times as many apples as Jill. How many apples did Mark pick? $6 \times 4 = \square$	Mark picked 24 apples. He picked 4 times as many apples as Jill. How many apples did Jill pick? $24 \div 4 = \square$	Mark picked 24 apples, and Jill picked only 6. How many times as many apples did Mark pick as Jill did? $24 \div 6 = \square$
	This month, Mark saved 5 times as much money as last month. Last month, he saved $7. How much money did Mark save this month? $5 \times 7 = \square$	This month, Mark saved 5 times as much money as he did last month. If he saved $35 this month, how much did he save last month? $35 \div 5 = \square$	This month, Mark saved $35. Last month, he saved $7. How many times as much money did he save this month as last? $35 \div 7 = \square$
Array and Area Problems			
	Product Unknown	Group Size Unknown	Number of Groups Unknown
(c) Array 	A carton has 3 rows of soup cans with 5 cans in each row. How many soup cans are there? $3 \times 5 = \square$	There are 15 soup cans placed in a carton in 3 equal rows. How many soup cans are in each row? $15 \div 3 = \square$	There are 15 soup cans placed in a carton in equal rows of 5 cans. How many rows are there? $15 \div 5 = \square$
(d) Area 	A garden is 4 feet by 7 feet. What is the area of the garden? $4 \times 7 = \square$ sq. feet	A garden has an area of 28 square feet. If one side is 4 feet long, how long is the side next to it? $28 \div 4 = \square$ feet	A garden has an area of 28 square feet. If one side is 7 feet long, how long is the side next to it? $28 \div 7 = \square$ feet
Combination Problems			
(e) Combinations 	Sam bought 4 pairs of pants and 3 jackets, and they all can be worn together. How many different outfits consisting of a pair of pants and a jacket does Sam have? $4 \times 3 = \square$	The combination structure is rarely used to divide.	

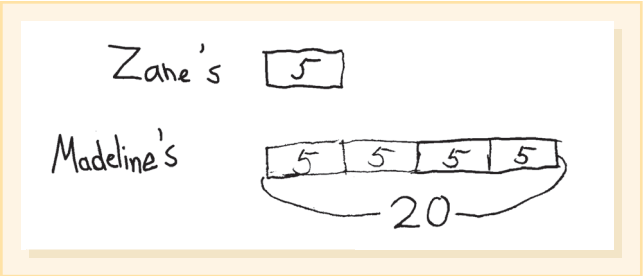


FIGURE 8.11 A student's work shows a model for multiplicative comparisons

is written as a division equation. When modeling multiplicative comparison problems, consider exploring them with a bar diagram. See Figure 8.11 for a bar diagram related to the following situation:

Zane has five small toy cars. Madeline has four times as many cars. How many does Madeline have?

Pause & Reflect

What you just read is complex yet important. Stop now and get a collection of about 35 counters and a set of paper plates to model the equal-groups examples starring Mark. Match the story with the structure model in Table 8.4(a). How are the problems alike and how are the problems different? Repeat for Jill problems.

Repeat the same process with the multiplicative comparison problems, using a colored plate different from your other plates to represent the reference set. Again, start with the first problem in each set and then the second problem. Reflect on how these problems are the same and different. ●

Array and Area Problems

Standards for Mathematical Practice
MP6. Attend to precision

The *array* represents an equal-group situation that can transition to an area model, so arrays and areas are combined as a multiplication structure (Table 8.4[c]). The first factor (the number of groups) represents the number of rows, and the second factor (the number of items in each equal group) represents the equal number found in each row (number of columns). An array can be modeled with counters, dots, or any items, as you can see in the sample problems in Table 8.4(c). Yet, if you begin to use the small square tiles for the array and move the tiles tightly together, the result can be recorded on grid paper and connected more easily to the area problem structure. Using the discrete items first prepares students for the more sophisticated use of continuous units with measuring area.

What distinguishes *area* from the other problem types is that the product is literally a different type of unit from the two factors. In a rectangular shape, the product of two lengths (length × width) is an area, usually measured in square units. Note in Table 8.4(d) how different the square units are from the two factors of length: 4 feet times 7 feet is not 28 feet but 28 square feet. The factors are each one-dimensional entities, but the product consists of two-dimensional units. Area is discussed in Chapter 18 (measurement).

Combination Problems

Combination problems involve counting the number of possible pairings that can be made between two or more sets (things or events). This structure is more complex and therefore not a good introductory point for multiplication, but it is important that you recognize it as another category of multiplicative problem structures. Students often start by using the model shown in Table 8.4(e), where one set is the row (pants) and the other the column (jackets) in a matrix format. Counting how many combinations of two or more things or events are possible is important in determining probabilities, the focus of Chapter 21.

Teaching Multiplication and Division

Learning Outcome 8.4 **Explain how to teach the meaning of the operations of multiplication and division.**

Multiplication and division are often taught separately, with multiplication preceding division. It is important, however, to combine multiplication and division soon after multiplication has been introduced, to help students see how these operations have an inverse relationship. In most curricula, these topics are first presented in second grade and become a major focus in third grade, with continued development in the fourth and fifth grades.

A major conceptual hurdle in working with multiplicative structures is understanding that although a group contains a given number of objects, these groups can also be considered as single entities (Blote, Lieffering, & Ouewhand, 2006; Clark & Kamii, 1996). Students can solve the problem, "How many apples in 4 baskets of 8 apples each?" without thinking multiplicatively simply by counting out 4 sets of 8 counters and then counting all. To think multiplicatively about this problem as *four sets of eight* requires students to conceptualize each group of eight as a single entity to be counted four times. Experiences with making and counting equal groups, especially in contextual situations, are extremely useful.

Contextual Problems. When teaching multiplication and division, it is essential to use interesting contextual problems instead of more sterile story problems (or naked numbers)—and yes, assessments focus on these contextual problems! Consider the following problem.

Yesterday, we discovered that it took 7 yards of paper to make a border for the bulletin board in the school's lobby. There are 25 more bulletin boards of the same size in the school's hallways. How many yards of paper will we need if we make borders for all the bulletin boards in the school hallways?

Recall the earlier discussion of the importance of contexts and note that this problem is based on students' experiences and builds on a familiar situation. When a familiar context is used, students are more likely to demonstrate a spontaneous and meaningful approach to solve the problem, because they are connected to it.

Rather than having students solve many problems in a single lesson with a focus on getting the answers, instead focus on sense-making. Solve a few problems, using multiple representations strategically, such as physical materials, drawings, and equations. Whatever students write on paper, they should provide enough detail for another person to follow their thinking. Leave enough space on an activity page to encourage multiple strategies so that students have a record of their thinking that (1) can be shared with their peers and (2) provide you with insights into their understanding.

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Choosing Numbers for Problems. When selecting numbers for multiplicative story problems or activities, there is a tendency to think that large numbers pose a burden to students or that 3×4 is somehow easier to understand than 4×17. An understanding of products or quotients is not affected by the size of numbers as long as the numbers are within your students' grasp. When given the challenge of using larger numbers, students may invent computational strategies (e.g., four 10s and then four more 7s) or model the problem with manipulatives.

As with addition and subtraction, one option is to provide a number choice task:

> ____ friends were sharing all their Lego pieces. They had a total of ____ pieces. How many pieces will each friend get?
>
> (5, 20) (4, 48) (8, 96) (7, 105)

Introducing Symbols. When students solve simple multiplication story problems before learning about multiplication symbolism, they will most likely begin by writing repeated addition equations to represent what they did. This moment is your opportunity to introduce the multiplication sign and explain what the two factors mean.

Standards for Mathematical Practice
MP6. Attend to precision

The U.S. convention is that 4×8 refers to 4 sets of 8, not 8 sets of 4. There is no reason to be so rigid about this convention that you would mark a student as incorrect (particularly because the convention is just the opposite in some countries where they say 4 taken 8 times). The important thing is that the students can tell you what each factor in their equations represents and that their equations match the context.

The quotient 24 divided by 6 is represented in three ways: $24 \div 6$, $6\overline{)24}$, and $\frac{24}{6}$. Students should understand each of these representations and know that these expressions are equivalent. The fraction notation is underused, yet it becomes important at the middle-school level and can support reasoning strategies for division. Students often mistakenly read $6\overline{)24}$ as "6 divided by 24" due to the left-to-right order of the numerals. Generally, this error does not match what they are thinking.

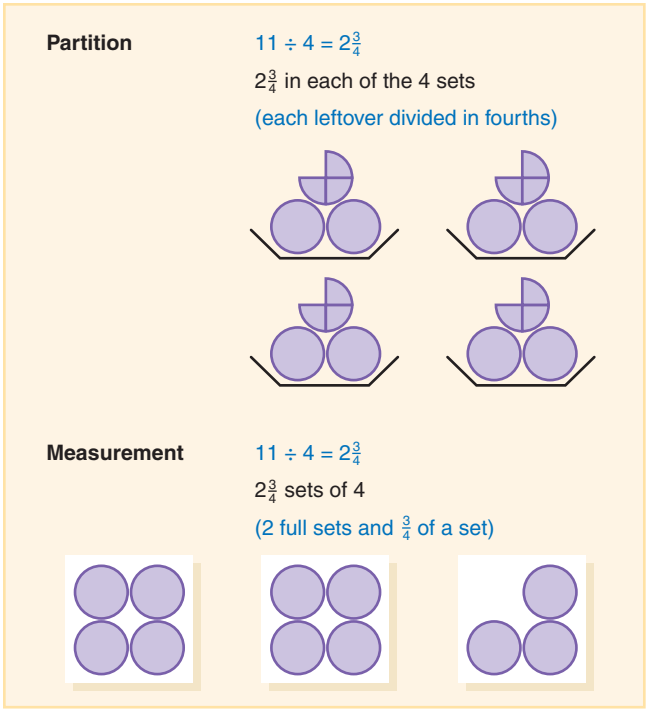

Partition
$11 \div 4 = 2\frac{3}{4}$
$2\frac{3}{4}$ in each of the 4 sets
(each leftover divided in fourths)
Measurement
$11 \div 4 = 2\frac{3}{4}$
$2\frac{3}{4}$ sets of 4
(2 full sets and $\frac{3}{4}$ of a set)

FIGURE 8.12 Remainders expressed as fractions

Compounding the difficulty of division notation is the unfortunate phrase "goes into," as in "6 goes into 24." This phrase carries little meaning about division, especially in connection with a fair-sharing or partitioning context. The *goes into* terminology is simply ingrained in adult parlance; it has not been in textbooks for years. Instead of this phrase, use appropriate terminology with students, such as "How many groups of 6 are in 24?"

Remainders. More often than not in real-world situations, division does not result in a whole number. For example, problems with 6 as a divisor will result in a whole number quotient only one time out of six. Note that it is technically incorrect to write the answer to a problem like $31 \div 4$ as 7 R 3 because this notation is not a number (a quotient should be a number). As written, the 3 is not well defined because it is really $\frac{3}{4}$. However, in the beginning, the form 7 R 3 may be the most appropriate. In the absence of a context, a remainder can be dealt with in only two ways: it can either remain a quantity left over or be partitioned into fractions (including decimal fractions). In Figure 8.12, the problem $11 \div 4$ is modeled to show the remainder as a fraction. Including when there are contexts, there are five ways the remainder can have an effect on the answer (see Table 8.5).

TABLE 8.5 THE EFFECT OF THE REMAINDER ON ANSWERS

Effect	Illustration
Left over	You have 30 strawberries to give in fair shares to 7 friends. How many strawberries will each child receive? Answer: 4 strawberries and 2 strawberries left over
Partitioned as a fraction (included in the quotient)	Each jar holds 8 ounces of liquid. If there are 46 ounces in the pitcher, how many jars will be filled? Answer: 5 jars filled and $\frac{6}{8}$ of another jar
Discarded (leaving a smaller, whole-number answer)	The rope is 25 feet long. How many 7-foot jump ropes can be made? Answer: 3 jump ropes
Forced answer to the next highest whole number	The ferry can hold 8 cars. How many trips will it have to make to carry 25 cars across the river? Answer: 4 trips
Rounded to nearest whole number (approximate result)	Six children are planning to share a bowl of 50 grapes. About how many grapes will each child get? Answer: About 8 grapes for each child

In real contexts, remainders must be interpreted.

Students should not just think of remainders as "R 3" or "left over." Addressing what to do with remainders or a quotient with a fraction must be central to teaching division. In fact, one of the most common errors students make on high-stakes assessments is to divide and then not pay attention to the context when selecting their answer. For example, in the problem, about the ferry, answering with $3\frac{1}{8}$ trips doesn't make any sense. One way to help students attend to the remainder (and in general make sense of their answer) is to remind them of the following categories to their problem solving:

- What I am looking for:
- My work:
- My answer:
- Interpreting my answer:

Pause & Reflect

It is useful for you to make up problems, using different contexts. See if you can come up with division problems whose contexts would result in remainders dealt with as fractional parts of the quotient, forced or rounded up, and rounded down or discarded. ●

Representing Multiplication and Division: Concrete Representations

Students' use of multiple representations is critical to a deep understanding of concepts and better performance in solving mathematics problems. Students have also shifted their thinking that mathematics is just memorizing procedures and fast computation when the focus shifts to connecting representations. This change results in more interested learners who participate more and see mathematics as a process of collaborative inquiry (Ni et al., 2018, p. 713). So, here we explore again the CSA model with an emphasis on multiplication and division.

In the beginning, students will be able to use the same models—sets of counters or cubes, bar diagrams, and number lines—for all four operations. A variety of concrete representations are shown in Figure 8.13.

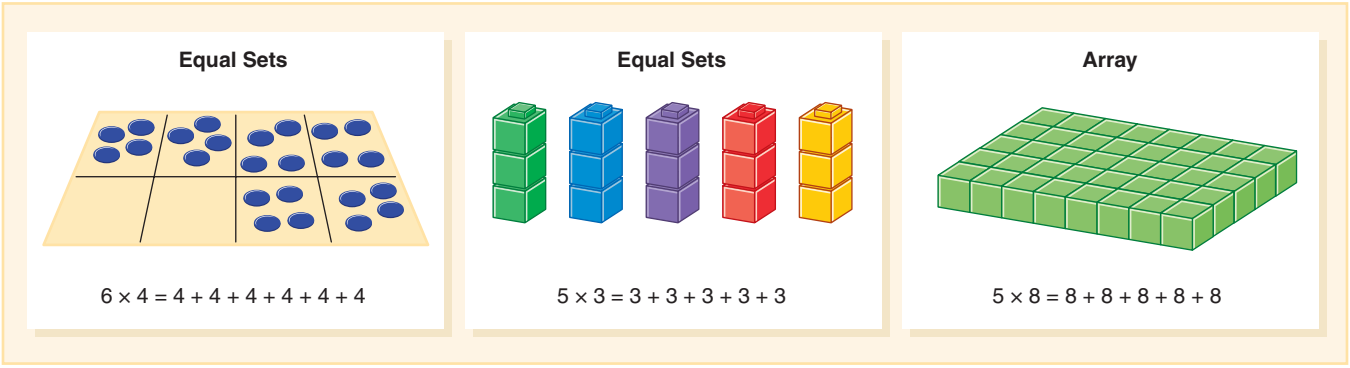


FIGURE 8.13 Concrete models for equal-group multiplication

Counting collections begins in preschool with counting individual objects but extends to multiplication when the collections are in equally grouped packages (Franke, Kazemi, & Turrou, 2018). The next four activities focus on equal groups and can be a full lesson or part of a learning center.

This next activity is based on students' experiences and builds on a familiar context. When a familiar context is used, students are more likely to demonstrate a spontaneous and meaningful approach to solve the problem, because they are connected to it.

Activity 8.6 Boxes and Bags

CCSS-M: 3.OA.A.1; 3.OA.A.3

To prepare for this activity, you will need one box (like a shoe box) for each group and many plastic bags and collections of small objects. Prepare each box with a different number of bags, each filled with the same number of objects. Number each box. For example:

Box 1: Craft pompoms: 12 in a bag, make 5 bags
Box 2: Pencil top erasers: 7 in a bag, make 7 bags
Box 3: Hair ties: 5 to a bag, 8 bags
Box 4: Shells: 9 to a bag, 6 bags

Use whatever inexpensive/available/safe items you can find!

Place students in groups of 2 or 3 and ask them to figure out how many items are in their box. They must figure it out by only opening and counting the items in one bag. After counting, ask students to write how they figured it out and then write an equation to represent the sets in their box on the Boxes and Bags Recording Sheet.

Pearson eText Activity Page: Boxes and Bags Recording Sheet

Activity 8.7 Chocolate Chip Pancakes

CCSS-M: 3.OA.A.1; 3.OA.A.3

This is a partner activity. Partners need a die and a set of paper plates (or pieces of blank paper for a semi-concrete drawing). Younger students can use regular dice, students who would like a challenge can use 10, 12, or 20-sided dice or a deck of number cards. The first partner rolls to figure out how many pancakes they are going to make. This student takes out that many plates or draws the plates on their paper. The partner then rolls the die to find out how many chocolate chips they will put in each pancake. Together they figure out how many chocolate chips they will need for these pancakes. On the Pancake activity page, they can record the totals. With chocolate chip pancake plates prepared, partners can trade plates, work together to figure out how many total chocolate chips, and then add the information to their recording sheet. This activity is ideal for a learning center, or students can use sticky dots and make plates that are reused. Consider other food ideas that reflect what your students like to eat or food from cultures represented in your classroom.

Children's Book Connections: *Pancakes for Breakfast* by Tomie DePaola; *Pancakes, Pancakes!* by Eric Carle; *Curious George Makes Pancakes* by H. A. Rey; *If You Give a Pig a Pancake* by Laura Numeroff; *Rhinos Don't Eat Pancakes* by Anna Kemp; *Pancakes in Pajamas* by Frank Asch.

Pearson eText Activity Page: Chocolate Chip Pancakes

Activity 8.8 It Stacks Up!

CCSS-M: 3.OA.A.1; 3.OA.A.3

Invite students to select an animal and then to identify what that animal eats (a resource is *What Do Animals Eat* by Stones, 2012). With their selection in mind, have them choose one (or for a challenge, assign them all) of the following problems to solve. Distribute centimeter counters and small paper cups, and have students build the food portions and stack them up. Use same-sized sets for a series of stories and then change the size of the set. Example stories are here:

Three _____ came to dinner. Each ate four _____. How many _____ are needed?

Six _____ came to dinner. Each ate four _____. How many _____ are needed?

Nine _____ came to dinner. Each ate four _____. How many _____ are needed?

Four _____ came to dinner. Each ate three _____. How many _____ are needed?

Seven _____ came to dinner. Each ate three _____. How many _____ are needed?

Ten _____ came to dinner. Each ate three _____. How many _____ are needed?

After students have solved the sets of similar problems, ask what patterns they noticed.

Pearson eText Activity Page: It Stacks Up

Activity 8.9 Divide and Conquer

CCSS-M: 3.OA.A.2

Using the context of a story about sharing, such as *Bean Thirteen* (McElligott, 2007), provide students with a supply of counters (beans) and a way to place them into small groups (small paper cups). Have students count out a number of counters for the whole or total amount and record this number. Next, specify either the number of equal groups to be made or the size of the groups: "Separate your counters into four equal-sized groups," or "Make as many groups of four as is possible." Have students write the corresponding multiplication equation for what their materials show; under that, have them write the division equation. For multilingual learners, be sure they know what *groups, equal-sized groups*, and *groups of four* mean. For students with special needs in math, start with a partition approach (i.e., separate your counters into three groups), in which they distribute the counters by placing each one at a time into each cup.

Explore the Expanded Lesson: Divide and Conquer for additional details about this activity.

Pearson eText Expanded Lesson: Divide and Conquer

Representing Multiplication and Division: Semi-concrete Representations

As with additive problems, students benefit from activities with visual models such as number lines, arrays, and bar diagrams, whether in an illustration or hand-drawn, to focus on the meaning of multiplication and division, as in Figure 8.14.

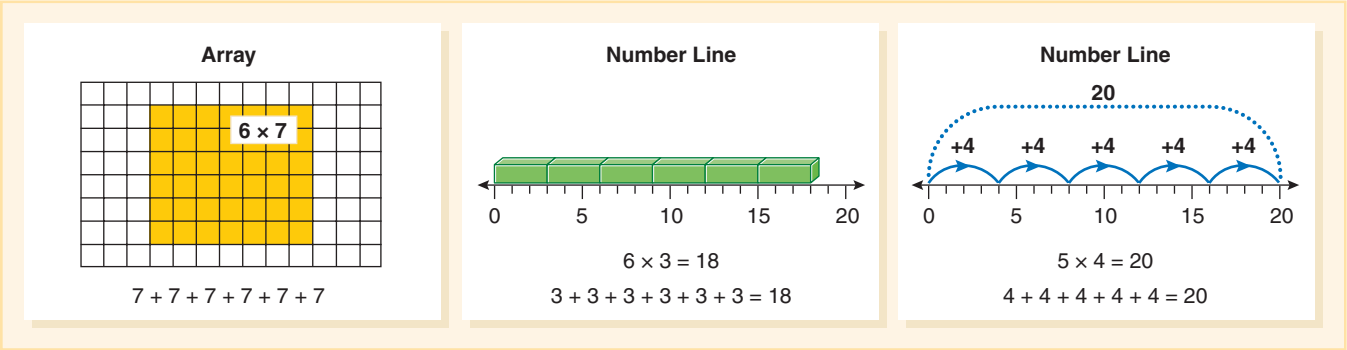


FIGURE 8.14 Semi-concrete representations for multiplication problems

Array Using Grids. A representation that is extremely important and widely used for multiplication and division is the array. An *array* is a rectangular arrangement using rows and columns. Sometimes, arrays can be modeled with square tiles (https://www.didax.com/apps/color-tiles/) or blocks and, other times, hand-drawn (semi-concrete) using a grid (see the 10 × 10 Array) or a hand-drawn, semi-concrete version.

Pearson eText Blackline Master: 10 × 10 Array

It is important that students first have many experiences with concrete arrays (tiles, for example) so they can internalize the relationship between the rows and columns before using grid paper to record the outline of an array. Let's look how the work with arrays with items such as tiles can be linked to semi-concrete representations, using grids with the following three activities.

Activity 8.10 Factor Quest

CCSS-M: 4.OA.B.4

Start by having students think about a context that involves arrays such as parade formations (see the end-of-chapter Literature Connections), seats in a classroom, or patches on a quilt. Then assign a number that has several factors—for example, 12, 18, 24, 30, or 36. Have students build as many rectangular arrays as they can (e.g., square tiles or cubes). Then have them record their arrays by drawing them on grid paper—see the Factor Quest Activity Page. For each, write the corresponding multiplication equations. For students who may have limited motor skills to manipulate the materials, try the Factorize applet on the NCTM Illuminations website.

Pearson eText Activity Page: Factor Quest

Draw students' attention to the dimensions of the arrays, having them make the connection between the factors in the multiplication expression they have written to the number of rows and columns (the length dimensions of the rectangle). Your class will undoubtedly want to decide whether a rectangle that is 3 by 8 should be counted differently from one that is 8 by 3. Leave the decision to the class, but take advantage of the opportunity to discuss how 3 rows of 8 are the same amount of units as 8 rows of 3. Note that if equal sets are made with counters on paper plates rather than arrays, 3 sets of 8 look very different from 8 sets of 3. So, as students begin to think about the commutative property of multiplication, arrays provide a helpful representation.

Standards for Mathematical Practice
MP7. Look for and make use of structure

Activity 8.11 extends Factor Quest in that students look for patterns, such as the number of factors, the type of factors, the shape of the resulting array, and so on. Rather than assigning numbers with several factors, this activity suggests numbers that have only a few factors so that differences between numbers become more distinct. This activity can support the exploration of numbers that are *prime* (numbers greater than one that only have a factor of itself and 1) and *composites* (numbers that can be made with two or more different arrays). Have students continue to consider the patterns that they notice as they classify different numbers into these categories.

Activity 8.11 Factor Patterns

CCSS-M: 4.OA.B.4

Tell students that their task is to find all the multiplication expressions and the corresponding rectangular array(s) for several numbers (e.g., 1 through 16 or 10 through 25). Have enough square tiles available that students can explore all possible arrays. For example, for the number 12, they can build 12×1, 6×2, 4×3, and the three matching pairs using the commutative property. Then they record their rectangles on the 1-Centimeter grid paper and label each rectangle with the corresponding multiplication expression (e.g., 6×2). This organization helps when students are comparing arrays across different numbers. After identifying the multiplication equations and the rectangular arrays, students should look for patterns in the factors and arrays. For example, which numbers have the fewest number of arrays and, therefore, the fewest number of factors? Which numbers have only a factor of 1 and itself? Which numbers have arrays that form the shape of a square? What can you say about the factors for even numbers? Do even numbers always have two even factors? What about odd numbers? Encourage students to think about why different patterns occur. See Feldman (2014) for another activity in which students conjecture about numbers that have 2, 3, 4 or 5 distinct factors.

Pearson eText Blackline Master: 1-Centimeter Grid Paper

Activities 8.10 and 8.11 can also include division concepts by changing the model. Have students build or draw arrays by specifying how many squares are to be in the array. Then specify the number of rows that should be made (partition) or the number in each row (measurement). When students have learned that 3 and 6 are factors of 18, they can write the equations $18 \div 3 = 6$ and $18 \div 6 = 3$ along with $3 \times 6 = 18$ and $6 + 6 + 6 = 18$ (assuming that three sets of six were modeled). Here we continue with another activity, using arrays to explore the distributive property.

TECHNOLOGY Note. Explore "Interactive Array Explorations" from Sine of the Times: http://www.sineofthetimes.org/interactive-array-explorations. This tool allows the user to

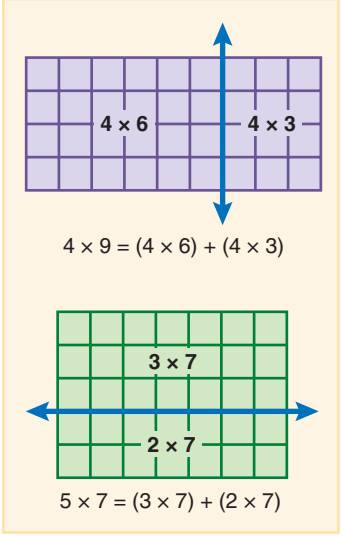


FIGURE 8.15 Array that demonstrates the distributive property of multiplication over addition

Activity 8.12 Split It Up

CCSS-M: 3.OA.B.5; 5.OA.A.1

Supply students with several sheets of 1-Centimeter grid paper (have color tiles available) to represent a small garden that will be planted with two kinds of vegetables. Assign each pair of students a garden plot size, such as 4×9. Garden sizes (products) can vary across the class to differentiate for students with special needs or students who want a challenge. Ask students to find all of the ways to make a single slice or cut through the garden to divide the plot for the two different vegetables. For each slice, students write an equation. For a slice that results in one row of 8, students would write $4 \times 9 = (4 \times 6) + (4 \times 3)$. Record students' suggestions, using these conventions even though students formally learn about grouping symbols such as parentheses in fifth grade and the order of operations in sixth grade. The individual expressions can be written inside the arrays as is shown in Figure 8.15. This slicing or partitioning of the array provides an opportunity for students to see the distributive property in action.

Pearson eText Blackline Master: 1-Centimeter Grid Paper

create multiple arrays on the same page, using different configurations of counters. Another tool demonstrates the relationship between multiplication and division, using equations and arrays: https://www.ictgames.com/mobilePage/arrayDisplay/index.html. Input a multiplication expression in the white box and then you can rotate it for the commutative property or view the associated division equation. ■

Representing Multiplication and Division: Abstract Representations

The next activity involves students in more abstract thinking because it engages them in using mental mathematics and reasoning with symbolic representations of the quantities to figure out how to solve either multiplication or division problems on a calculator without a functioning operation key.

Activity 8.13 The Broken Division Key

CCSS-M: 4.NBT.B.6

Have students find methods of using a calculator to solve division exercises without using the divide key. "Find at least two ways to figure out 135 ÷ 5 or 61 ÷ 14 without pressing the divide key." If the problem is put in a story context, one method may actually match the problem better than another. Good discussions may follow different solutions with the same answers. Are they both correct? Why or why not?

Consider ways to explore a broken multiplication key as well.

Pause & Reflect

Can *you* find three ways to solve 61 ÷ 14 on a calculator without using the divide key? For a hint, see the footnote.*●

Why Not Division by Zero? Sometimes students are simply told, "Division by zero is not allowed," often because teachers were never taught how best to explain this concept (Ojose, 2014; Quinn, Lamberg, & Perrin, 2008). Some students may think that the answer should be either zero or the number itself. To avoid merely sharing an arbitrary rule, pose problems to be modeled that involve zero: "Take 30 counters. How many sets of 0 can be made?" or "Put 12 blocks in 0 equal groups. How many are in each group?" or "Can you show me how to share 5 oranges with 0 children?" Then move students toward reasoned explanations (Crespo & Nicol, 2006) that consider the inverse relationship of multiplication and division and put the answer back into a multiplication problem as a check. Then, with the orange problem, you would ask, "What, when multiplied by 0, produces an answer of 5?" Right; there is no answer. Therefore, division by 0 is undefined; it just doesn't make sense when we use our definition of division and its inverse relationship to multiplication.

Strategies for Solving Word Problems

Learning Outcome 8.5 **Develop methods for helping students solve word problems.**

We have suggested the use of contextual problems or story problems to help students develop meanings for multiplication and division, but often, students still need more support for what to do when given a word problem. In this section, you will learn some techniques for helping them.

*There are two measurement approaches to find out how many 14s are in 61 (by repeatedly adding or subtracting). A third way is related to partitioning by finding 14 times what number is close to 61.

Analyzing Word Problems

Consider the following problem:

> **In building a road through a neighborhood, workers filled in large holes in the ground with dirt brought in by trucks; 638 truckloads of dirt were required to fill the holes completely. The average truck carried $6\frac{1}{4}$ cubic yards of dirt, which weighed 7.3 tons. How many tons of dirt were used to fill the holes?**

Typically, in fifth grade, problems of this type are found as part of a series of problems revolving around a single context or theme. Data may be found in a graph or chart or, perhaps, a short news item or story. Students may have difficulty deciding on the correct operation and are often challenged to identify the appropriate data for solving the problem. Sometimes they will find two numbers in the problem and guess at the correct operation. Instead, students need tools for analyzing problems. At least two strategies can be taught that are very helpful: (1) thinking about the answer before solving the problem, or (2) working a simpler problem.

Activity 8.14 Word Problem Sort

CCSS-M: 3.OA.A.1; 3.OA.A.2

Give each student a collection from the Multiplicative Word Problem cards, with access to appropriate physical materials (e.g., counters, square tiles, grid paper) for testing ideas. Have students select two or three of the problems that they think are similar. Then they use a glue stick to paste the problems they choose onto the top of the Word Problem Sort activity page. Have students use the bottom of the page to write an explanation of their thinking and tell how they decided the problems are alike. This sorting process (based on Caldwell, Kobett, & Karp, 2014) can also be used to assess how students analyze the meaning of the story problem sentence by sentence and look for structure.

Pearson eText Activity Page: Multiplicative Word Problem
Pearson eText Activity Page: Word Problem Sort

Think about the Answer before Solving the Problem

Students need to spend adequate time thinking about the problem and imagining the situation (recall the Three Reads routine in Chapter 4). In addition, multilingual learners need to comprehend both the contextual words (like *dirt*, *filled*, and *road*) and the mathematical terminology (*cubic yards*, *weighed*, *tons*, and *how many*). Instead of rushing in and beginning to do calculations, believing that number crunching is what solves problems, students should spend time talking about (and, later, thinking about) what the answer might look like. In fact, one great strategy for differentiation is to pose the problem with the numbers missing or covered up (Bushart, 2019; Holbert & Barlow, 2012/2013). This approach eliminates the tendency to number crunch. For our sample problem, it might look like this:

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

What is happening in this problem? Some trucks were bringing dirt to fill up holes.

Is there any extra information we don't need? We don't need to know about the cubic yards in each truck.

What will the answer tell us? It will give us how many tons of dirt were needed to fill the holes. The answer will be some number of tons.

Will that be a small number of tons or a large number of tons? There were 7.3 tons on each truck, but there were a lot of trucks, not just one. The answer is probably going to be a lot of tons.

About how many tons do you estimate it will be? If there were 1000 trucks, it would be 7300 tons. So, it will be less than that. But it will be more than half of 7300, so the answer is more than 3650 tons.

In this type of discussion, three things are happening. First, students are focusing on the problem and the meaning of the answer instead of on numbers. The numbers are not

important in thinking about the structure of the problem. Second, with a focus on the structure of the problem, students identify the numbers that are important as well as numbers that are not important. Third, the thinking leads to a rough estimate of the answer and the unit of the answer (tons in this case). In any event, thinking about what the answer tells and about how large it might be is a useful starting point.

Work a Simpler Problem

The reason that models are rarely used with problems such as the dirt problem is that the large numbers are very challenging to model. Distances in thousands of miles and time in minutes and seconds—data likely to be found in the upper grades—are sometimes difficult to model. The general problem-solving strategy of "try a simpler problem," as discussed in Chapter 3 (see Table 3.1), can almost always be applied to problems with unwieldy numbers.

A simpler-problem strategy has the following steps:

1. Substitute small whole numbers for all relevant numbers in the problem.
2. Model the problem (with counters, drawings, number lines, bar diagrams, or arrays), using the new numbers.
3. Write an equation that solves the simpler version of the problem.
4. Write the corresponding equation, substituting back the original numbers.
5. Calculate!
6. Write the answer in a complete sentence and decide whether it makes sense.

Figure 8.16 shows how students simplified the numbers in the dirt problem. It also shows an alternative in which only one of the numbers is made smaller and the other number is illustrated symbolically. Both methods are effective.

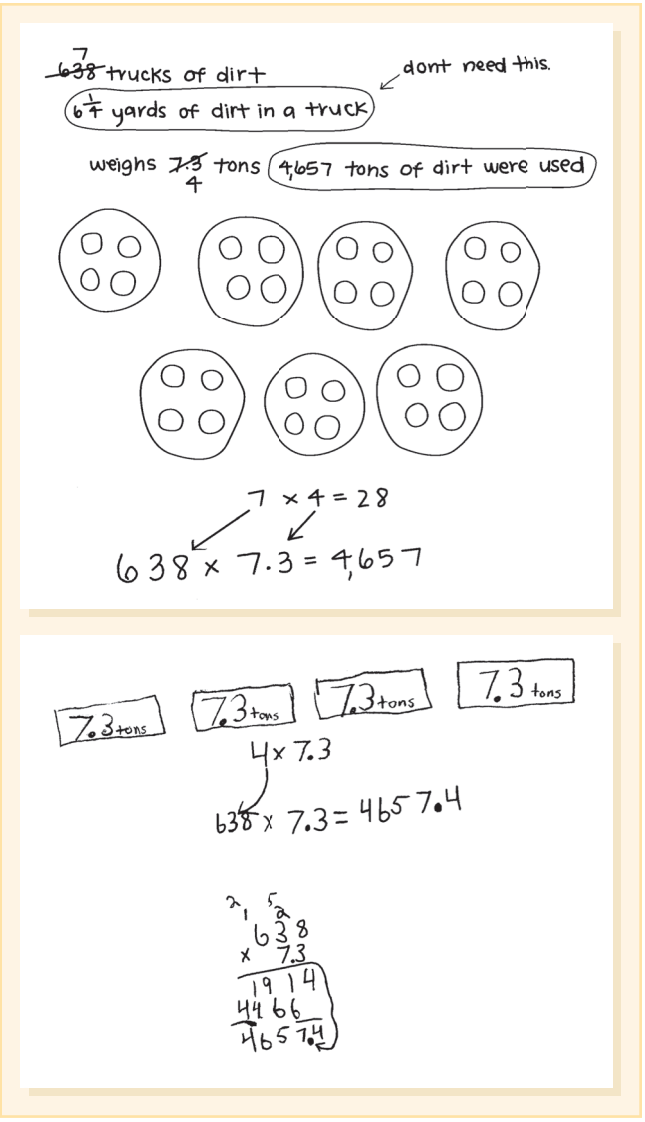


FIGURE 8.16 Two ways students created a simpler problem

Students need strategies they can consistently use to analyze a problem instead of using the word-problem game (De Corte & Verschaffel, 1985), where they are basically circumventing the process of making sense of the situation and instead guessing which operation to use. It is much more useful to have students do a few problems in which they must use a model or a drawing to justify their solution than to give them a lot of problems in which they guess at a solution but don't use reasoning and sense-making.

Caution: Avoid the Key-Word Strategy!

In the past, it was often suggested that students should be taught to find *key words* in story problems to use to decide whether to add, subtract, multiply, or divide. Some students are still encouraged to use lists of key words with the corresponding operation linked to the word. For example, this strategy suggests that if you see the words *altogether* and *in all* in a story problem, that means you should add, and *left* and *fewer* indicate you should subtract. The word *each*, they would say, suggests multiplication. This approach is unfortunately reported in the research as the most commonly identified method teachers say they use to help students solve word problems (Pearce, Bruun, Skinner, & Lopez-Mohler, 2013).

In contrast to the belief that this approach is useful, researchers and mathematics educators have long cautioned against the key-word strategy (e.g., Clement & Bernhard, 2005; Heng & Sudarshan, 2013, Karp, Bush, & Dougherty, 2019; Powell et al., 2022). Instead, students should be making sense of the operations (Dixon, Nolan, Adams, Tobias, & Barmoha, 2016).

Here are four arguments against presenting the key word approach:

1. The key-word strategy sends a wrong message about doing mathematics. The most important approach to solving any contextual problem is to analyze it and make sense of it by using all the words. The key word approach encourages students to ignore the meaning and structure of the problem. Mathematics is about reasoning and making sense of situations. Sense-making strategies always work!
2. Key words are often misleading. Many times, the key word or phrase in a problem suggests an operation that is incorrect. The following problem shared by Drake and Barlow (2007) demonstrates this possibility.

> **There are three boxes of chicken nuggets on the table. Each box contains six chicken nuggets. How many chicken nuggets are there in all? (p. 272)**

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Drake and Barlow found that a student generated the answer of 9, using the words *how many in all* as a suggestion to add $3 + 6$. Instead of making sense of the situation, the student used the key word approach as a shortcut in making a decision about which operation to select.

3. Many problems do not have key words. A student who has been taught to rely on key words is then left with no strategy. Here's an example:

> **Aidan has 28 goldfish. Twelve are orange and the rest are yellow. How many goldfish are yellow?**

4. Key words don't work with two-step problems or more advanced problems, so using this approach on simpler problems sets students up for failure with more complex, multistep problems because they are not learning how to read for meaning.

Multistep Word Problems

Multistep problems appear for the first time when second-graders are expected to solve two-step addition and subtraction word problems. Two-step word problems in all four operations are part of the third-grade standards, and multistep problems in all four operations with whole numbers start in fourth grade, including problems with remainders. Fifth graders use multistep problems with measurement scenarios, and logically, multistep problems continue in middle school and beyond with a variety of numbers and contexts. Yet students may initially have difficulty with the multistep problems, particularly students with special needs in math (Hord & Marita, 2014). To start, be sure students can analyze the structure of one-step problems in the ways we have discussed. The following ideas, adapted from Huinker (1994), are designed to help students see how two problems can be linked to think about multistep problems.

1. Give students a one-step problem and have them solve it. Before discussing the answer, have the students use the answer to the first problem to create a second problem. Then they can pass the problem to another classmate who is asked to solve the second problem. Here is an example:

> ***Given problem:*** **It took 3 hours for the Morgan family to drive the 195 miles to Washington, DC. What was their average speed?**
>
> ***Second problem:*** **The Morgan children remember crossing the river at about 10:30 a.m., or 2 hours after they left home. How many miles from home is the river?**

2. Make a *hidden question*. Repeat the previous approach by giving groups different one-step problems. Have them solve the first problem and write a second problem. Then they should write a single combined problem that leaves out the question from the first problem. That question from the first problem is the *hidden question*, as in this example:

> ***Given problem:*** **Toby bought three dozen eggs for 89 cents a dozen. How much was the total cost?**
>
> ***Second problem:*** **How much change did Toby get back from $5?**
>
> ***Hidden-question problem:*** **Toby bought three dozen eggs for 89 cents a dozen. How much change did Toby get back from $5?**

Have other groups identify the hidden question. Because all students are working on a similar task (be sure to mix the operations), they will be more likely to understand what is meant by a *hidden question*.

3. Pose standard two-step problems and have the students identify and answer the hidden question. Consider this problem:

> **The Marsal Company bought 275 widgets wholesale for $3.69 each. In the first month, the company sold 205 widgets at $4.99 each. How much did the company make or lose on the widgets? Do you think the Marsal Company should continue to sell widgets?**

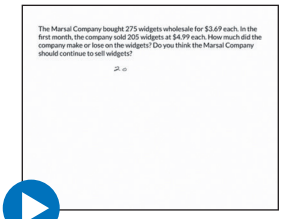
Begin by considering these questions: "What's happening in this problem?" (something is being bought and sold at two different prices). "What will the answer tell us?" (how much profit or loss there was). If students are stuck, you can ask, "Is there a hidden question in this problem?" Although the examples given here provide a range of contexts, for multilingual learners, using the *same* (relevant and familiar) context across this three-step process would reduce the linguistic demands and therefore make the stories more comprehensible—and the mathematics more accessible. Another approach is to use a table with each of the questions to support students' working memory (Hord & Marita, 2014). Have students repeat this process until all of the problem's multiple steps are addressed.

FORMATIVE ASSESSMENT Notes. One of the best ways to assess knowledge of the meaning of the operations is to have students generate story problems for a given equation or result (Drake & Barlow, 2007). Use the Translation Task Activity Page (discussed in Chapter 5) as an assessment. Ask students to record and answer an expression such as 5×7 in the upper left-hand quarter and then write a story problem representing the expression in the upper right, draw a picture (or model) in the lower left, and describe their solution process in the last section. (Students with special needs in math can audiotape their story problem and the description of their solving process.) Students successfully completing all four sections demonstrate their depth of understanding, or they may reveal individual areas of strength and areas where further experiences are needed. Adapt the assessment by providing the result (e.g., 24 cents) and asking students to write a division problem (or another operation) that will generate that answer, along with models and word problems written in the remaining quarters. Having students create their own stories and representations is a powerful assessment (McCormick & Essex, 2017). ■

Pearson eText Activity Page: Translation Task

Common Challenges Related to the Meaning of the Operations

Knowing common challenges identified through the research is a critical part of planning and can greatly influence how a lesson is structured and what problems you use. See Table 8.6 for possible ways to support students.


The Marsal Company bought 275 widgets wholesale for $3.69 each. In the first month, the company sold 205 widgets at $4.99 each. How much did the company make or lose on the widgets? Do you think the Marsal Company should continue to sell widgets?

Pearson eText
Video Example 8.6
In this video a student solves the Marsal Company problem. How does the student solve the problem? What do you think of the student's solution method and response? If you were this teaching this student, what questions would you pose and why?

Pearson eText
Video Example 8.7
In this video the teacher has students solve a word problem. Pay attention to how the students solve the problem, the questions the teacher asks the students, and how the teacher extends the problem.

TABLE 8.6 COMMON CHALLENGES RELATED TO THE MEANING OF THE OPERATIONS

Common Challenge	What It Looks Like	How to Help
1. The equal sign is interpreted as an operation symbol or as a signal to compute, rather than indicating an equivalence relationship.	For $5 + 4 = \_\_\_ + 3$, says the _____ should be 9 because $5 + 4 = 9$. When asked if $6 = 6$, says no, because there is no computation to do. When asked if $7 = 3 + 4$, says no, because "you can't write it that way because there is no computation to do. It's backward."	• Use a number balance to illustrate the relational meaning of the equal sign. • Read the equal sign in an equation as "is the same as" or "equals." • Avoid saying $5 + 3$ "makes" 8 because the word "makes" sounds like an operation or an action. • Pose true/false number sentences, using a variety of equation formats.
2. The commutative property of addition is overgeneralized to subtraction.	In solving $24 - 7$, subtracts $4 - 7$ as $7 - 4$ for an answer of 23.	• Have students model problems with single-digit numbers and show how you cannot subtract 7 cubes from 4 cubes.
3. Adding zero is understood as making a number bigger and subtracting zero makes a number smaller.	$6 + 0 = 7$ or $12 - 0 = 11$	• Use story problems that introduce adding and subtracting zero in a meaningful context that students can act out.
4. Rather than counting the units between the numbers (or intervals) on a number line, tick marks or numbers are counted.	The count is consistently off by 1. Points to tick marks or numbers as they count.	• Use the sequence of number-line models in Figure 8.7 to emphasize that the unit is what is counted on a number line. • Use units of lengths, steps, and arrows to show what is being counted.
5. The inverse relationship between addition and subtraction and/or multiplication and division is not recognized.	Addition isn't used to solve subtraction situations. Multiplication isn't used to solve division problems.	• Avoid the rote use of fact families to teach the inverse relationship, because that emphasizes procedures without showing students the inverse relationship. • Have students act out a series of multiplication/division problems with the same three numbers, using concrete materials—showing relationship between the number in each group, the number of groups, and the product.
6. Addition and multiplication are perceived as always making numbers bigger (and the corollary that factors in multiplication problems are always smaller than the product).	There is the thinking that when adding or multiplying, the answer should always be a larger number.	• Point out that these patterns are only true for some numbers sets (Karp, Bush, & Dougherty, 2014). • Point out counterexamples that are within their reach, such as that $54 + 0 = 54$ or that $15 \times 0 = 0$.
7. Subtraction and division are perceived as always making numbers smaller. **The divisor must be less than the dividend.**	There is the thinking that with a subtraction or division problem, the answer should always be a smaller number. This idea is used to check for a reasonable answer or to justify thinking.	• Provide examples as well as counterexamples, such as $8 \div 1 = 8$. • Have the student equally share 3 cookies with 5 friends. This approach will help them act out the division and see that the divisor can be greater than the dividend. • Avoid using generalizations that may appear to be rules, unless the students are clear about the circumstances in which the rule works.
8. A key-word strategy is used to determine which operation to use in a word problem.	When solving a word problem, the problem is scanned for the numbers and any key words to help determine which operation to use.	• Take out the numbers in the word problem and ask the student to read the problem. Then ask the student what questions might be asked. • Have the student use a variety of representations to help make more explicit the relationship between the quantities. • Suggest recommendations for analyzing contextual problems (e.g., imagine the situation; think about the answer before solving the problem; also see Table 3.1).

Common Challenge	What It Looks Like	How to Help
9. The wrong operation is chosen in word problems.	Rather than using a problem-solving approach, there is guessing or use of the operation studied that week to answer word problems.	• Focus on the structure of the problems in Table 8.2, using graphic organizers or the story problem sorting activity.
10. The remainder is considered left over and not part of the answer.	The remainder is written as 9 R 2 and is not used in deciding how to answer the question. The remainder is forgotten or ignored and is not included in the answer.	• Use multiple problems that show the various ways remainders are interpreted and how that analysis affects the answer. • Discuss student work samples where the remainder is interpreted improperly. Have students critique the work in a class discussion.
11. There is uncertainty about whether there is division by zero.	Problems such as: $5 \div 0 = 0$ are written.	• Focus on division as having an inverse relationship with multiplication. What number when you multiply it by zero will equal 5? • Ask students to take out 5 counters. How many sets of 0 can be made? Or put 5 blocks in 0 equal groups. How many blocks are in each group?

RESOURCES FOR CHAPTER 8

WRITING TO LEARN

Write your answers to assess your understanding and application of chapter content.

1. Why is it important to know about different types of problem situations for the operations?
2. Make up a compare story problem. Alter the problem to provide examples of all six possibilities for compare problems.
3. Explain how missing-part activities help students build a connection between addition and subtraction or between multiplication and division.
4. Make up multiplication story problems to illustrate the difference between equal groups and multiplicative comparison. Then create a story problem involving area or arrays.
5. What is the role of contexts, manipulatives, and representations in learning about the operations?
6. In your own words, in a way you would explain to a colleague or a student's family member, why is the use of a key-word strategy not effective?

LITERATURE CONNECTIONS

Books with stories or pictures about collections, the purchase of items, measurements, and so on can be used to pose problems or, better, to stimulate students to invent their own problems. A favorite for this purpose is *The Doorbell Rang* by Pat Hutchins (1986). Here are others:

Bedtime Math

Overdeck (2013)

This book (and www.bedtimemath.org) helps family members and caretakers incorporate math problems into the evening routine with three levels of problem difficulty: wee ones (pre-K), little kids (K–2), and big kids (grade 2+). Each problem set revolves around high-interest topics such as roller coasters, foods, and animals.

One Hundred Hungry Ants

Pinczes (1999)

A Remainder of One

Pinczes (2002)

These two books focus on multiplication and division. The first tells of 100 ants going to a picnic that move from a single file of 100 to two rows of 50, then four rows of 25, and so forth. This story uses visual representations of arrays, and students can try different numbers of ants to explore factors and products. The second book describes a parade formation of 25 bugs. As the queen views the rectangular array of parading bugs, she notices that 1 bug is trailing behind. The bugs try to create different numbers of rows and columns, but 1 bug is always leftover. Give students different numbers of bugs to generate formations that will leave 1, 2, or none out of the parade.

RECOMMENDED READINGS

Articles

Champagne, Z., Schoen, R., & Riddell, C. (2014). Variations in both-addends-unknown problems. *Teaching Children Mathematics*, *21*(2), 114–121.

The focus of the article is on the problem type of both addends unknown. The authors share examples of problems, students' thinking about their solution methods, and student work samples.

Karp, K. S., Bush, S. B., and Dougherty, B. J. (2019). Avoiding the ineffective key-word strategy. *Teaching Children Mathematics*, *25*(7), 428–435.

This article explores the use of sense-making in solving word problems as a way to avoid the use of a key-word strategy. The emphasis is on the meanings of the operations as a variety of other strategies are shared.

Books

Carpenter, T. P., Franke, M. L., Johnson, N.C., Turrou, A. C., & Wager, A. A. (2017). *Young children's mathematics: Cognitively guided instruction in early childhood education*. Portsmouth, NH: Heinemann.

This book emphasizes understanding word-problem structures for all operations. The structures are explained in detail along with methods for using these problems with students.

Caldwell, J., Kobett, B., & Karp, K. (2014). *Putting essential understanding of addition and subtraction into practice in grades prekindergarten–2*. Reston, VA: NCTM.

Lannin, J., Chval, K., & Jones, D. (2013). *Putting essential understanding of multiplication and division into practice in grades 3–5*. Reston, VA: NCTM.

These two books offer the big ideas about how to teach the four operations, including the mathematical content teachers need to know and connections to topics that come before and after these critical ideas.

CHAPTER

9

Developing Basic Fact Fluency

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities, you should be able to:

- **9.1** Describe the developmental process and research-based approach for teaching basic facts.
- **9.2** Illustrate and explain strategies for helping students develop addition and subtraction fact fluency.
- **9.3** Illustrate and explain strategies for helping students develop multiplication and division fact fluency.
- **9.4** Justify limitations of timed tests and explain effective assessment alternatives.
- **9.5** Explain effective methods for ongoing practice and intervention for basic fact fluency.

Basic facts for addition and multiplication are the number combinations in which both addends and both factors, respectively, are less than 10. Basic facts for subtraction and division are the corresponding combinations. Thus, $15 - 8 = 7$ is a subtraction fact because the corresponding addition parts are less than 10, and $64 \div 8 = 8$ is a division fact. As you will read, the key to effective teaching of basic facts is focusing on *fluency* not just *mastery/automaticity*. These terms are defined in the following sections. Developing basic fact fluency requires avoiding practices such as rote drill and timed-tests and replacing them with activities that focus on reasoning strategies. Beyond building fluency and automaticity more effectively, such an approach builds student number sense and confidence!

BIG IDEAS

- Students move through three phases in developing fluency with basic facts: counting, reasoning strategies, and automaticity. Instruction and assessment must help students progress through these phases without rushing them.
- Number relationships provide the basis for strategies that help students remember basic facts or figure out unknown facts. For example, when solving $8 + 6$, students can decompose 6 into $2 + 4$, add the 2 to the 8 to make a 10, and then solve $10 + 4 = 14$ (the Make 10 strategy).

- The most effective way for students to learn their basic facts is through explicit strategy instruction; this includes learning foundational facts first and then derived fact strategies.
- Assessment of basic facts must reflect effective assessment practices, which do not include timed tests.

Teaching Basic Facts Developmentally

Learning Outcome 9.1 **Describe the developmental process and research-based approach for teaching basic facts.**

The goal for basic facts instruction is to develop fluency *and* automaticity. Basic fact fluency means that students are efficient, flexible, and accurate in solving basic fact expressions (Bay-Williams & Kling, 2019; NRC, 2001). Automaticity (or mastery) is the quick recall of facts, also called automaticity. Knowing basic facts well leads to gains in mathematics achievement overall (Nelson, Parker, & Zaslofsky, 2016). *Addition and subtraction* fact fluency begins in kindergarten with decomposing numbers and understanding one more and one less and continues through grade 2, when students are expected to have automaticity or mastery of their addition facts. Similarly, *multiplication and division* fact fluency begins in grade 2 as students explore doubles, and is a major focus in grade 3, where automaticity is often expected.

Developmental Phases for Learning Basic Facts

Developing fluency with basic facts is a developmental process—just like every topic in this book! Students progress from counting to eventually just knowing that $2 + 7$ equals 9 or that 5×4 equals 20. This developmental learning progression takes time and many experiences. Baroody (2006) describes three phases of learning facts. Here, each phase is described, along with an example of how a student would think through $4 + 7$ or 4×7 if they are operating at that phase.

Phase 1: Counting strategies: Using object counting (e.g., blocks or fingers) or verbal counting to determine the answer.

$4 + 7$: Student starts with 7 and counts on verbally 8, 9, 10, 11.
4×7: Student skip counts by 7s, saying 7, 14, 21, 28.

Phase 2: Reasoning strategies: Using known information to determine an unknown combination logically.

$4 + 7$: Student knows that $3 + 7$ equals 10, so $4 + 7$ is one more, 11 (Make 10 strategy).
4×7: Student knows 2×7 equals 14 and doubles that answer to get 28 (Doubling strategy).

Phase 3: Mastery or Automaticity: Producing answers efficiently (quickly and accurately).

$4 + 7$: Student quickly responds, "It's 11; I just know it."
4×7: Student uses doubling automatically and says, "It's 28."

Notice that two things happen at Phase 3, a student may just know the answer, having thought about it so many times, or they may be enacting a strategy very adeptly, hardly noticing they are doing it. Both options are efficient and appropriate. Because Phase 1 is addressed in Chapters 7 and 8, this chapter primarily focuses on Phase 2 and Phase 3. Phase 2 is often underemphasized or neglected entirely, yet it is an essential bridge between inefficient counting strategies and automaticity. Phase 2 uses, and further develops, students' relational understanding, and students can become so quick at applying a strategy that it becomes as automatic as recall (Baroody, Purpura, Eiland, Reid, & Paliwal, 2016).

FORMATIVE ASSESSMENT Notes. When are students ready to work on reasoning strategies? When they are able to (1) use counting-on strategies (start with the largest and count up) and (2) decompose numbers within 10 (e.g., decompose 6 into $5 + 1$).

Quick Interviews can determine student readiness. A quick interview takes less than a minute. Prepare the tasks in writing and say them aloud to the student. A few ideas are here related to readiness:

1. Counting On: $7 + 2$ $1 + 6$ $10 + 2$ $2 + 9$ [Do students count on from the larger number or just know the answer?]
2. Decompose: How can 6 children be sitting at two tables? [Probe for more than one way.]
3. Write addition expressions that equal 7.

You can informally ask these questions as students are engaged in doing activities, or have them rotate to a station where you are asking these questions. ■

Approaches to Teaching Basic Facts

There are essentially two approaches to teaching the basic facts: memorization and explicit strategy instruction. Memorization has been a common approach for over a century, despite the fact that research conducted over recent decades consistently finds a strategy approach to be more effective. The pros and cons of each approach are briefly discussed in this section.

Memorization. This approach goes straight from counting to memorization of facts, without devoting time to developing reasoning strategies or using number relationships (Baroody, Bajwa, & Eiland, 2009). There is strong evidence from research that this method is significantly less effective than a strategy-based approach (e.g., Baroody et al., 2016; Brendefur, Strother, Thiede, & Appleton, 2015; Henry & Brown, 2008; Locuniak & Jordan, 2008; Jordan, Kaplan, Ramineni, & Locuniak, 2009; Purpura, Baroody, Eiland, & Reid, 2016; Tournaki, 2003). A heavy focus on memorizing strategies results in students with lower number sense (Henry & Brown, 2008).

Why doesn't memorization work? It skips Phase 2 of the developmental process, resulting in three major limitations:

- *Inefficiency.* There are too many facts to memorize. Without using strategies, students must memorize 100 addition facts (just for the addition combinations 0–9) and 100 multiplication facts (0–9), along with the 200 subtraction and division facts. Yikes!
- *Inflexibility.* Students don't learn strategies, so they are left with resorting to less sophisticated counting strategies.
- *Inappropriate applications.* Students misapply the facts and don't check their work (they don't have strategies to use to confirm whether the sum or product is reasonable or correct).

Despite the fact that memorization is a weak learning strategy, it is still the most widely used in the United States. It could be why we have a nation of people who don't feel they are good at math or don't like it. You can change that. Avoid this approach.

Explicit Strategy Instruction. For more than three decades, explicit strategy instruction has been used in many classrooms. Students learn a strategy (e.g., Make 10). Students then explore and practice the strategies (e.g., using a double ten-frame to see how to move counters over to make a 10). Extensive research supports the use of explicit strategy instruction in helping all students learn (and remember) their basic facts (e.g., Baroody et al., 2009; Baroody et al., 2016; Fuson, 1992; Rathmell, 1978; Thornton, 1978; Thornton & Toohey, 1984).

Explicit strategy instruction is intended to *support* student thinking and building generalizations rather than giving the students something new to remember. The key is to help students learn to use the possible strategies and then *choose* one that helps them solve the problem without counting. The big idea of using reasoning strategies is that students use

Pearson eText
Video Example 9.1
In this video John Van de Walle discusses the importance of using a strategy approach to learning basic facts.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Pearson eText
Video Example 9.2

Pearson eText
Video Example 9.3

Pearson eText
Video Example 9.4
In these videos, Connor (9.2), Myrna (9.3) and Miguel (9.4) each solve $6 + \_\_\_\_ = 13$. What do you notice about the different strategies each student employs?

known facts and relationships to *derive unknown facts*. For example, students must know their Combinations of 10 addition facts and the $10 + \_\_\_\_\_$ facts before they are ready to learn the Make 10 strategy. They must know the doubles multiplication facts to do the "double and one more group" strategy.

Build on Conceptual Understanding

Teaching basic facts includes working with subsets within the facts in a developmentally appropriate sequence (discussed in the next section). As you are working on a subset of facts, your goal is to help students progress through the three phases described previously. For example, ask students how they solved $7 + 4$. Some will have counted all, some will have counting on (Phase 1), and others will use the Make 10 or other reasoning strategy. Help the students who are counting to count on, and for those counting on to apply a reasoning strategy. Through number talks, basic fact games, and other experiences, provide engaging and diverse experiences for students to develop the strategies. Students will move to more efficient strategies over the course of the year, become quicker at applying them, and eventually will develop automaticity.

Story Problems. Research has found that when a strong emphasis is placed on solving problems, students not only become better problem solvers but also develop automaticity with more basic facts than do students in a drill program (NRC, 2001). In fact, posing a story problem each day, followed by a brief discussion of the strategies that children used, can improve students' accuracy and efficiency with basic facts (Rathmell, Leutzinger, & Gabriele, 2000). Story problems provide context that can help students understand the situation and apply flexible strategies for doing computation. For example, if you wish to elicit the Make 10 strategy, you might use this story:

Rachel had 9 ponies in one barn and 6 ponies in the pasture. How many ponies did she have altogether?

Thinking about how the numbers and situation in this story lend students to think of $9 + 6$ as equivalent to $10 + 5$ (one pony could be moved to the barn). Another option is to *pair* story problems (Bay-Williams & Kling, 2019) as shown in the next activity.

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Activity 9.1 Paired Story Problems

CCSS-M: 1.OA.A.1; 1.OA.C.6; 2.OA.B.2

Prepare two corresponding stories, one that uses a foundational fact that students know and a second one that uses a reasoning strategy. For example, consider these two paired multiplication stories, posed to students who already know their 2s facts and are learning the doubling strategy to work on their 4s facts:

Story 1: In 2 weeks, we will go to the zoo. How many days until we go to the zoo?
Story 2: In 4 weeks, we will go to a farm. How many days until we go to the farm? Use your thinking from Story 1 to help you solve this problem.

Display or distribute story 1 and give students time to solve. Then present the second story and read it aloud, emphasizing that they are to use what they did in the first problem to solve the second problem. A student might refer to the calendar as an array and see $14 + 14$, or they may reason abstractly and double 14 to get 28. After giving students time on this question, the teacher can help students see that single facts can be broken apart into known facts (e.g., that 4×7 is the same as $2 \times (2 \times 7)$).

Quick Looks. The purpose of quick looks is to move students beyond counting to seeing the way numbers can be composed or decomposed. For young children, this is a high-leverage routine (McCoy, Barnett, & Combs, 2013; Shumway, 2011). You can use dot cards, five-frames, and ten-frames for the images (see the "Relationships between Numbers 1 through 10" section in Chapter 7 for many ideas). Ten-frames help students see number relations and develop strategies. The quick look routine focuses on using number relationships to solve facts. It includes three quick looks and then discussion:

1. Flash an image (e.g., Quick Image 1) for a few seconds and hide it.
2. Repeat step 1 with the same image.
3. Show the image a third time. Ask students, "How many did you see?" Collect answers.
4. Then ask, "How did you see it?" Listen to different ways students saw the total. Draw or point to the aspects described so that other students can see what the speaker is sharing.

Pairing two images can help students see relationships that lead to the strategies. For example, in Figure 9.1(a) the two images help students connect a combination of 10 to the Make 10 strategy, and in Figure 9.1(b) the two images focus on doubling (the same problem used for the stories in the previous section). When pairing two images, you still do the full quick look routine with Quick Image 1, though there will likely be much less discussion. Then, repeat the routine for the second image. In Step 4, you can hold the two next to each other for students to see how they compare. In a number talk, you can do this routine for three to four pairs of quick images and conclude by asking students what pattern they are noticing and how that pattern can help them solve basic fact problems.

Pearson eText
Video Example 9.5
Consider the ways the teachers in this video are building conceptual foundations for fact fluency.

Standards for Mathematical Practice
MP8.

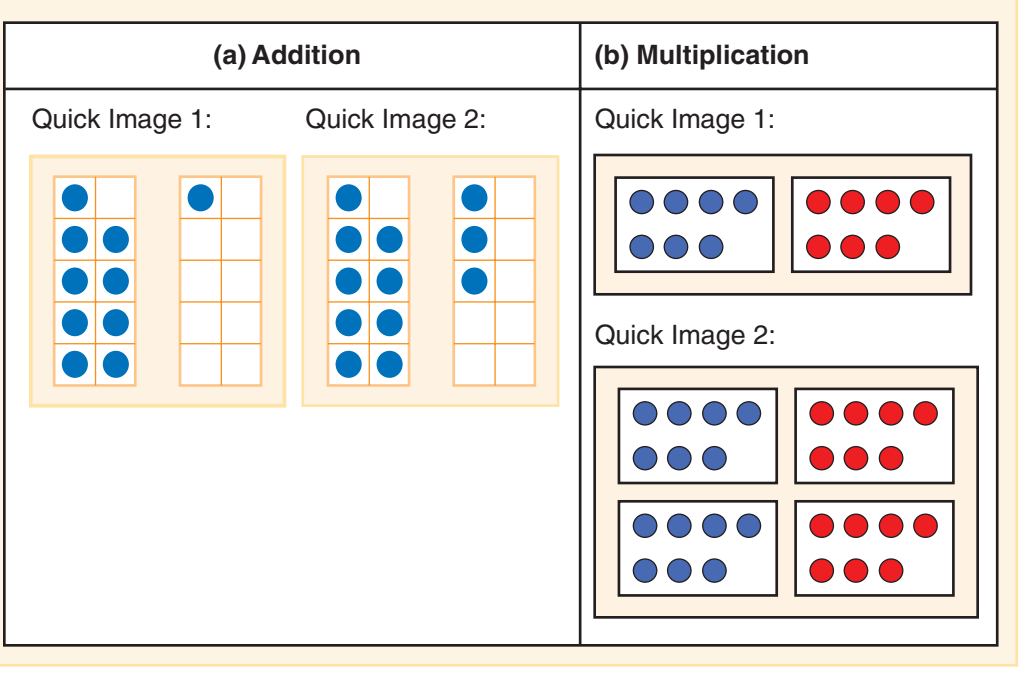


FIGURE 9.1 Paired quick images to highlight strategies

Pearson eText Blackline Master: Dot Cards
Pearson eText Blackline Master: Five-Frames
Pearson eText Blackline Master: Ten-Frames

Pearson eText Activity Page: Relationships between Numbers 1 through 10

Addition and Subtraction Facts

Learning Outcome 9.2 **Illustrate and explain strategies for helping students develop addition and subtraction fact fluency.**

Reasoning strategies build upon each other. Therefore, it doesn't make sense to learn facts in sequential order— +0, +1, +2 —but, rather, to learn foundational facts in an order based on conceptual understanding and then use those foundational facts to work on reasoning strategies that will lead to automaticity with all the facts. Bay-Williams and Kling (2019) propose the flexible learning progression illustrated in Figure 9.2.

Recall that basic fact fluency depends on progressing through three phases (Baroody, 2006). Reasoning strategies for addition facts are directly related to one or more number relationships that were developed in Chapter 7. It takes many experiences over many months for students to move from using strategies to just knowing their facts (Phase 3).

Notice that no memorization is needed—just many activities like the ones shared here (and in Chapters 7 and 8)! The first five strategies listed are foundational strategies, so they should be developed through phase 3 (mastery/automaticity) before developing the other strategies.

Addition Foundational Fact Sets

+/−0, 1, 2. A good beginning for this fact set is to use story problems in which one of the addends is a 0, a 1, or a 2. For example, "Seven children were waiting for the slide. Then 2 more children got in line. How many children were waiting for the slide?" Ask different students to explain how they got the answer of 9. Some will count all, some will use a counting on strategy from 7, and some will just know that 2 more than 7 equals 9. Help students see the connection between counting on and adding 2; for example, show them a number path or number line.

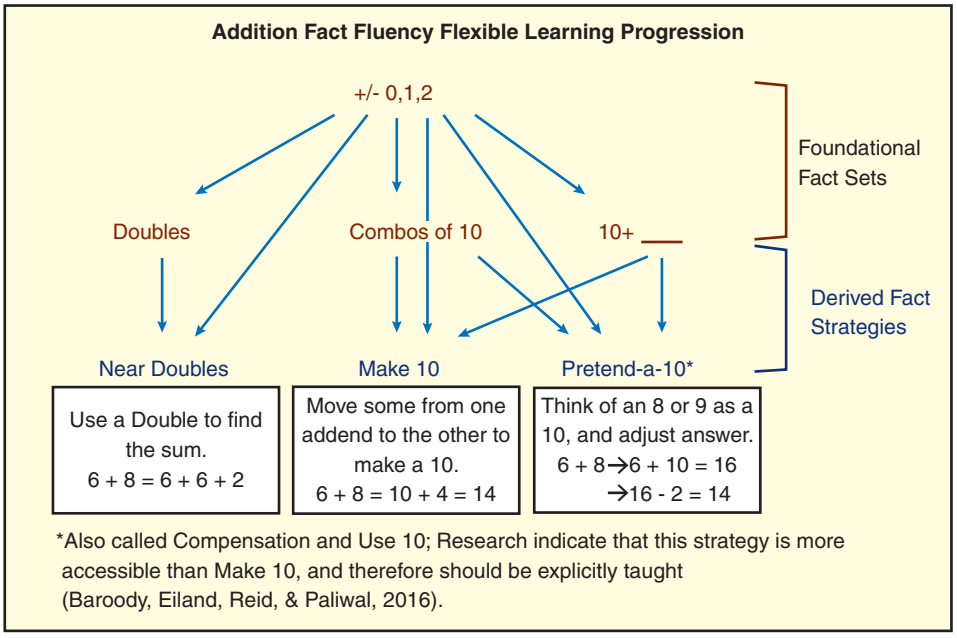

Addition Fact Fluency Flexible Learning Progression
+/- 0,1,2
Doubles
Combos of 10
10+
Foundational Fact Sets
Derived Fact Strategies
Near Doubles
Make 10
Pretend-a-10*
Use a Double to find the sum.
6 + 8 = 6 + 6 + 2
Move some from one addend to the other to make a 10.
6 + 8 = 10 + 4 = 14
Think of an 8 or 9 as a 10, and adjust answer.
6 + 8→6 + 10 = 16
→16 - 2 = 14
*Also called Compensation and Use 10; Research indicate that this strategy is more accessible than Make 10, and therefore should be explicitly taught (Baroody, Eiland, Reid, & Paliwal, 2016).

FIGURE 9.2 Flexible learning progression for addition facts

©ASCD. Adapted from *Math Fact Fluency*. Permission required.

Activity 9.2 How Many Feet in the Bed?

CCSS-M: 1.OA.A.1; 1.OA.C.6; 2.OA.B.2

Read *How Many Feet in the Bed?* by Diane Johnston Hamm (1991). On the second time reading the book, ask students how many more feet are in the bed each time a new person gets in. Ask students to record the equation (e.g., $6 + 2$) and tell how many feet in total. Two less can be considered as family members get out of the bed. Find opportunities to make the connection between counting on and adding using a number line. For multilingual learners, be sure that they know what the phrases "two more" and "two less" mean (and clarify the meaning of *foot,* which is also used for measuring). Acting out this situation with students in the classroom can be a great illustration for both multilingual learners and students with special needs in math.

In the context of this activity, the expression is set up with the larger number first. However, when the expression is reversed (e.g., $2 + 6$), students may initially count on from the first number rather than from the larger number. An explicit focus on strategies (e.g., counting on from the larger number) helps students, particularly those with special needs in math, select the more efficient strategy (Dennis, Sorrells, & Falcomata, 2016).

Students must be able to add 1 or 2 automatically *through 20*. Even though $14 + 1$ is not considered a basic fact, students must know it to use the Near Doubles strategy (e.g., $7 + 8 = 7 + 7 + 1 = 14 + 1$). Activity 9.3 engages students in adding 0, 1, or 2 to be the first one to get a bean (or counter) to 19.

Activity 9.3 Game: First Bean to 19

CCSS-M: 1.OA.C.5; 1.OA.C.6; 2.OA.B.2

Materials: A die labeled +0, +1, +2, +0, +1, +2 (or a spinner with these three options), two beans or counters (one for each player), and a First Bean to 19 game board (see Figure 9.3)

Students take turns. On their turn, they roll the die and move their bean to the new spot on the number path. They say and record the addition sentence. For example, if a student is on 11 and rolls a 1, they say, "Eleven plus 1 equals twelve" or "One more than eleven is twelve." They can record equations in their notebook or a recording page.

Pearson eText Activity Page: First Bean to 19 Game Board

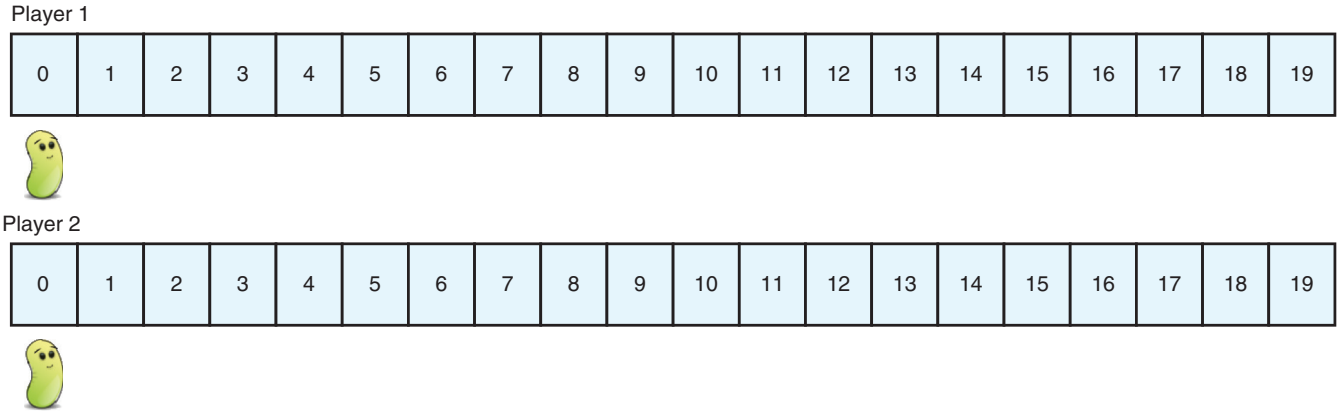


FIGURE 9.3 First Bean to 19 game board

Activity 9.4 Game: Dart to the Start

CCSS-M: 1.OA.C.5; 1.OA.C.6; 2.OA.B.2

Materials: A die labeled −0, −1, −2, −0, −1, −2 (or a spinner with these three options), two beans or counters (one for each player), and a First Bean to 19 game board (see Figure 9.3)

This game is played just like First Bean to 19 (Activity 9.3), only in reverse, with the beans placed at 19 and the first reaching 0 winning the game. As always, encourage students to say aloud the subtraction move they make. If they are on 16 and roll –2, they may say "Sixteen minus 2 equals fourteen."

Standards for Mathematical Practice
MP7. Look for and make use of structure

While zeros can be easy to compute, the concept of adding zero must be developed through concrete experiences. Asking students to generalize from a set of problems is a good way to reinforce reasoning and avoid overgeneralization.

Activity 9.5 Zero Has an Identity

CCSS-M: 1.OA.C.5; 1.OA.C.6; 2.OA.B.2

Give students a set of expressions to solve, such as the ones in the Zero Has an Identity activity page. Have students model each one on their Part-Part-Whole graphic organizer, using counters. Ask students what they notice about the problems, and what they notice across problems. Ask them to create their own stories and/or to illustrate the problems to go with each problem.

Pearson eText Activity Page: Zero Has an Identity
Pearson eText Activity Page: Part-Part-Whole Graphic Organizer

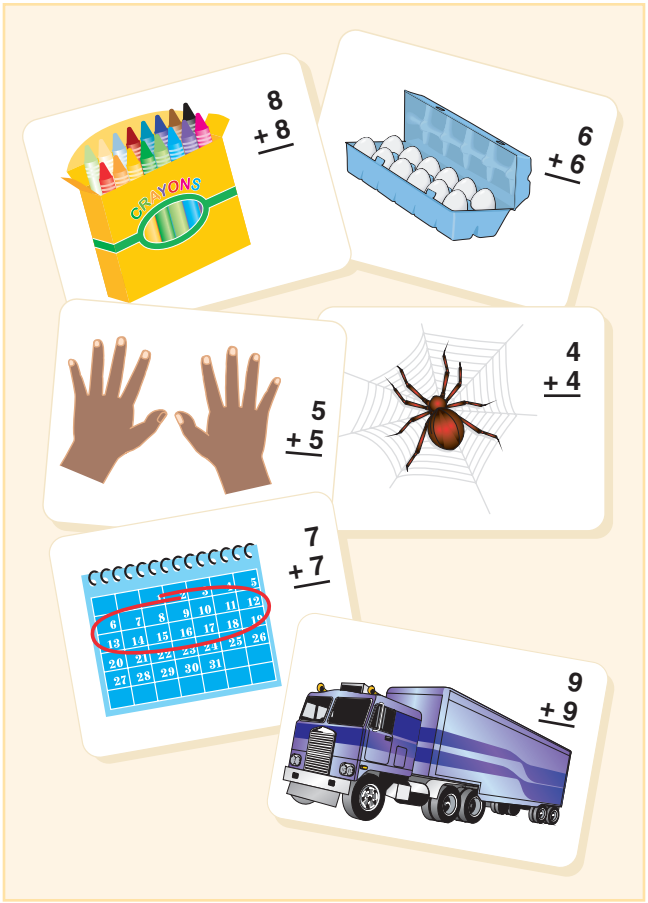


FIGURE 9.4 Real-life situations for doubles facts

Doubles

There are ten doubles facts from $0 + 0$ to $9 + 9$. These facts are foundational and provide the basis for the Near Doubles strategy. Many students find doubles easy, perhaps from reciting doubles. Picture cards that show real-life doubles can help students learn doubles if they do not know them (see Figure 9.4). Story problems can focus on pairs of the same addends: "Alex and Zack each found 7 seashells at the beach. How many did they find together?" And quick images with doubles, using ten-frames, can help students see that $7 + 7$ equals $5 + 5 + 2 + 2$, or $10 + 4$.

Activity 9.6 Double the Ducks

CCSS-M: 1.OA.C.5; 1.OA.C.6; 2.OA.B.2

Read *Double the Ducks* (Murphy, 2002), a story that begins with 5 ducks, each bringing home a friend. Have students work with a partner. Give each pair a set of counters. The first partner selects some of those counters to be the ducks. Their partner tells how many ducks there will be when each duck brings a friend. To support students with special needs in math and to make strategies more visible for all students, ducks (counters) can be placed on a ten-frame.

Activity 9.7 Calculator Doubles

CCSS-M: 1.OA.C.6; 2.OA.B.2

Students work in pairs with a calculator. Students enter the double maker (2×) in the calculator. One student says a double—for example, "Seven plus seven." The other student presses 7, says what the double is, and then presses = to see the correct double (14) on the display. The students then switch roles and reset the calculator (2×). For multilingual students, you can encourage saying the double in their native language or in both their native language and in English. (Note that this activity with the calculator can easily be adapted to other fact sets.)

Combinations of 10. Perhaps the most important strategy for students to know is the combinations that equal 10. It is a foundational fact from which students can derive many other facts. Consider story situations such as the following and ask students to tell possible answers.

There are 10 fishing boats, some at the dock and some at sea.
How many boats might be at the dock and at sea?

Pearson eText Activity Page: Fishing Boats

There are many children's books focused on the concept of 10. While they might be counting books, you can incorporate questions like, "How many more to equal 10?" Such books could be used as a context for Activity 9.8.

Activity 9.8 How Many More to Equal 10?

CCSS-M: 1.OA.B.4; 1.OA.C.6; 2.OA.B.2

Display counters on one ten-frame (see Figure 9.5) and ask, "How many more to equal 10?" This activity can be repeated using different start numbers. Eventually, display a blank ten-frame and say a number less than 10. Students start with that number and complete the "10 fact." If you say, "four," they say, "four plus six equals ten." This can be completed as whole class or with students working with a partner. Students who are still in Phase 1 of learning the facts (using counting strategies) or students with special needs in math can cover the open spaces on the ten-frame to confirm the missing part. This activity can be implemented in a learning center to provide additional experiences with learning Combinations of 10.

Pearson eText Blackline Master: Ten-Frame

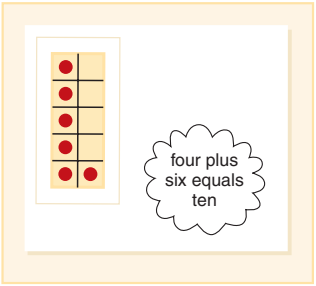


FIGURE 9.5 Combinations of 10 on a ten-frame

Activity 9.9 Game: Go Fish for 10s

CCSS-M: 1.OA.B.4; 1.OA.C.6; 2.OA.B.2

Materials: Mini-ten-frames as a deck of cards, or a regular deck of cards, removing kings and jacks (queens = 0, aces = 1)

The classic game of Go Fish is a great fit for Combinations of 10. Place students in groups of 3 or 4. Deal five cards to each player. In the classic game, on a player's turn, they ask one friend if they have a card that matches one of theirs (e.g., a 6). If the friend has a 6, they have to give it to the player. The player has a match, and lays it down in front of them and they get another turn. In this Fish for 10s version, when the child has a 6, they select a friend and ask, "Do you have a 4?" If the friend does, they hand it over, and the player puts down the match (a combination of 10). If the friend does not have the card, they say, "Go Fish." The player draws a card. If it is not a 4, their turn is over. If it is a 4, they get to go again. Play continues until all the cards are gone. If a player runs out of cards, they draw three new cards.

Pearson eText Blackline Master: Little Ten-Frames

10 + _____. Adding onto 10 is not a basic fact, but it is a foundational fact set because it is necessary for both the Make 10 and Pretend-a-10 strategies. Using a full ten-frame and a partial ten-frame can help students with this group of facts. Just like the foundational facts of one more and two more, the goal is to move from counting on to just knowing. A number line or hundred chart can help students notice the relationship in these sums that will help them move to automaticity.

Activity 9.10 Game: Teen Time

CCSS-M: 1.OA.B.4; 1.OA.C.6; 2.OA.B.2

Materials: Ten-sided die (or deck of cards), 15 counters per player and a Teen Time game board per player

Students can play in groups of 2–5. Students place all 15 counters on whichever numbers they want on their game board. On their turn, a player rolls the die (e.g., 7), adds that number to 10, saying "Ten plus seven is seventeen," and removes a counter from that sum (17). If there is no counter on that number, they lose their turn. First player to remove all their counters wins!

Teen Time game board:

10	11	12	13	14	15	16	17	18	19

Pearson eText Activity Page: Teen Time Game Boards

***TECHNOLOGY* Note.** Several websites offer virtual visuals to help students with Combinations of 10 and 10 + __ facts. For example, DreamBox (dreambox.com) teacher tools include quick looks for numbers within 10, using ten-frames and math racks, and an app for finding combinations that equal 10 (and other totals as well). Math Learning Center (https://www.mathlearningcenter.org/) apps include ten-frames, number lines, and math racks (number racks), where you can create the visuals you want to share with students. ■

Reasoning Strategies for Addition

Once students have their foundational fact sets to automaticity (Phase 3), they are ready to learn strategies (Phase 2) for the rest of the facts. These strategies eventually become the foundation for reasoning with multi-digit addition, fraction addition, and decimal addition. Learning these strategies is not only to help students learn their addition facts; it helps them develop procedural fluency in general. Whereas in the preceding section games were used to develop automaticity, in this section, games will be used to develop decision-making skills. Students need to decide when they want to use a Near Doubles strategy, for example, and when they want to use a Pretend-a-10 strategy. Therefore, in this section, we briefly explain each strategy and then share games that will engage students in choosing and using the strategies.

Near-Doubles. The Near-Doubles strategy involves using a double fact to figure out the given problem. For instance, to solve $6 + 7$, a student might use either their $6 + 6$ or their $7 + 7$ fact to solve the problem: $6 + 6 + 1$ or $7 + 7 - 1$. This strategy is therefore sometimes called "doubles-plus-one" or "doubles-minus-one," but using the more general phrase communicates that there are options. Also, a student might solve $6 + 8$ by thinking $6 + 6 + 2$—also a Near Doubles strategy. A good way to develop this strategy is with paired quick looks (see Figure 9.1). Begin with a doubles fact quick image, such as $6 + 6$, and follow it with a quick image of $6 + 7$ or $8 + 6$. Transition to using a string of expressions, pairing doubles facts with facts that lend to the Near Doubles strategy. For example:

4 + 4	5 + 5	8 + 8	7 + 7
4 + 6	5 + 7	6 + 8	6 + 8

Another way to help students see a double in a non-double fact is to do a matching activity, as illustrated here in Activity 9.11.

Activity 9.11 Game: Flip and Find

CCSS-M: 1.OA.B.4; 1.OA.C.6; 2.OA.B.2

MATERIALS: A doubles reference mat and a set of fact equation cards that lend to using the Near Doubles strategy (see activity pages). Another option is to cut up a practice page to make the cards.

Students play with a partner. They turn the cards face-down next to the doubles reference mat (see Figure 9.6). On their turn, a player *flips* the card face-up and *finds* a double they will use to add. The player shares their thinking out loud and then places the card under that double fact. A way to increase the challenge is for one partner to share one way, and the other partner to find a different double to solve it. For example, for $7 + 6$, one student might use $6 + 6 + 1$, and another might use $7 + 7 - 1$. After all cards are placed, players examine the facts that are in the same pile and discuss what the facts have in common.

Pearson eText Activity Page: Doubles Reference Mat
Pearson eText Activity Page: Addition Facts that Fit Near Doubles

Make 10. All of the basic facts with sums between 11 and 18 can be solved by using the Make 10 strategy. Students use their known facts that equal 10 and then add the rest of the number to 10. For example, to solve $6 + 8$, a student might start with the larger number (8) and see that 8 is 2 away from 10; therefore, they take 2 from the 6 to equal 10 and then add on the remaining 4 to get 14. Make 10 is also aptly called *break apart to make ten (BAMT)* (Sarama & Clements, 2009).

This reasoning strategy is extremely important and is heavily emphasized in high-performing countries (Korea, China, Taiwan, and Japan), where students learn facts sooner and more accurately than U.S. students do (Henry & Brown, 2008). Yet this strategy is not emphasized enough in the United States. A study of California first graders found that this strategy contributed more to developing fluency than using doubles did (even though using doubles had been emphasized by teachers and textbooks in the study) (Henry & Brown, 2008).

The Make 10 strategy can also be applied to larger numbers. For example, for $28 + 7$, students can make 30, seeing that $28 + 7 = 30 + 5$. Thus, this reasoning strategy deserves significant attention when teaching addition (and subtraction) facts.

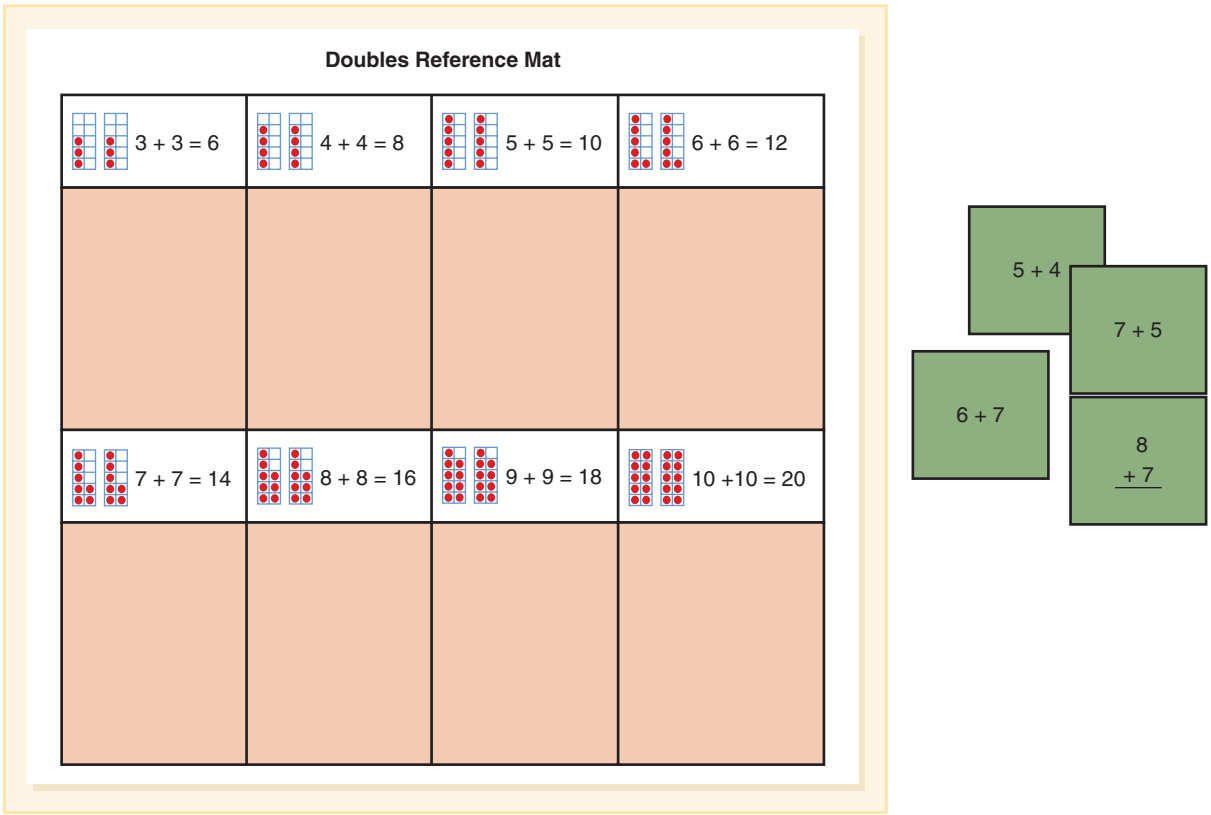


FIGURE 9.6 Doubles reference mat and example cards

Quick images or manipulating double ten-frames can help students develop the Make 10 strategy, the focus of Activity 9.12.

Pearson eText Blackline Master: Double Ten-Frames

Activity 9.12 Move It, Move It

CCSS-M: 1.OA.B.3; 1.OA.C.6; 2.OA.B.2

Distribute the Move It, Move It activity gage and a mat with a double ten-frame. Flash cards with facts are placed next to the double ten-frame, or a fact can be given orally. The students cover each frame with counters to represent the problem ($9 + 6$ would mean covering nine places on one frame and six on the other). Ask students to "move it"—to decide a way to move the counters so that they can find the total without counting. Ask students to explain what they did and connect to the new equation. For example, $9 + 6$ may have become $10 + 5$ by moving one counter to the first ten-frame. Emphasize strategies that are working for that student (5 as an anchor and/or Combinations of 10 and/or Make 10).

Pearson eText Activity Page: Move It, Move It
Pearson eText Blackline Master: Double Ten-Frames

Pearson eText
Video Example 9.6
Listen as John Van de Walle describes reasoning strategies for subtraction facts.

Pretend-a-10. Pretend-a-10 (also called Use 10) is a compensation strategy. The student pretends one of the addends is a 10, adds, and then compensates for how they changed one addend. For example, for $9 + 6$, a student thinks, $10 + 6$ equals 16, 9 is one less than 10, so the sum is also one less, 15. Notice this does not require decomposing or recomposing a number, only knowing how far away one addend is from 10. Therefore, it can be less difficult to implement than Make 10. Students who learned this strategy and practiced it while playing games did substantially better than their peers on addition facts involving adding 8s or 9s (Baroody et al., 2016).

Paired stories again work well to develop this strategy. For example:

Story 1: Ten horses are in the red barn and 6 horses are in the brown barn. How many horses are in the barns?
Story 2: Today, 9 horses are in the red barn and 6 horses are in the brown barn. How many horses are in the barns? *Use your work from the first problem to help you solve this problem.*

Reasoning Strategies for Subtraction

Pearson eText
Video Example 9.7
In this video consider what strategy Andrew uses to solve a missing-part problem by thinking "8 plus what amount equals 14?"

Subtraction facts prove to be more difficult than addition, perhaps because they require knowing the related addition fact strategies and perhaps because students do not have sufficient opportunity to learn and practice these strategies. Without such opportunities, however, students will continue to rely on counting strategies, a slow and often error-prone method.

Think-Addition. As the label implies, in this strategy students use known addition facts to produce the unknown quantity or part of the subtraction. For $13 - 8$, a student thinks, "What plus 8 equals 13?" Students taught this strategy do significantly better than their peers on subtraction (Baroody et al., 2016). Considering subtraction as think-addition rather than take-away is conceptually different, because students are really comparing the two numbers to see how far apart they are. A student might think aloud, saying: "What do I add to the part I see to get to the whole?" For example, for $14 - 8$, rather than take away 8, a student thinks, "What do I add to 8 to get to 14?"

One way to build the relationship between addition and subtraction is to illustrate facts on triangle or T cards as illustrated in Figure 9.7. The T is particularly useful, because

it mimics the visual for the part-part-whole relationship (Baroody et al., 2016). Both cards help students see the relationship between addition and subtraction, the focus of Activity 9.13.

Activity 9.13 What's Under My Thumb?

CCSS-M: 1.OA.B.4; 1.OA.C.6; 2.OA.B.2; 3.OA.B.5; 3.OA.B.6; 3.OA.C.7

Create a set of triangle or T cards with fact families for each pair of students or have students create their own based on the facts they are learning (see Figure 9.7). You can begin this activity as a whole class to model the directions and then move to partners. Hold up a card with your thumb over one number. Ask, "What is under my thumb?" Call on several students to share how they reasoned to get the answer. Place students in partners to continue play.

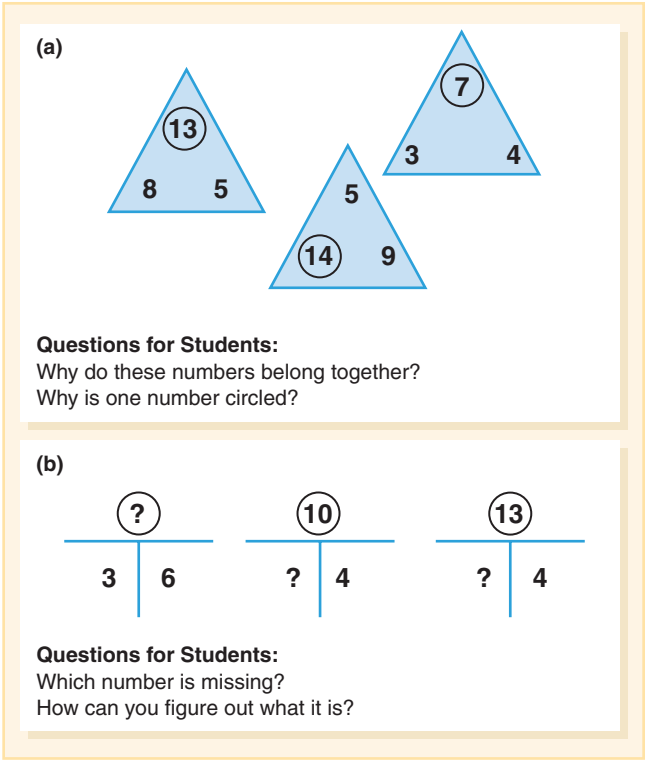


FIGURE 9.7 Triangle cards and T cards show part-part-whole relationship

Pause & Reflect

Before reading further, look at the three subtraction facts shown here and reflect on what thought process you use to get the answers. Even if you just know them, think about what a likely process might be.

11 −9	12 −6	15 −7

What stories might you tell that will help students think-addition? ●

Story problems that promote think-addition are those that sound like addition but have a missing addend: *join, initial part unknown*, *join*, and *change unknown*; and *part-part-whole*, *part unknown* (see Chapter 8). As with addition facts, it is helpful to begin with facts that have totals of 10 or less (e.g., 8 − 3, 9 − 7) before working on facts that have a total (minuend) higher than 10 (e.g., 13 − 4). Here is an example:

Janice had 9 fish in her aquarium. Grandma gave her some more fish. Then she had 12 fish. How many fish did Grandma give Janice?

Notice that the action is *join*, which suggests addition. There is a high probability that students will think, "Nine and how many more equals 12?" Connect this thought process with the subtraction fact, 12 − 9. Students may use 10 as a benchmark to solve this, thinking, "It takes 1 to get to 10 and 2 more to get to 12, which equals 3." A sentence frame can help students articulate and keep track of their thinking, for example as they use a number line:

Standards for Mathematical Practice MP2.

It takes _____ to get to 10 and _____ more to get to _____.

That is a total distance of _____.

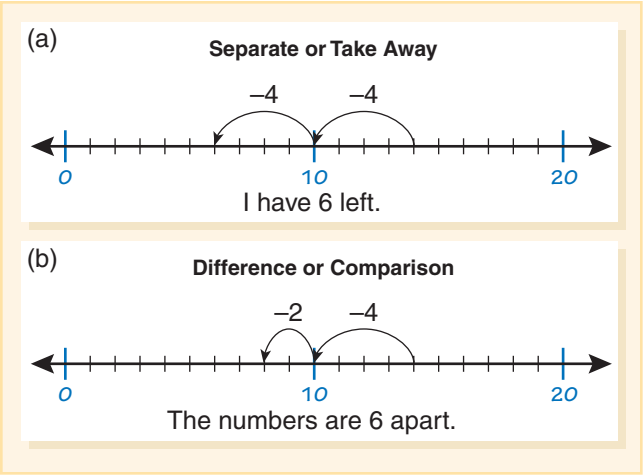


FIGURE 9.8 Down under 10 illustrated on the number line for 14 – 8

Down under 10. The Down under 10 strategy is the reverse of Make 10, and there are two ways to think about it—separate and comparison. Let's look at 14 – 8. For separate, you are taking away 8, so move down 4 to get to 10 and then move down 4, and the answer is 6 (see Figure 9.9[a]). For comparison, you are finding the difference or distance. How far apart are 14 and 8? Jump down 4 to the 10 and two more to the 8—they are 6 apart (see Figure 9.8[b]).

To develop the Down Under 10 strategy, write pairs of facts in which the difference for the first fact is 10 and the second fact is either 8 or 9: 16 – 6 and 16 – 7; 14 – 4 and 14 – 6, and so on. Have students solve each problem and discuss their strategies. If students do not naturally see the relationship, ask them to think about how the first fact can help solve the second. Illustrate on a number line. Use story problems that vary the situation types, such as these:

Becky walks 16 blocks to school. She has walked 7. How many more blocks does Becky have left? (Separate)
Becky walks 16 blocks to school; Corwin walks 9 blocks. How many more blocks does Becky walk? (Comparison)

Activity 9.14 Apples in the Trees

CCSS-M: 1.OA.B.4; 1.OA.C.6; 2.OA.B.2

Project a double ten-frame as a display with chips (apples) fully covering the first ten-frame and some of the second (e.g., for 16, place 10 in the first frame and 6 in the second frame). Tell students some apples have fallen to the ground—you will tell them how many, and they will tell you how many are still in the trees. Ask them to share how they thought about it. Repeat with different values.

Pearson eText Blackline Master: Double Ten-Frame

Take from 10. This excellent strategy is not as well known or commonly used in the United States but is consistently used in high-performing countries. It takes advantage of students' knowledge of the combinations that make 10, taking the initial value apart into 10 + _____. This is how it works for 15 – 8:

1. Break apart the whole: 15 = 10 + 5
2. Take 8 from the 10: 10 – 8 = 2
3. Add 5 back on: 2 + 5 = 7

Try Take from 10 on these examples:

12 – 8 =	17 – 9 =	14 – 6 =

If you have students from other countries, they may know this strategy and can share it with others. It can be used for all subtraction facts having minuends greater than 10 (the toughies) by just knowing how to subtract from 10 and knowing addition facts.

Activity 9.15 Apples in Two Trees

CCSS-M: 1.OA.B.3; 1.OA.B.4; 1.OA.C.6; 2.OA.B.2

Adapting Activity 9.11, explain that each ten-frame is a different tree. Tell students you will tell them how many apples fall out of the full tree, and they will tell you how many apples are left (on both trees). Connect to the related equations, as illustrated previously by recording students' ideas and eventually having students record their equations.

Games to Practice Addition and Subtraction Strategies

Games are fun to play over and over again and therefore are an excellent way to provide repeated experiences for students to learn and practice their facts. Playing games that infuse reasoning strategies helps students be able to select strategies flexibly, decide which strategy is most appropriate for the given problem, and become more efficient and accurate in finding the answer (Bay-Williams & Kling, 2014, 2019; Godfrey & Stone, 2013). Games increase student involvement, encourage student-to-student interaction, and improve communication—all of which are related to improved academic achievement (Forbringer & Fahsl, 2010).

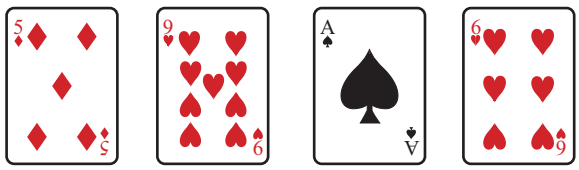
The following games provide students opportunities to practice strategies for addition and/or subtraction facts. Coach students to think aloud. It is important to encourage students *not* to worry about how fast they are, but instead to commit to using their newly learned strategies (rather than counting). As students play, you observe and listen to their thinking. The first game, Lucky 13, is adapted from Bay-Williams & Kling (2019) and is available free online, with a Jamboard and instructional video at the Kentucky Center of Mathematics (http://kcm.nku.edu/mathfactfluency/game11.php)

Activity 9.16 Lucky 13

CCSS-M: 1.OA.B.3; 1.OA.C.6; 2.OA.B.2

Materials: Deck of ten-frame cards or regular cards (kings and jacks removed; queens = 0, aces = 1)

Students play in groups of 2–4. Each player is dealt 4 cards face-up. From their four cards, each player picks two cards, which, when added together, produce a sum as close to 13 as possible (over or under). Players determine how far their sum is away from Lucky 13. That is their score for Round 1. For example, if a student draws:



they might pick 6 and 5; 6 + 5 = 11. Eleven is 2 away from 13, so their score is 2. Or, they may have picked 9 and 5, which is 1 away from 13, so they would score 1. After they make their selections, they say their answer and how they added (their strategy) to their partners. After each player goes, they discard the extra cards, and the dealer gives each player 4 new cards. Play 5 rounds. Lowest score wins!

Activity 9.17 Pick 2 to Connect 4

CCSS-M: 1.OA.B.3; 1.OA.C.6; 2.OA.B.2

Materials: Deck of cards; Pick 2 to Connect 4 game board

This game is based on the classic game of Connect 4, a partner game in which you try to get 4 in a row while preventing your partner from getting four in a row by blocking them. Player 1 deals 3 cards to both players. Player 2 picks 2 of those cards to form a sum or difference and take that place on the game board. Player 1 then picks 2 of their 3 cards and takes a place on the game board. The deck goes to Player 2. Player 2 then deals 3 cards to both players and repeats these steps. The winner is the first to get 4 in a row.

Pearson eText Activity Page: Pick 2 to Connect 4 Game Board

Like with addition, it is important to have subtraction games in which students choose from among their strategies. Activity 9.18 is such a game.

Activity 9.18 Game: Take It Away

CCSS-M: 1.OA.B.4; 1.OA.C.6; 2.OA.B.2

Materials: Deck of ten-frame cards or regular cards (kings and jacks removed, queens = 0, aces = 1); four each of two different-colored chips

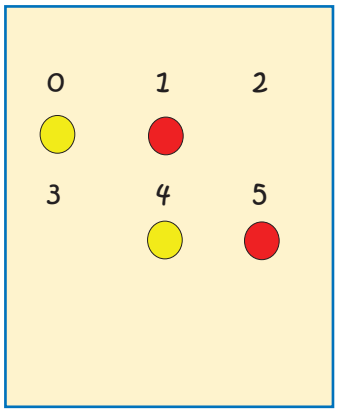
Students play with a partner. Begin by having the students record these numbers spread out on a piece of paper or use the Take It Away game board. The goal of this game is to have four numbers before your partner does. On a player's turn, they are dealt three cards. They pick two and subtract to take a number. Player places their color chip below the number.

If the number is unoccupied, the player places their color chip below it.

If their opponent has a chip there, the player gets to take it away and replace it with their own chip.

If the player cannot make a difference other than what they already have, they lose their turn.

Play continues until a player has 4 numbers on the game board.



Pearson eText Activity Page: Take it Away Game Board
Pearson eText Activity Page: Little Ten-Frame Cards

Multiplication and Division Facts

Learning Outcome 9.3 **Illustrate and explain strategies for helping students develop multiplication and division fact fluency.**

Focusing on reasoning strategies for multiplication and division facts is just as important as it is with addition and subtraction facts (Baroody, 2006; Wallace & Gurganus, 2005). Traditional approaches to learning the multiplication facts have started with students memorizing 0s, 1s, 2s, and so on until students have all 100 facts mastered, typically through memorization. There are several flaws to this approach. First, it doesn't work for many students (which is why there is so much fact remediation happening in grades 4 and 5, middle school, and even high school). Second, memorizing misses the opportunity to learn strategies that are important to procedural fluency in general. Third, students are not given the opportunity to make sense of and use number relations, which will further develop their number sense and confidence in doing mathematics.

A strategy-based approach provides opportunities for students to engage in discourse about how to solve a particular fact. Thompson & Gay (2020) found that combining intentional discourse with explicit strategy instruction positively influenced students' learning of their facts and the related reasoning strategies. Like with addition, a strategy-based approach begins with learning foundational fact sets, as illustrated in Figure 9.9.

Multiplication Foundational Facts: 2, 10, 5, 0, 1, and Squares

Starting with 2s, 10s, and 5s connects to students' experiences with addition doubles and skip counting. Students can count collections of various sizes to build a foundation for multiplication facts and reasoning strategies (Franke, Kazemi, & Turrou, 2018). Tens are not considered basic facts (because ten is two digits), but they are a foundational fact because knowing 10s facts is an important way to derive other facts (e.g., 9s and 8s). As with addition and subtraction facts, start with story problems and visuals, such as arrays, to develop reasoning strategies. Additionally, the commutative property of multiplication is critical to learning facts but more challenging to understand than the commutative property of addition. For example, visualize a restaurant with 2 groups of 8 people compared to the same restaurant with 8 groups of 2 people—it is hard to see that this is the same number of people. To build understanding of

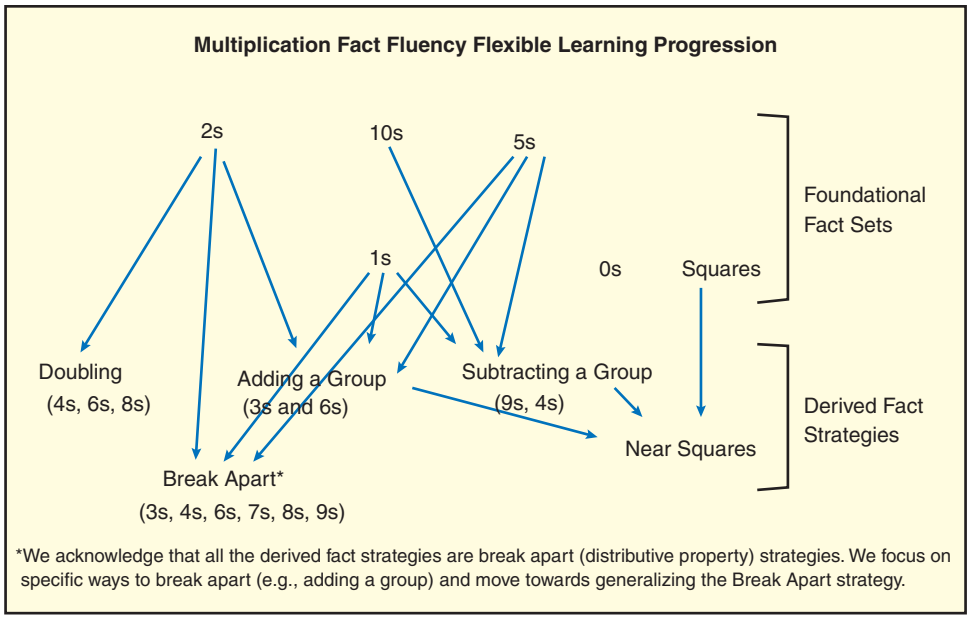
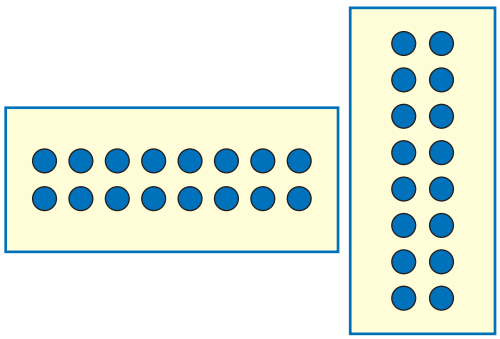

Multiplication Fact Fluency Flexible Learning Progression
2s
10s
5s
1s
0s
Squares
Foundational Fact Sets
Doubling (4s, 6s, 8s)
Adding a Group (3s and 6s)
Subtracting a Group (9s, 4s)
Near Squares
Derived Fact Strategies
Break Apart* (3s, 4s, 6s, 7s, 8s, 9s)
*We acknowledge that all the derived fact strategies are break apart (distributive property) strategies. We focus on specific ways to break apart (e.g., adding a group) and move towards generalizing the Break Apart strategy.

FIGURE 9.9 Flexible learning progression for multiplication fact fluency

©ASCD- Permission required.

the commutative property for multiplication, use arrays. For example, multiplication array cards can be rotated to show that a 2×8 array is the same quantity as an 8×2 array.



Pearson eText Activity Page: Multiplication Array Cards

Twos. Facts that have 2 as a factor are equivalent to the addition doubles and should already be known by students. Help students realize that 2×7 is the same as $7 + 7$. The activities provided in the Doubles activity in the earlier "Addition Foundational Fact Sets" section can be used here, with students asked to represent the situations as a sum and a product. For example, in the Ducks activity (Activity 9.4), if 4 ducks bring home friends, that is $4 + 4$ and it is the same as 2×4. Use story problems in which 2 is the number of sets. Later, use problems in which 2 is the size of the sets, helping students recognize the commutative property of multiplication.

George was making sock puppets. Each puppet needed 2 buttons for eyes. If George makes 7 puppets, how many buttons will he need for the eyes?

Tens. Students also bring a lot of experience with tens, having worked on place value. They can continue to build tens (see Ch. 11 for grouping activities with 10s) to think about 4 groups (sets) of 10 as 4×10. They also need to understand the commutativity of this fact (i.e., that $4 \times 10 = 10 \times 4$). Look at the illustration in Figure 9.10 and see if you can see both 4 groups of 10 and 10 groups of 4.

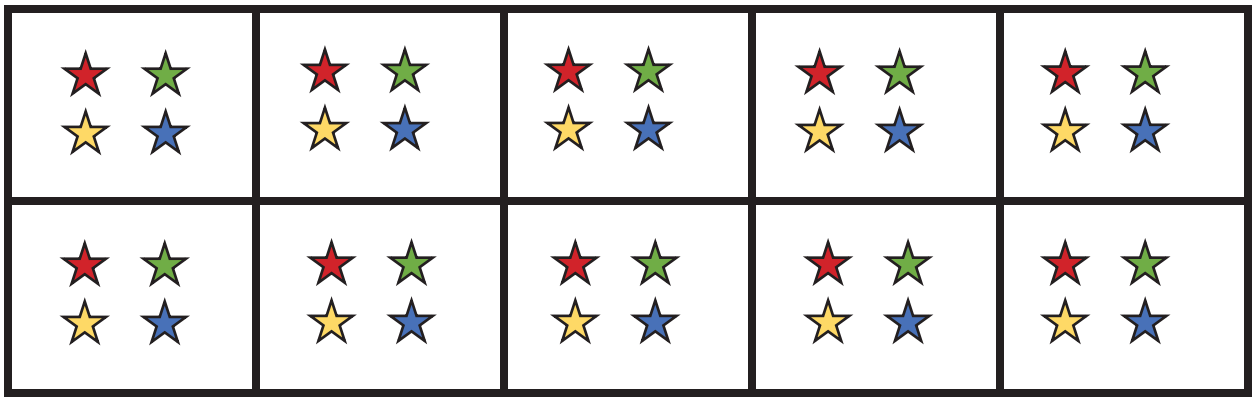


FIGURE 9.10 Can you see 4 groups of 10 and 10 groups of 4?

Fives. Fives may seem easy because students are good at skip counting by fives. They keep track of how many fives have been counted (if we skip count by 5s four times on the number line, where will we land?), and that is the answer. This is a Phase 1 strategy and a good beginning. The context of time is also a good connection for story situations, as in Activity 9.19.

Activity 9.19 Clock Facts

CCSS-M: 3.OA.A.1; 3.OA.C.7

Focus on the minute hand of the clock. When it points to a number, how many minutes after the hour is it? See Figure 9.11(a). Connect this idea to multiplication facts with 5. Hold up a clock fact card as in Figure 9.11(b) and then point to the number on the clock corresponding to the other factor. In this way, the fives facts become the clock facts.

Pearson eText Activity Page: Clock Fact Card

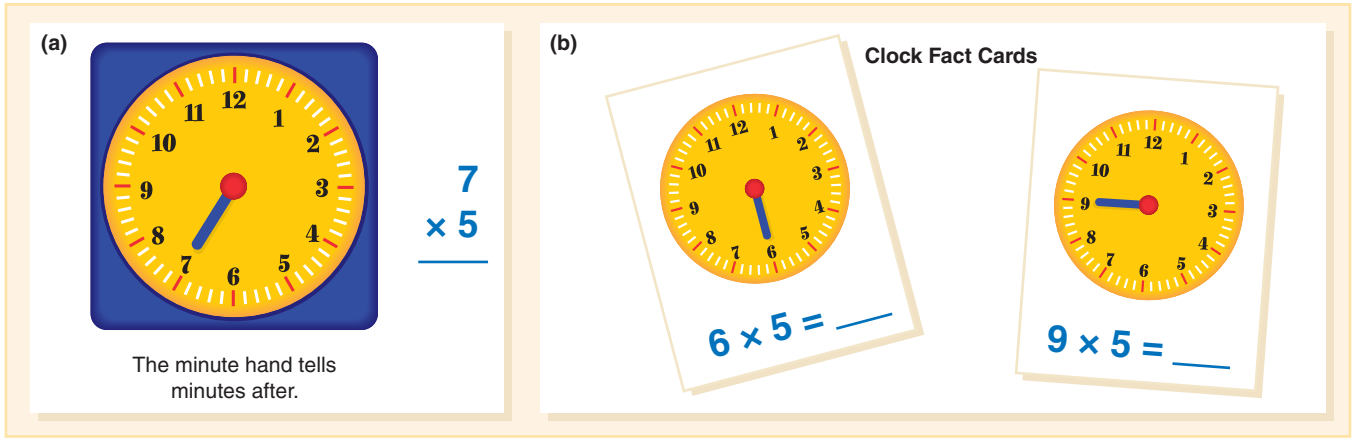


FIGURE 9.11 Using clocks to help learn fives facts

Standards for Mathematical Practice
MP7. Look for and make use of structure

But students must move beyond skip counting to gain automaticity. This can be extra challenging with 5s because they are good at counting by 5s. Quick looks and stories can help students use their 10s facts to learn their 5s facts. Number strings (a series of related problems) can be effective in illuminating strategies (Bray & Maldonado, 2018). Consider this number string, which provides an opportunity for students to think about the relationship between multiplying by 5s and multiplying by 10s:

1. $2 \times 10 =$ **3.** $4 \times 10 =$ **5.** $6 \times 5 =$
2. $4 \times 5 =$ **4.** $8 \times 5 =$ **6.** $5 \times 5 =$

Zeros and Ones. These facts, though apparently easy on a procedural level, tend to get confused with rules that some students learned for addition—for example, the fact $6 + 0$ stays the same, but 6×0 is always zero, or that $1 + n$ is the next counting number, but $1 \times n$ stays the same. The concepts behind these facts can be developed best through story problems. For example, invite students to tell stories to match a problem.

$6 \times 0 =$ _____. There are six bowls for raisins. Each bowl is empty. How many raisins?
$0 \times 6 =$ _____. You worked 0 hours babysitting at $6 an hour. How much money did you make?

Avoid telling students rules that aren't conceptually based, such as "Any number multiplied by zero is zero."

Illustrate the one facts, using arrays to show commutativity $(8 \times 1 = 1 \times 8)$, and use stories like those for zero. With 0 and 1, help students notice patterns through exploring stories and using concrete tools and then have them generalize what it means to have $n \times 0$, $0 \times n$, $1 \times n$, and $n \times 1$.

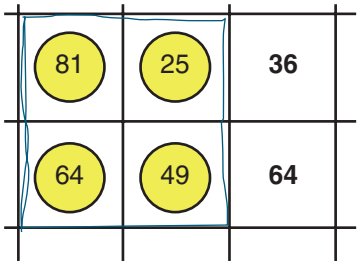
Squares. The square facts are 1×1, 2×2, 3×3,...10×10. They are called the square facts because their arrays (or areas) form a square. These facts are ones that students often remember well (somewhat like they seem to know doubles). The squares facts are useful as foundational facts, because they are right next to the facts that are the toughest for students. For example, 6×7 and 7×8 are both one group away from 7×7. The following activity is an adaptation of the classic game Connect 4. Rather than get four in a row, students try to get 4 in a square.

Activity 9.20 Game: Squares Score

CCSS-M: 1.OA.B.4; 1.OA.C.6; 2.OA.B.2; 3.OA.B.5; 3.OA.B.6; 3.OA.C.7

Materials: Squares Score game board (See Figure 9.12); 10-sided die or deck of cards with kings, jacks, and 10s removed and queens = 0, aces = 1; two different-colored counters (or a laminated game board and different-colored dry-erase markers for each player)

Pair students (trios can also work). On a player's turn, they roll the die or flip a card, square the number, and claim one place on the game board. For example, if they roll a 9, they claim one of the available 81 squares and mark it as theirs (with a counter, or an X with their colored marker). When they claim four spaces that form a 2×2 square, they score 1 point (as shown here).



Players can also choose to block another player from making a square. If a player rolls a 9 and all the 81 places are taken, they lose their turn. When time is up, or the game board is full, the player with the most 2-by-2 squares (highest score) wins.

Pearson eText Activity Page: Squares Score Game Board

Squares Score
Game Board

1	81	16	4	49	16
9	25	81	25	36	64
81	0	64	49	64	16
16	25	9	4	1	0
36	9	0	49	81	16
1	64	81	25	36	49

FIGURE 9.12 Squares Score game board

Reasoning Strategies for Multiplication and Division

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

The foundational facts comprise most of the multiplication facts. The following multiplication fact chart shows the foundational facts in white, and the remaining 30 multiplication facts shaded in light green. When students recognize the commutative property, there are only 15 facts left. But, these remaining facts are the tougher ones for students to learn, and they are the ones that will be mastered through reasoning strategies. As a reminder, the goal is to move students from Phase 1 (skip counting) to Phase 2 (reasoning strategies) to reach Phase 3 (automaticity) eventually.

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

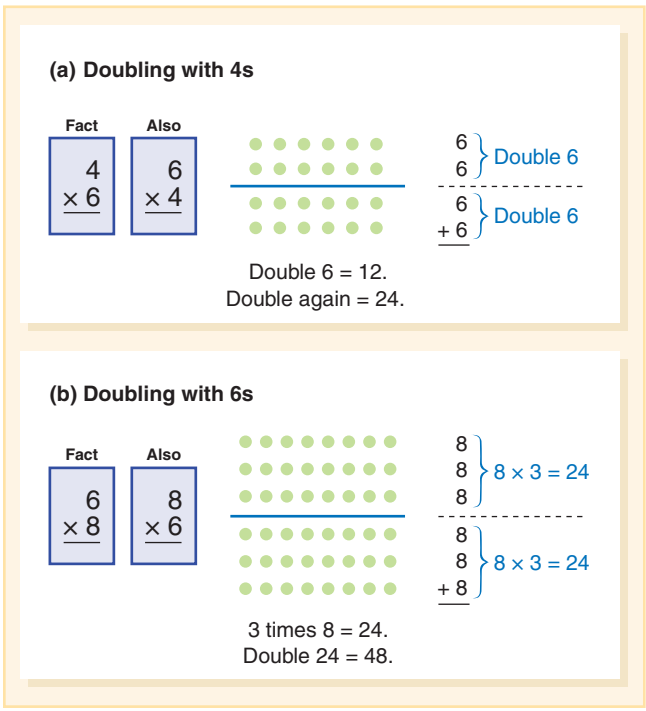


FIGURE 9.13 Illustrating the Doubling strategy

Major reasoning strategies are described here, followed by games that enable students to learn the strategies and develop automaticity in a non-stressful way.

Doubling. Doubling extends the idea of ×2 and is a very effective reasoning strategy in helping students learn the difficult facts (Flowers & Rubenstein, 2010/2011). Doubling is a great way to learn the 4s facts. A look back at Figure 9.1 shows how quick looks can be used to develop the Doubling strategy. Figure 9.13(a) provides illustrations of how to use arrays to build the conceptual foundation. Once students know their 3s facts, they can return to doubling when working on the 6s facts (Figure 9.13(b)). And once 4s facts are automatic, they can be used to solve 8s facts. The following number string focuses on developing the Doubling strategy (Bray & Maldonado, 2018):

$2 \times 6 \quad 4 \times 6 \quad 6 \times 4 \quad 3 \times 4 \quad 2 \times 7 \quad 4 \times 7 \quad 6 \times 8$

As you can see, doubling can address many of the remaining facts in the multiplication facts table.

Add a Group. This strategy helps with several facts groups and becomes a strategy that is used for multiplying multi-digit numbers. To begin, 3s facts can be thought of as ×2 plus one more group of _____ (see Figure 9. 14(a)). Next, 6s facts can be thought of as ×5 and 1 more group of _____ (see Figure 9.14(b)).

When students know their 5s well, they can generalize this strategy to solve 7s facts and 8s facts. For example, students who know 5×6 can start with that and add two more groups of 7 to solve 7×6.

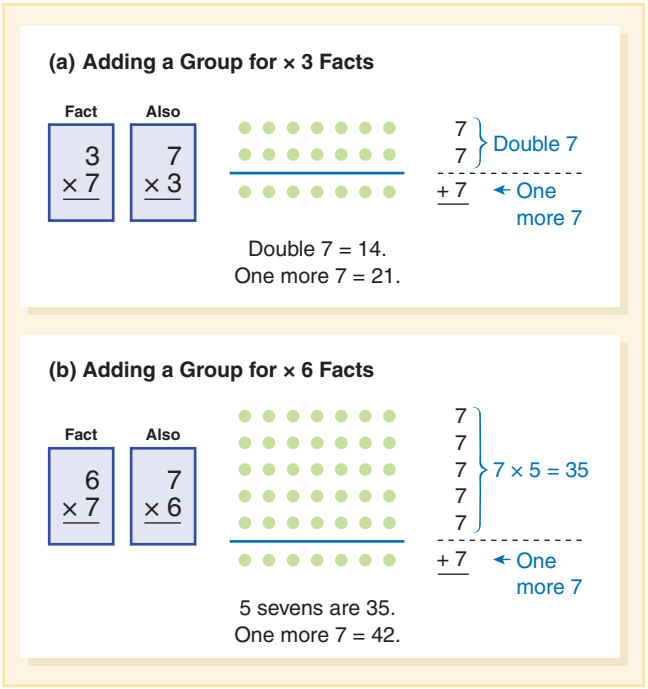


FIGURE 9.14 Adding a group reasoning strategy

Subtract a Group. This strategy is very useful for learning the ×9 facts in a conceptual way. Of course, there are a number of ways to remember the ×9 facts and interesting patterns to explore in the products: (1) the tens digit of the product is one less than the non-nine factor, and (2) the sum of the two digits in the product equals 9. But seeing how 9s can be found by using 10s builds a conceptual relationship that can be used to solve problems like 19×5 (think $20 \times 5 - 5$). Introduce this idea by showing a set of bars such as those in Figure 9.15 with only the end cube a different color. After explaining that every bar has 10 cubes, ask students if they can think of a good way to figure out how many are yellow.

Subtracting a group can also be used to figure out 4s facts by knowing the 5s facts, but typically, students prefer to use doubling to generate their 4s facts.

Near Squares. This strategy is both Adding a Group and Subtracting a Group. A student sees a fact like 7×6 and can choose whether they want to start with 6×6 (six groups of 6) and add a group of 6 or if they want to start with 7×7 (seven groups of 7) and subtract a group of seven. Notice that in both cases, it is clear how big the group is because both factors are 6 (or both are 7).

Break Apart. The strategies described so far are what are called break-apart strategies. After students have learned different ways they can break apart (e.g., doubling or adding a group), they need to see that they can break a factor apart in a way that makes the problem manageable. For example, for 8×6, you can decompose 8 into $5 + 3$ to multiply the parts by 6: 5×6 equals 30 and 3×6 equals 18. The sum of 30 and 18 equals 48. Break-apart in this way is the distributive property: $6(5 + 3) = 6 \times 5 + 6 \times 3 = 30 + 18$. A multiplication array with lines through a 10-by-10 array to show fives (known facts) can help students see how they might apply the distributive property and decompose a factor.

Pearson eText Blackline Master: Multiplication Array

Pause & Reflect

How might students use break-apart to solve the facts illustrated in the array cards in Figure 9.16? ●

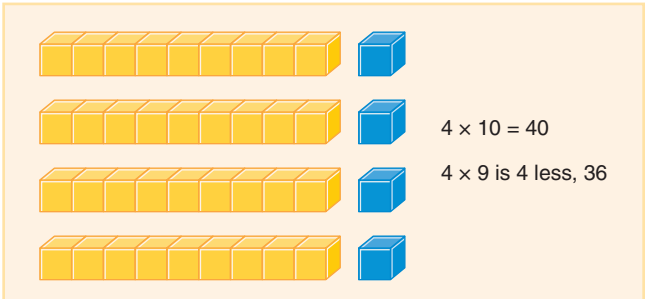


FIGURE 9.15 Using 10s to determine 9s facts

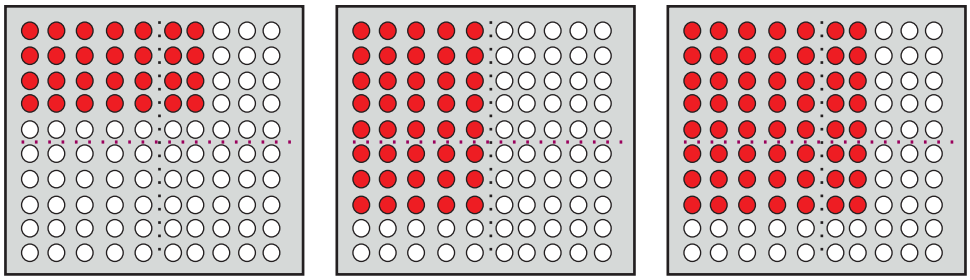


FIGURE 9.16 A multiplication array can illustrate how to decompose a factor (apply the distributive property)

Division Facts. Automaticity with multiplication facts and understanding the relationship between multiplication and division are key to learning division facts. For example, to solve 36 ÷ 4, we tend to think, "Four times what number is thirty-six?" In fact, because of this approach, the reasoning facts for division are to (1) think-multiplication and then (2) apply a multiplication reasoning fact, as needed. Missing factor stories can assist in making this connection.

Aaron and Michael are creating bags of mini-muffins for a bake sale. They baked 36 muffins and are putting four in a bag. How many bags will they be able to fill?

Notice this story can be represented as $4 \times$ _____ $= 36$ (missing factor) or in an equivalent equation as $36 \div 4 =$ _____ (division).

Beyond division facts, students benefit from practicing near-division facts (e.g., $50 \div 6$), because near-division situations are more likely to happen in real life, required for estimating and required for solving larger division problems. To determine the answer to $50 \div 6$, students employ a guess-and-check strategy: 6 times 7 (low), 6 times 8 (within 2; that must be it). Students should be able to do these near-fact problems mentally and efficiently.

Activity 9.21 How Close without Going Over?

CCSS-M: 3.OA.A.2; 3.OA.A.4; 3.OA.B.6

Division often involves figuring out the nearest basic fact. This activity engages students in figuring out what that nearest fact is and writing the remainder. Begin by posting missing factor equations such as these:

$4 \times \square \rightarrow 23$, _____ left over

$6 \times \square \rightarrow 27$, _____ left over

$\square \times 7 \rightarrow 55$, _____ left over

Ask students to decide mentally on the missing factor and be ready to share. Then ask several students to share their factor and how they figured it out. Finally, ask what the left-over will be for that factor. This can be repeated with equations written as division:

$40 \div 9 \rightarrow \square$ _____ left over

Equations like these can be cut out from worksheets or recorded quickly on notecards, and partners can flip the cards over and take turns finding the missing values.

Games to Practice Multiplication and Division Facts and Strategies

As noted several times in this chapter, students need ample time in Phase 2 (using reasoning strategies) before they will move to automaticity. Games with no time pressure provide students the time to select a strategy, think through that strategy, and listen to their peers describe their thinking about strategy choices. All year long, learning centers can provide opportunities for ongoing practice. Here are a few popular games that can be played over and over again. These games help students look for known facts in an unknown fact problem. Activity 9.22 uses arrays, helping students see how to break apart factors.

Activity 9.22 Game: Strive to Derive

CCSS-M: 3.OA.A.1; 3.OA.B.5; 3.OA.C.7

Materials: For each pair of students: multiplication array cards; a straight, thin stick (like uncooked spaghetti or a coffee stirrer); and two dice, one labeled with 3, 3, 6, 6, 9, and 9; one with 0, 1, 4, 6, 7, and 8

Students play with a partner. They spread array cards on the table so they can be seen. Player 1 rolls the dice and selects the related array card. Player 1 then places the stick to partition the array into known facts. For example, Player 1 rolls a 3 and a 6, she finds the 3-by-6 or 6-by-3 array card. She places the piece of spaghetti to show a partition. One option is to partition it as 2×6 and 1×6. Another option is to partition the array as 3×5 and 3×1. If Player 1 can solve the fact using derived facts, she scores 1 point. Array card is returned to the table. Player 2 repeats the process. Continue to 10 points. Initially, or to modify for students who struggle, focus on one foundation set of facts, for example, Strive to Derive by 5.

Pearson eText Activity Page: Multiplication Array Cards

Activity 9.23 focuses on one fact set. Students can be focusing on a strategy (e.g., doubling) and then play Bump for their 4s facts. The key is that students are encouraged to use a strategy (not skip count) as they are moving toward automaticity. The more they play, the better students become at using strategies. Encourage students to talk about their strategies and to self-monitor, working toward using efficient strategies.

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

Activity 9.23 Game: Bump Multiplication

CCSS-M: 3.OA.A.1; 3.OA.B.5; 3.OA.C.7

Materials: Deck of cards, piece of blank paper

This is a partner game. The players begin by writing all the multiples of a fact set (e.g., 4s) on a piece of paper (mix up the order of products to discourage skip counting as you play; see below). On a player's turn, they draw a card and multiply the card by the selected fact (e.g., 4) and place a counter under that product. For example, if they draw a 6, they can claim 24. If the player already has that number, they lose a turn, and play goes to their opponent, but if their opponent has that product, they get to bump them off the game board and claim that number. The first player to claim 5 of the multiples wins the game.

Multiplication games that involve a game board with the products on it (like Bump) actually engage students in thinking about multiplication and division, because students are thinking, "I need to get 28. What times 4 equals 28? 7." The classic game in Activity 9.24 engages students in both multiplication and division (and can also be played with addition and subtraction).

Activity 9.24 Game: Salute!

CCSS-M: 1.OA.B.4; 1.OA.C.6; 2.OA.B.2; 3.OA.B.5; 3.OA.B.6; 3.OA.C.7

Materials: Deck of ten-frame cards or regular cards (kings and jacks removed, queens = 0, aces = 1)

Salute! is a popular, fun game that can be played over and over. It practices both addition and subtraction facts (or multiplication and division). Place students in groups of three and give each group a deck of cards. Two students draw a card without looking at it and place it on their forehead facing outward (so the others can see it). The student with no card tells the sum (or product). The first of the other two to correctly say what number is on their forehead wins the card set. Each student explains how they found the sum/product or figured out the card on their forehead. After each student explains, rotate roles and continue play. For many learners, including students with special needs in math and multilingual learners, speed can inhibit participation and increase anxiety, so remove the speed component of winning the cards and just take turns. This game can be differentiated by including only certain cards (e.g., addition facts using only the numbers 1 through 5). For game instructions and a Jamboard, go to http://kcm.nku.edu/mathfactfluency/game19.php.

Pearson eText Activity Page: Little Ten-Frame Cards

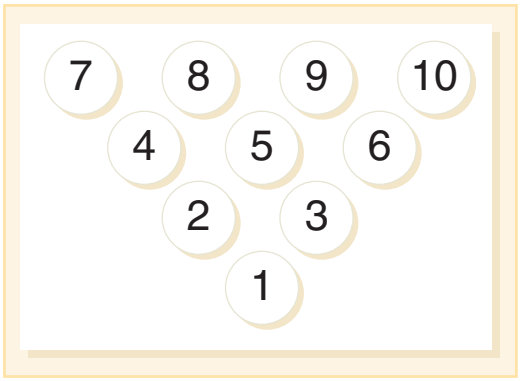
Mixed Facts and More Games

As students are learning their multiplication and division facts, they need continued opportunities with their addition and subtraction facts. Bowl-a-Fact (Shoecraft, 1982) is an excellent activity to do this, and it can be used as a routine, a learning center, or a game.

Activity 9.25 Bowl-a-Fact

CCSS-M: 2.OA.B.2; 3.OA.C.7; 5.OA.A.1

You will need three regular dice. Use the Activity page or just draw circles placed in triangular fashion to look like bowling pins, as illustrated here. For culturally diverse classrooms, be sure that students are familiar with bowling. (If they are not, show a video clip.)



Player(s) roll the three dice one time. Students use all the numbers on the three dice to come up with equations that result in answers that are on the pins. For example, if you roll 4, 2, and 3, they can knock down the 5 pin with $4 \times 2 - 3$. If they can produce equations to knock down all 10 pins, they get a strike. If they get stuck, they roll again and see whether they can knock the rest of the pins down for a spare.

Pearson eText Activity Page: Bowl-a-Fact

Importantly, games that involve game boards, like Pick 2 to Connect 4 (Activity 9.17), Take It Away (9.18), Squares Score (Activity 9.19), and Bump Multiplication (Activity 9.23), are easy to adapt to other fact sets or strategies. For example, Squares Score could be a game board with all the multiples of 8. Pick 2 to Connect 4 and Squares Score are based on the classic game, Connect 4. Table 9.1 offers additional ideas for how classic games can be adapted to focus on basic facts. (See also Forbringer & Fahsl, 2010 and Bay-Williams & Kling, 2019).

***TECHNOLOGY* Note.** There are many online resources to support the teaching and practicing of facts. However, many are timed and only focus on drill. These resources can work against real fact fluency. Look for resources that include visuals and incorporate reasoning strategies. The Math

TABLE 9.1 CLASSIC GAMES ADAPTED TO BASIC FACT FLUENCY

Classic Game	How to Use It with Basic Fact Fluency	Suggestions for Differentiation
Bingo	Give each student a 5×5 blank Bingo board. Write or post the answers for the facts you are going to use, and have students record them to create their own Bingo card. For example, if you are playing 5s Bingo, write all the multiples of 5 on the board. Roll a 10-sided die. When you roll a 4, students can cover *one* of the 20s they wrote down.	Any fact set or strategy can be used; simply list the possible answers on the board for students to make their Bingo card.
Concentration	Create cards that have a fact problem (e.g., 3×5) on one half and the answers (e.g., 15) on the other half. Shuffle the cards and turn them face-down in a 6×4 grid. (If you like, you can make the grid larger to use more cards.)	Select cards that focus on a particular cluster of facts (e.g., $+1$ and $\times 5$ facts) for each round of the game. Multiple groups can play the game simultaneously—each group will use the parts of the deck that contain the facts they are working on.
Dominoes	Create (or find online) dominoes that have a fact on one side and an answer (not to that fact) on the other. Each student gets the same number of dominoes (around eight). On their turn, students can play one of the dominoes in their hand only if they have an answer or a fact that can connect to a domino on the board.	As with other games, select the dominoes that focus on a particular cluster of facts.

Learning Center (https://www.mathlearningcenter.org), mentioned earlier, and Brainingcamp (brainingcamp.org) have apps to help you create images to help students visualize fact strategies. The Kentucky Center for Mathematics (http://kcm.nku.edu/mathfactfluency/) hosts a *Math Fact Fluency* companion website (see book in Recommended Readings). The site includes about 20 games ready to download and use or to play online, and assessment ideas, too! ■

Assessing Basic Fact Fluency

Learning Outcome 9.4 Justify limitations of timed tests and explain effective assessment alternatives.

A glance back at Chapter 5 will illustrate many formative assessment strategies: observations, interviews, performance tasks, and writing. Why do we use formative assessment strategies? To figure out what students know and what they do not know so that we can focus instruction on what students need to know, do, or understand. Formative assessment improves student achievement (Wiliam, Lee, Harrison, & Black, 2004; Wiliam & Thompson, 2007). Learning basic facts must use such effective assessment practices and *not* timed tests or Mad Minutes.

The Problem with Timed Tests

There are a number of issues with timing basic fact tests. First, timed tests do not assess fluency. Recall that fluency includes efficiency, flexibility, and accuracy. A test gives no insight on whether students chose an efficient strategy or were flexible. Second, they are designed to measure automaticity (answering quickly), but you don't know how much time was spent on each question, so you don't know whether a student is automatic. Third, the scores rarely provide the data you need to ensure that the students know all their facts: What does a score of 85% on a Mad Minute mean? Fourth, timed tests negatively affect development of number sense and recall of facts (Henry & Brown, 2008). Fifth and foremost, there is growing evidence that young children are experiencing anxiety specific to mathematics in the early grades, and timed tests cause stress for students (Boaler, 2012, 2014; Ramirez, Gunderson, Levine, & Beilock, 2013).

Timed tests create stress even for students who perform well (Boaler, 2012, 2014). Perhaps a surprising finding, students with the highest working memory capacity (i.e., stronger reasoners) show the strongest negative relation between math anxiety and math performance (Beilock & Carr, 2005; Ramirez et al., 2013; Vukovic, Kieffer, Bailey, & Harari, 2013). When those first graders and second graders with strengths in reasoning face anxiety-inducing situations, they tend to rely on inefficient strategies that use less working memory or they make mistakes when using strategies (Ramirez, Chang, Maloney, Levine, & Beilock, 2016). This means that when we put students in high-stress situations, like timed tests, we are pushing

them back to Phase 1 rather than supporting their progress to Phase 3. Not surprisingly, then, those high-performing and highly anxious students perform worse (Ramirez et al., 2013).

In sum, there are many reasons why *not* to use timed tests, and there is *no good reason* to use timed tests. Instead, use high-quality assessments, such as observations, interviews, journal writing, and student self-assessments.

Observing Basic Fact Fluency

Perhaps your biggest role as a teacher is noticing student thinking. For basic facts, you are looking to see whether (1) they have learned the facts I am focusing on and (2) are adept at using the strategies we are learning. EQUATIONObservation tools like those featured in Figure 9.17 can help you keep a record of what facts students are learning and what strategies they are using.

Option 1: Focus on One Fact Set **Students are:** Playing Squares Score (Activity 9.20)

Squares Facts	1 × 1	2 × 2	3 × 3	4 × 4	5 × 5	6 × 6	7 × 7	8 × 8	9 × 9
P. Arnold									
J. Booth									

Option 2: Focus on Addition Foundational Facts **Students are:** Playing games with fact sets

Foundational Facts	±0, 1, 2	Doubles	Combos of 10	10 + __
P. Arnold				
J. Booth				

Option 3: Focus on Multiplication Strategies **Students are:** Playing strategy games

Foundational Facts	Doubling	Adding a Group	Subtracting a Group	Near Doubles	Other
P. Arnold					
J. Booth					

FIGURE 9.17 Observation tool possibilities

Interviews and Journals

Whether students are talking or writing, you are gaining insights into how they are solving the problems. When interviewing, you are able to notice all components of fluency, so you can see whether the student is automatic with a fact. Let's say you wanted to know whether a student knows the foundational facts listed in the aforementioned observation tool. You do not need to ask all of those facts, just a sampling! So you might pick three facts from each category and just ask the student to tell you the answer and how they know. If they are automatic, this will take less than one minute. If you want to know whether a student can use a strategy, then you can give a few facts that align with that strategy's use and ask them to give you the answer and tell you how they thought about it. Activity 9.26 is good for assessing students' flexibility and could be used as an interview or a journal-writing prompt.

Activity 9.26 If You Didn't Know

CCSS-M: 1.OA.C.6; 2.OA.B.2; 3.OA.B.5; 3.OA.C.7

Pose the following task: If you did not know the answer to 8 + 5 (or any fact that you want students to think about), how could you figure it out without counting? Encourage students to come up with more than one way (hopefully using the strategies suggested previously). Multilingual learners, and many other students, benefit from first sharing their ideas with a partner before sharing with the whole class.

Additional ideas are provided in Table 9.2.

TABLE 9.2 ASSESSING BASIC FACT FLUENCY

	Assessment Strategy		
Fluency Components	**Observation**	**Interview Probes**	**Writing (Journals or Tests)**
Flexibility	Does the student use different strategies as they play games or do activities?	Nicolas solved $6 + 8$ by changing it in his mind to $4 + 10$. What did he do? Is this a good strategy? Tell why or why not. Solve 6×7 using one strategy. Ask: Is there another way you could solve it?	Explain how you think through solving these two problems: $13 - 3 =$ $12 - 9 =$ Review the multiplication table. Write which facts are your toughies. Next to each one, tell a strategy that you want to remember to use.
Efficiency	Does the student pick a strategy that makes sense for that fact? For example, for $9 + 2$ they might count on, but not for $9 + 6$. How long does it take to select and enact a strategy?	Go through this stack of cards and sort by the ones you *just know* and the ones you *are still working on*.	Solve these (basic fact) problems. (Provide a set of 10.) If you *just knew* the answer, circle it. If you *used a strategy,* write the strategy's name (e.g., Close Fact).
Accuracy	Is the student applying the strategies correctly? Which facts is the student consistently getting correct?	What is the answer to 7×8? How do you know it is correct (how might you check it)?	Have students solve five facts then explain how they know they are correct.

Addressing Challenges Related to Basic Fact Fluency

Learning Outcome 9.5 Explain effective methods for ongoing practice and intervention for basic fact fluency.

Students must learn their facts and remember them! Even if you teach fifth grade and basic facts are not in your curriculum, you will need to provide interventions to ensure that your students develop automaticity. Here are some tips to support you in ongoing support of fact fluency.

Ongoing Support for Fact Fluency

Drill is not an intervention! Students who do not know their facts may be stuck back in counting strategies (Phase 1) and likely need stronger number sense and reasoning strategies (Phase 2). Effective intervention attends to the three phases, with a strong focus on strategy development. These tips can help:

1. **Inventory the known and unknown facts and reasoning strategies**. Find out which facts are known quickly and comfortably and which are not (see assessment ideas discussed previously). Invite students to do this for themselves as well. Listen and watch for which reasoning strategies they know and are using. Provide sheets of mixed multiplication facts and ask students to answer the ones they just know and circle the facts where they have to pause to count or use a strategy. Review the results with them and discuss which strategies and facts to work on.
2. **Explicitly teach reasoning strategies**. When a student cannot recall their facts, more drill is not effective. Students need instruction on the reasoning strategies. Use visuals and stories and provide opportunities for students to think aloud as they apply the strategies. For example, if a student does not know the sums from 11 to 19, then use ten-frame quick looks to illustrate how Make 10 and Pretend-a-10 strategy work. Then provide opportunities for them to practice in an enjoyable way (e.g., play a game like the ones in this chapter).

3. **Build in success**. Begin with easier and more useful reasoning strategies like Combinations of 10 for addition. Success builds success! Have students find all the facts that can be solved with a newly learned strategy. Use fact charts to show the set of facts you are working on. It is surprising how the chart quickly fills up with mastered facts.
4. **Provide hope by talking about their strengths**. As noted in the discussion of timed tests, students' confidence can be affected if we only focus on their weaknesses and what they don't know. Students may feel they are doomed to use finger counting forever. Let these students know that they will explore strategies that will help them with the facts. Turn off the timers. Shorten (or eliminate) the quizzes. Spot their strengths and share them!

Common Challenges and How to Help

Table 9.3 shares some of the common challenges as well as ways to help students avoid or overcome them.

TABLE 9.3 COMMON CHALLENGES RELATED TO BASIC FACT FLUENCY

Common Challenge	What It Looks Like	How to Help
1. Noticing and using the commutative property	Student is slower to add $2 + 8$ than $8 + 2$ (they may be counting on from the first number). Student can solve 3×8, but not 8×3.	For addition, Use a ten-frame for each addend and reverse them. Add contexts that can be reversed (part-part-whole stories). For multiplication, use areas and arrays that can be rotated to show that $3 \times 8 = 8 \times 3$.
2. Recognizing inverse relationships	Student is not able to think of $14 - 9$ as "What plus 9 equals 14?" or $36 \div 4$ as "How many groups of 4 are in 36?"	Use the triangle cards and/or T cards (see Activity 9.13). Explicitly practice the language used here.
3. In multiplication, applying adding (or subtracting) a group, students lose track of the group size.	For 9×8, student uses 7×6 but instead of subtracting one *group* of 8, they subtract 1, getting the answer of 79.	Provide examples of the problems, using an array. Show how the known fact is used to add or subtract a row (or column) from the array. Emphasize the group or row.

RESOURCES FOR CHAPTER 9

WRITING TO LEARN

1. Describe the phases of the learning progression for basic facts, using your own examples to illustrate the thinking at each phase.
2. How can you develop meaning for the basic fact strategies?
3. What are the three major reasoning strategies for addition facts, and how might a student use them to add $7 + 9$?
4. Describe two reasoning strategies a student might use to multiply 7×6.
5. Why are games and interactive software important in supporting basic fact fluency?
6. This chapter suggests an approach to assessing facts that may be quite different from your own experience. Describe 3–5 ideas from this chapter that you will use and/or avoid using.

LITERATURE CONNECTIONS

There are so many wonderful books to link to basic facts! The books in Chapters 7 and 8 also fit basic facts. The Columba (2013) article, "So, Here's the Story," from *Teaching Children Mathematics* describes literature links and activities for each of the multiplication facts. Here are just some of the many great books:

A Place for Zero

LoPresti (2003)

This clever, enjoyable picture book focuses on the meaning of zero in addition and multiplication.

Arithmechicks Add Up and Arithmechicks Take Away

Stephens (2019, 2020)

These engaging books focus on addition or subtraction facts and include a discussion of reasoning strategies at the end of the books.

One Is a Snail, Ten Is a Crab: A Counting by Feet Book

Sayre & Sayre (2006)

The feet in this book are different-sized groups. This book is so clever and leads to wonderful opportunities to explore skip counting, strategies for multiplying, and even exploration of numbers beyond the basic facts!

One Less Fish

Toft and Sheather (1998)

This beautiful book with an environmental message starts with 12 fish and counts back to 0 fish. On a page with 8 fish, ask, "How many fish are gone?" and "How did you figure it out?" Encourage students to use the Down Over 10 strategy. Any counting-up or counting-back book can be used in this way!

Sheep Won't Sleep

Cox (2019)

These sheep need to be counted in 2s, 5s, and 10s. A great connection to foundation facts for multiplication.

Ten Magic Butterflies

McKellar (2019)

Written by the actress Danica McKellar, this colorful book focuses on Combinations of 10 and includes teaching suggestions.

The Twelve Days of Summer

Andrews and Jolliffe (2005)

You will quickly recognize the style of this book with five bumblebees, four garter snakes, three ruffed grouse, and so on. The engaging illustrations and motions make this a wonderful book. Students can apply multiplication facts to figure out how many of each item appears by the end of the book. (For example, three ruffed grouse appear on days 3, 4, 5, and so on.)

What's New at the Zoo? (addition) and What's the Difference? (subtraction), Multiply on the Fly (multiplication), and The Great Divide (division)

Slade (2009, 2010, 2011, 2012)

Each of these books blends insect facts about animals with basic (math) facts. Readers will explore a variety of contexts that help students make sense of each operation.

RECOMMENDED READINGS

Articles

Baroody, A. J. (2006). Why children have difficulties mastering the basic fact combinations and how to help them. *Teaching Children Mathematics*, *13*(1), 22–31.

Baroody suggests that basic facts are developmental in nature and contrasts "conventional wisdom" with a number-sense view. Great activities are included as exemplars.

Boaler, J. (2014). Research suggests that timed tests cause math anxiety. *Teaching Children Mathematics*, *20*(8), 469–474.

This wonderful article challenges the long-standing practice of timed tests. Just because we have always done this, it doesn't mean it's a good idea!

Buchholz, L. (2016). A license to think on the road to fact fluency. *Teaching Children Mathematics*, *22*(9), 557–562.

This is a beautifully written reflection on how this second-grade teacher had departed from her daily strategy focus (due to a new curriculum), realized the error in her ways, and her return to daily strategy use. She describes pragmatic ways she ensures that every student develops strategies and fact fluency. (See also Buchholz (2004) for great strategy ideas).

Kling, G., & Bay-Williams, J. M. (2014). Assessing basic fact fluency. *Teaching Children Mathematics*, *20*(8), 489–497.

Whereas the previously mentioned Boaler article describes the problem with timed tests, these authors reinforce those concerns but go on to offer a rich collection of alternatives to assess students' journey to fluency better.

Books

Bay-Williams, J., & Kling, G. (2019). *Math fact fluency: 60+ games and assessment tools to support learning and retention.* Alexandria, VA: ASCD.

Full of games and assessment tools, this book provides a plan for implementing a research-based, engaging fact fluency plan in your classroom, including ideas for family engagement.

O'Connell, S., & SanGiovanni, J. (2011). *Mastering the basic math facts in multiplication and division: Strategies, activities and interventions to move students beyond memorization.* Portsmouth, NH: Heinemann.

Both this book and the partner book about addition and subtraction (same authors and parallel title) are excellent resources for planning and teaching basic facts effectively!

CHAPTER

10 Developing Whole-Number Place-Value Concepts

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **10.1** Recognize the foundational ideas of place value as an integration of three components: base-ten concepts through groupings and counting, numbers written in place-value notation, and numbers that are spoken aloud.
- **10.2** Demonstrate how to develop students' skills in place value through the use of base-ten models.
- **10.3** Explain how students can use grouping activities to deepen their understanding of place-value concepts.
- **10.4** Explain strategies to support students' ability to write and read numbers.
- **10.5** Recognize that there are patterns in our number system that provide the structure for computational strategies.
- **10.6** Describe how the place-value system extends to large numbers.

Understanding and using place value is fundamental to mathematics, particularly for the operations. Consider the sum $49 + 18$. How would you add it in your head? Many people will take one from the 18 and move it to the 49 to create the equivalent expression $50 + 17$ (or move two in the other direction to get $47 + 20$). In other words, they are making tens to add more readily. A significant part of place value development includes students putting numbers together (*composing*) and taking them apart (*decomposing*) in a wide variety of ways as they begin operations with two-digit numbers.

Place value is also a way for students to think about larger quantities (Mix, Smith, & Crespo, 2019). Place-value knowledge is a critical milestone in students' understanding of many topics in mathematics. Yet, Rodgers (2012, p. 648) notes, "[D]espite the unchanging and recursive nature of our base-ten system, it seems some students never manage to fully unravel the hidden code that underlies place value." Let's change that!

The following big ideas are the foundational concepts that will lead students to a full understanding of place value and its importance in computation.

BIG IDEAS

- Sets of 10 (and tens of tens) can be perceived as single entities or units; for example, three units of size 10 and two units of size one describes 32 single objects.
- The positions of digits in numbers determine what they represent and which size unit they count. This is the major organizing principle of place-value numeration and is central for developing number sense.
- There are patterns to the way that numbers are formed. For example, each decade has a symbolic pattern reflective of the 0-to-9 sequence (e.g., 20, 21, 22 . . . 29).
- The groupings of ones, tens, and hundreds can be taken apart in different but equivalent ways. For example, beyond the typical way to decompose 256 of 2 hundreds, 5 tens, and 6 ones, it can be represented as 1 hundred, 14 tens, and 16 ones, or 25 tens and 6 ones. Decomposing and composing multi-digit numbers in flexible ways is a necessary foundation for computational estimation and exact computation.
- Really big numbers are difficult to conceptualize and are best understood in terms of familiar real-world referents. The number of people who will fill the local sports arena is, for example, a meaningful referent for those who have experienced that crowd.

Developing Whole-Number Place-Value

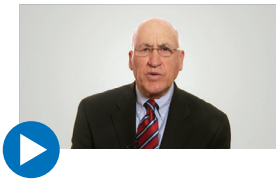
Learning Outcome 10.1 **Recognize the foundational ideas of place value as an integration of three components: base-ten concepts through groupings and counting, numbers written in place-value notation, and numbers that are spoken aloud.**

Students develop these big ideas by:

- Moving from counting forward and backward in place value parts (i.e., tens, hundreds, etc.)
- Representing the value of a number using materials, reading and writing numbers including knowing the value of a given digit
- Regrouping and partitioning numbers in a variety of ways such as there are 43 tens in 437
- Putting numbers in order by magnitude, and
- Using their knowledge of magnitude to estimate (Rogers, 2012).

Let's start by looking at place-value parts made through grouping.

Place-value understanding requires an *integration* of new and sometimes difficult-to-construct concepts of grouping by tens (the base-ten concept) with procedural knowledge of how groups are recorded in our place-value system and how numbers are written and spoken. Importantly, learners must understand the word *grouping*, especially multilingual learners who may become confused because the root word *group* is frequently used for instructing students to work together. Table 10.1 provides a progression of how grouping strategies develop (based on the work of Fuson (2006)).



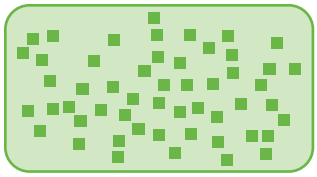
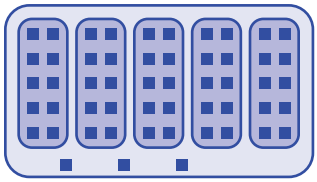
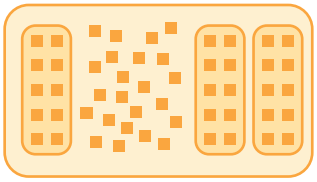
Pearson eText
Video Example 10.1
In this video former NCTM President Francis "Skip" Fennell discusses the big ideas of place value.

Unitary: Counting by Ones

Kindergarteners know a lot about numbers with two digits (10 to 99) as they learn to count to 100 and count out sets with 20 or more objects. They count students in the room, turn to specific page numbers in their books, and so on. However, initially their understanding is based on a count-by-ones approach to quantity, so the number 18 to them means 18 ones. For example, after counting 18 teddy bears, if you ask a student what the 1 and 8 mean in 18, they might say the 1 stands for 1 teddy bear and the 8 stands for 8 teddy bears. Such students are considered pre–place value because they have not had enough experiences to realize we are always grouping by tens. Let's look at a diagnostic interview to assess where students are in this progression.

FORMATIVE ASSESSMENT Notes. Ask students to count out 53 tiles from a collection on the table. Have the students write the number that tells how many tiles they just counted.

TABLE 10.1 PLACE VALUE GROUPING PROGRESSION.

Grouping Stage	Visual Representation	Counting Approach	Students' Look-Fors:
UNITARY Perceives 10 as ones Counts by ones		1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and so on	• Name a quantity or "tell how many" by counting each piece. • Not yet able to think of 10 as a single unit. • Use counting by ones as the only way they are convinced that different sets have the same amount.
BASE-TEN Recognizes 10 as a unit Counts by groups of tens and ones		1, 2, 3, 4, 5 groups of 10 and 1, 2, 3, ones (singles) or 10, 20, 30, 40, 50, 51, 52, 53	• Count a group of 10 objects as a single item (unitizing). • Coordinate the base-ten approach with a count by ones as a means of telling "how many."
EQUIVALENT Fluently works with groups of 10 Can count by using nonstandard base-ten groupings		Before counting, students would trade and then count 10, 20, 30, 40, 50, 51, 52, 53 but can make flexible groupings	• Group the pieces flexibly into versions that include tens and ones, but all possible trades have not been carried out. • Use these alternate groupings to relate to computation by being able to trade or regroup numbers in a variety of ways.

Then ask the students how they know they have that number of tiles. Here are some questions to consider to help identify students' strengths and next instructional steps:

- Did they write a reversal of "35" instead of "53?"
- Do they show evidence that they are thinking of ten as a unit by automatically grouping them or are they still in a pre–place value stage and counting them by ones?
- Do students explain their thinking about how they know they have 53 by creating groups of 5 tens, and so on, or do they just say "because I counted them."
- Do they provide evidence that they know more than the names of the positions and show understanding that the 5 in the tens place represents five groups of ten and the 3 represents three single things?

Fuson refers to students' pre–base ten understanding of number as *unitary*. That is, there are no groupings of ten, even though a two-digit number is associated with the quantity. Once students can count out a set of 53 by ones, encourage them to see that making groupings of tens and leftover ones is another way of counting that same quantity. Each of the sets in Table 10.1 has 53 tiles, yet students move through three distinct place-value grouping stages to construct the idea that all of these sets are the same amount. ■

Base-Ten Grouping

There is a subtle yet profound difference between students at the stages in the learning progression: Where *unitary* stage students count only by ones, students in the *base-ten* grouping stage interpret ten as a unit and thereby understand the idea that 5 groups of ten and 3 more is the same amount as 53 counted by ones. The students who understand this relationship will see no need to count the base-ten grouping by ones. They understand the fifty-threeness of the *unitary* and *base-ten* grouping to be the same. But students in the *unitary* stage may be unsure of how many they will get if the groups were ungrouped.

A priority is to help students integrate what they already know about numbers from counting by ones with the grouping-by-tens concept. If they still are only counting by ones, ask, "What will happen if we count these items by tens and ones (or groups and singles)?" If a set has been grouped into tens and singles and counted, then ask, "How can we be really

certain that there are 53 things?" or "How many do you think we will get if we count by ones?" You cannot just *tell* students that these counts will all be the same and expect that it will make sense to them—it is a relationship they must construct through continued experiences.

Equivalent Grouping

The third stage includes groupings with fewer than the maximum number of tens and is referred to as *equivalent groupings*. Understanding the equivalence of *base-ten grouping* and *equivalent grouping* indicates that grouping by tens is not just a rule that is followed, but also that any grouping by tens, including all or some of the singles, can help tell how many. Many computational techniques (e.g., regrouping in addition and subtraction) are based on equivalent representations of numbers.

Standards for Mathematical Practice
MP7. Look for and make use of structure

Integrating Base-Ten Groupings with Words

The *standard language* we use to say a number such as fifty-three must be connected with the grouping-by-tens concept. The count by tens and ones results in saying the number of groups and singles separately: "five tens and three" is called *base-ten language*. Students benefit from connecting base-ten language with the standard language: "five tens and three—fifty-three," as shown in Figure 10.1.

There are several variations of the base-ten language for 53: 5 tens and 3, 5 tens and 3 ones, 5 tens and 3 singles, and so on. Each may be used interchangeably with "fifty-three." If you have multilingual learners, it is best to select one approach (e.g., 5 tens and 3 ones) and consistently connect it to the standard language. Other languages often use the base-ten format (e.g., 17 in Spanish is *diecisiete*, literally meaning "ten and seven"), so a way to be culturally responsive is to use the base-ten variation that best connects to the languages known by your students.

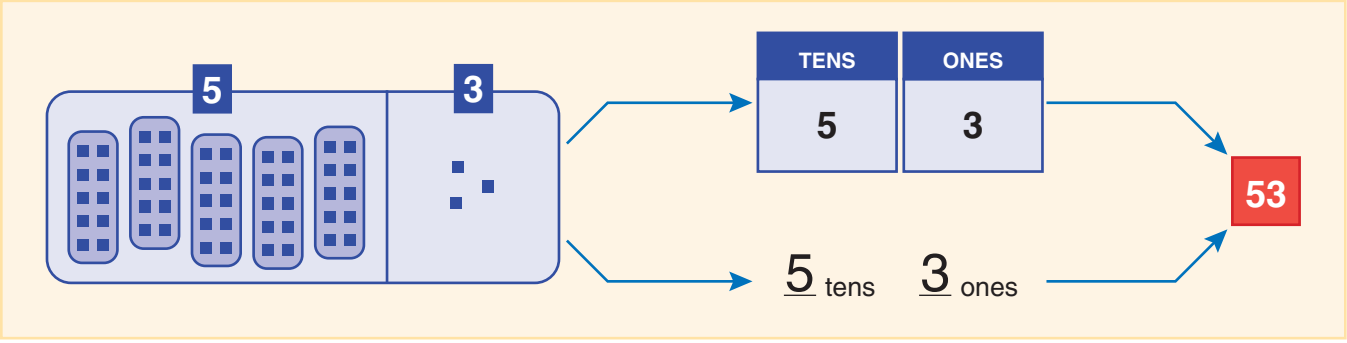


FIGURE 10.1 Connecting quantity, base-ten language, and standard language

Be precise in your language. Whenever you refer to a number in the tens, hundreds, or thousands (or beyond), make sure you do not just say "six," but instead refer to it with its place-value location, such as "6 tens" (or 60). Students are often confused when numbers are discussed as digits rather than as their actual value.

Standards for Mathematical Practice
MP6. Attend to precision

Integrating Base-Ten Groupings with Place-Value Notation

Notation of numbers evolved in different cultures in different ways. Howe (2020) refers to five stages of historical progression of notation related to place value:

1. Tallies
2. Grouping
3. Iterated grouping, (i.e., establishing a multiplicative hierarchy of units or a base)
4. Inventing digits (Not all cultures invented their own; some borrowed from other cultures.)
5. Inventing zero, which, together with the earlier stages, enables place value notation.

The symbolic scheme that we use for writing numbers (ones on the right, tens to the left of ones, and so on) must be coordinated with the grouping scheme. The explicit counting

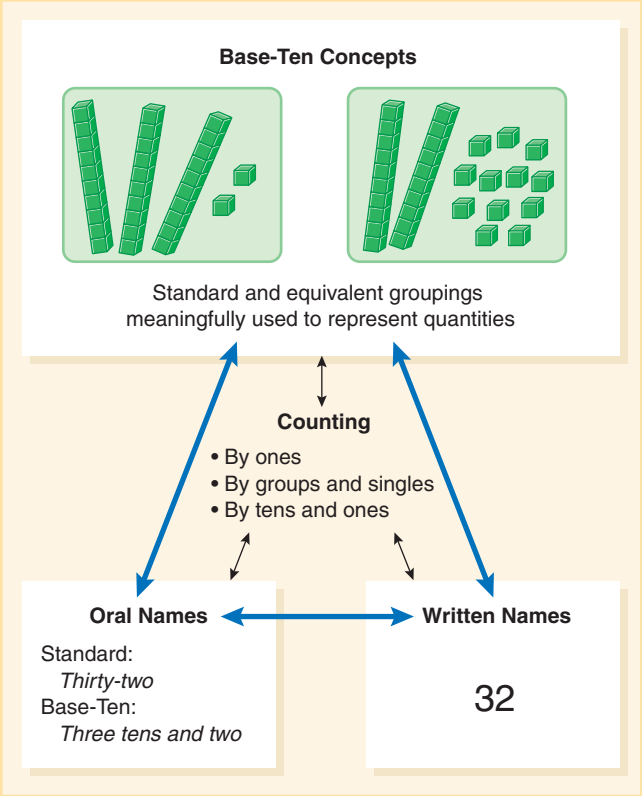


FIGURE 10.2 Integrated place-value understanding

by groups and singles matches the individual digits as the number is written in the usual left-to-right manner. A similar coordination is necessary for hundreds and other place values. Students will initially find it challenging to see "ten" as both 10 ones and 1 ten, which is why many concrete experiences are necessary.

Figure 10.2 summarizes the ideas of the integrated place-value understanding discussed so far. Note that all three methods of counting shown in the corners of the triangle support relational understanding when students can flexibly integrate base-ten concepts, written names, and oral names. Language again plays a key role in making these connections.

The mathematician Roger Howe developed another progression of five stages of place-value notation (Howe & Reiter, 2012; Howe, 2019; Howe, 2020) that become more sophisticated over time. This progression goes beyond the historical development of numerals to students' understanding of how numbers are built in the base-ten system we use. These stages reflect the underlying structure that supports students' ability to take a number apart down to its base-ten components. Let's look at this progression, using examples (see Table 10.2)

These stages need to be reinforced through experiences with concrete and semi-concrete materials as linked to the more abstract notation. Translating between these representations is essential in computing in our number system.

TABLE 10.2 HOWE'S FIVE STAGES OF WHOLE-NUMBER PLACE VALUE

Stage	Notation	Description
1	517	Common notation.
2	$517 = 500 + 10 + 7$	Numbers can be uniquely written as a sum of their place-value parts in expanded form.
3	$517 = 5 \times 100 + 1 \times 10 + 7 \times 1$	Multiplicative structure of the base-ten units with each digit multiplied by the corresponding base-ten unit.
4	$517 = 5 \times (10 \times 10) + 1 \times 10 + 7 \times 1$	Each unit is a repeated product of ten (each to the left is ten times the unit to the right).
5	$517 = 5 \times 10^2 + 1 \times 10^1 + 7 \times 10^0$	Linking to the operations of algebra, using exponents and powers.

Base-Ten Models for Place Value

Learning Outcome 10.2 **Demonstrate how to develop students' skills in place value through the use of base-ten models.**

When students are learning base-ten concepts, they combine multiplicative understanding (each place is ten times the value of the place to the right) with a positional system (each place has a value)—something hard to do prior to learning about multiplication! Physical models for base-ten concepts play a key role in helping students develop the idea of "a ten" as both a single entity and as a set of 10 units. Remember, though, that the models do not show the concept to the students; the students must mentally construct the "ten makes one relationship" through multiple experiences and impose that connection on the model.

The most effective base-ten model for ones, tens, hundreds, and thousands is one that is *proportional*. That is, a model for ten is physically 10 times larger than the model for a one, and a hundred model is 10 times larger than the ten model. Proportional materials allow students to confirm that ten of any given piece is equivalent to one piece in the column to the left (10 units equals 1 ten, and so on). Base-ten proportional models can be categorized as either *groupable* or *pregrouped*.

Groupable Models

Models that most clearly reflect the relationships of ones, tens, and hundreds are those for which students can build a ten from single pieces or units and verify its value. When students put 10 beans in a cup, the cup of 10 beans literally *is the same as* the 10 single beans. Bundles of wooden craft sticks or coffee stirrers can be grouped with rubber bands. Plastic connecting cubes or virtual versions (https://www.didax.com/apps/unifix/) can be built into rods of ten and thereby provide a logical transition to pregrouped rods because they form a similar shape. Examples of these groupable models are in Figure 10.3(a). Virtual versions of these tools are effective, connect well to virtual learning environments, and can be more efficient when working with larger quantities.

As students begin to make groupings of ten, start introducing the language of "tens" by matching the objects, such as "cups of tens and ones" or "bundles of tens and singles." Then graduate to a general phrase, such as "groups of tens and ones." Eventually, you can abbreviate this language simply to "tens," such as "four tens and seven."

When students become more familiar with these models, collections of tens can be made in advance by the students and kept as ready-made tens (e.g., craft sticks can stay in bundles, connecting cubes left connected). This approach is a transition to the pregrouped models described next.

Pregrouped Models

Models that are pregrouped commonly appear in curricular materials and are used in corresponding instructional activities. Pregrouped base-ten materials, such as those in Figure 10.3(b) and the paper Base-Ten Materials at our Pearson eText location, cannot be taken apart or put together. When 10 single pieces are accumulated, they must be exchanged or traded for a ten, and likewise, ten tens must be traded for a hundred. The advantage of these physical or virtual models (https://apps.mathlearningcenter.org/number-pieces/) is their ease of use and the efficient way they model large numbers. Compared to commercial base-ten blocks, digital materials are easily grouped and ungrouped, available in endless supply, and can be manipulated by students and displayed by projector or smart board. Remember, though, that virtual models are no more conceptual than physical models and as such are only a representation for students who already understand the relationships involved.

Pearson eText Blackline Master: Base-Ten Materials

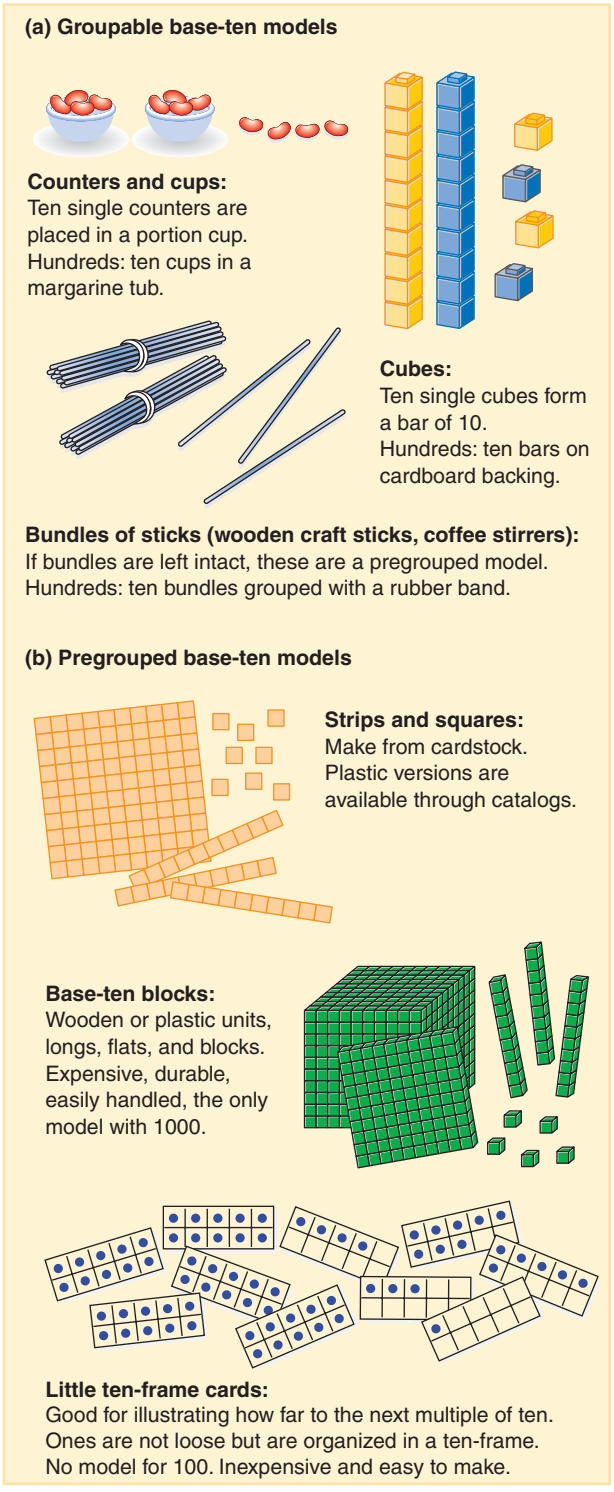

(a) Groupable base-ten models
Counters and cups:
Ten single counters are placed in a portion cup.
Hundreds: ten cups in a margarine tub.
Cubes:
Ten single cubes form a bar of 10.
Hundreds: ten bars on cardboard backing.
Bundles of sticks (wooden craft sticks, coffee stirrers):
If bundles are left intact, these are a pregrouped model.
Hundreds: ten bundles grouped with a rubber band.
(b) Pregrouped base-ten models
Strips and squares:
Make from cardstock.
Plastic versions are available through catalogs.
Base-ten blocks:
Wooden or plastic units, longs, flats, and blocks.
Expensive, durable, easily handled, the only model with 1000.
Little ten-frame cards:
Good for illustrating how far to the next multiple of ten.
Ones are not loose but are organized in a ten-frame.
No model for 100. Inexpensive and easy to make.

FIGURE 10.3 Groupable and pregrouped base-ten models

Standards for Mathematical Practice
MP7. Look for and make use of structure

With pregrouped models, make an extra effort to confirm that students understand that a ten piece really is the same as 10 ones. Although there is a large pregrouped cube to represent 1000, have students group 10 hundred pieces and attach them together with a rubber band as a cube to show how it is formed. Otherwise, some students may only count the square units they see on the surface of the six faces and think the cube represents 600.

The little ten-frames are also pregrouped and effectively link to the familiar ten-frames students used early on to think about numbers within 10 and to learn basic facts. As such, they may initially be more meaningful than base-ten materials made from paper strips and squares. Little ten-frames show the distance to the next decade. For example, when 47 is shown with 4 ten cards and a seven card, a student can see that three more will make five groups of 10, or 50. This can help students reason about the relative size of the number (closer to 50 than to 40), as well as give ideas for how to compute flexibly.

Pearson eText Blackline Master: Little Ten-Frames

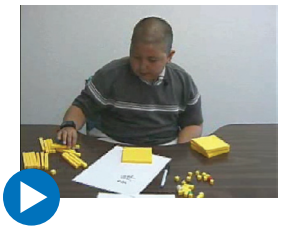
A limitation of pregrouped physical is that students can use them without reflecting on the ten-to-one relationships. For example, if students are just *told* to trade 10 ones for a ten, it is possible for them to make this trade without attending to the ten-ness of the piece they call a ten. Similarly, students can "make the number 42" by simply picking up 4 tens and 2 ones pieces without understanding that if all the pieces were broken apart, there would be 42 ones. Therefore, effective teaching of place value must first provide adequate experiences with groupable models.

***TECHNOLOGY* Notes.** Using electronic versions of base-ten manipulatives, students (including those with special needs in math) can place materials such as ones, tens, hundreds, or thousands on the screen with simple mouse clicks. Number pieces (https://www.mathlearningcenter.org/resources/apps/number-pieces or base-ten materials http://www.didax.com/apps/base-ten-blocks) help learners explore place value while helping them visualize operations with multi-digit numbers. Consider using the objects at https://www.coolmath4kids.com/manipulatives/base-ten-blocks, because students can draw a loop around the materials and group them. They can use these pieces to represent multi-digit numbers and regrouping, adding, subtracting, multiplying, and dividing. ■

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Non-proportional Models

Non-proportional models, or models in which the ten is not physically 10 times larger than the one, are *not* used for introducing place-value concepts. They may be used when students already have a conceptual understanding of the numeration system and need additional reinforcement, or by older students who may need to return to place-value concepts because they are struggling with content that requires revisiting place-value understanding. Examples of non-proportional models include an abacus that has same-sized beads on wires in different columns, money, or chips that are given different place values by color or a number on the chip.



Pearson eText
Video Example 10.2
In this video pay attention to how the student solves the problem using each different method. What is the student's source of confusion? If you were working with this student, what problem would you follow up with and why?

Activities to Develop Base-Ten Concepts

Learning Outcome 10.3 **Explain how students can use grouping activities to deepen their understanding of place-value concepts.**

Now that you have a sense of the important place-value concepts, we turn to activities that assist students in developing these concepts. This section focuses on base-ten concepts or grouping by tens as they connect to the oral and written names for numbers.

Grouping Activities

Reflect for a moment on how strange it must sound to hear "seven ones." Certainly, students have never said they were seven ones years old. The use of the word *ten* as a singular group name can also be mysterious. Consider the phrase "Ten ones makes one ten." The first *ten* carries the usual meaning of 10 things, but the other *ten* is a singular noun, a thing. How can something the student has known for years as the name for a collection of things suddenly

become one thing? And if you think this idea is confusing for native speakers, imagine the potential difficulty for multilingual learners.

Because students come to their development of base-ten concepts with a count-by-ones idea of number, you must begin there. Students need to experiment with showing amounts in groups of like size and perhaps come to an agreement that ten is a very useful size to use. The following activity could be done as an example of a beginning effort at developing grouping concepts.

Activity 10.1 Counting in Groups

CCSS-M: K.NBT.A.1; 1.NBT.B.2a

Find a collection of between 25 and 100 items that students might be interested in counting—perhaps the number of shoes in the classroom, a container of cubes, a long chain of plastic links, or the number of crayons in the classroom crayon box. Then ask, "How could we count our shoes in some way that would be easier than counting by ones?" Whatever suggestions students suggest, try them. After testing several methods, discuss what worked well and what did not. If no one suggests counting by tens, you might casually suggest that as an idea to try.

One teacher challenged her students to find a good way to count all the connecting cubes being held by the students after the students collected a cube for each of their pockets. The first suggestion was to count by sevens. That was tried but did not work very well because none of the students could easily count by sevens. In search of a more efficient way, the next suggestion was to count by twos. This approach did not seem to be much better than counting by ones. Finally, they settled on counting by tens and realized that this method was a good and fast way of counting.

This activity and similar ones (see *Choral Counting and Counting Collections* by Franke et al. (2018)) provide you with the opportunity to develop the idea that materials can actually be arranged into groups of tens before the fast way of counting is begun. Remember that students may count "ten, twenty, thirty, thirty-one, thirty-two" but not fully realize the thirty-two-ness of the quantity. To connect the count-by-tens method with their understood method of counting by ones, the students need to count both ways and discuss why they get the same result.

The idea in the next activity is for students to make groupings of ten and record or say the amounts. Number words are used so that students will not mechanically match tens and ones with individual digits.

Activity 10.2 How Many in the Bag?

CCSS-M: K.NBT.A.1; 1.NBT.B.2a; 1.NBT.B.2b; 1.NBT.B.2c

Prepare baggies of different objects such as toothpicks, buttons, beans, plastic chips, connecting cubes, craft sticks, and so on. Students should use the Bag of Tens activity page as a recording sheet similar to the top of Figure 10.4. The bags can be placed at stations around the room or given to pairs of students. Students empty the bags and count the contents. The amount is recorded as a number word. Then the objects are grouped in as many tens as possible and recorded on the form. After returning the objects to the bags, bags are traded, or students move to another station. Note that students with special needs in math may initially need to use a ten-frame to support their counting.

Pearson eText Activity Page: Bag of Tens

Variations of this activity are suggested by the three other recording sheets in Figure 10.4. On the Get This Many Activity Page, students count the dots and then count out the corresponding number of counters. Provide small cups to put the groups of ten in. Notice that students first count the set in a way they understand (e.g., count by ones), record the amount in words, and then make place-value groupings. The Fill the Tens and Loop This Many activity pages begin with a verbal name (number word), and students must count the indicated amount and then make groups of ten.

Pearson eText Activity Page: Get This Many
Pearson eText Activity Page: Fill the Tens
Pearson eText Activity Page: Loop This Many

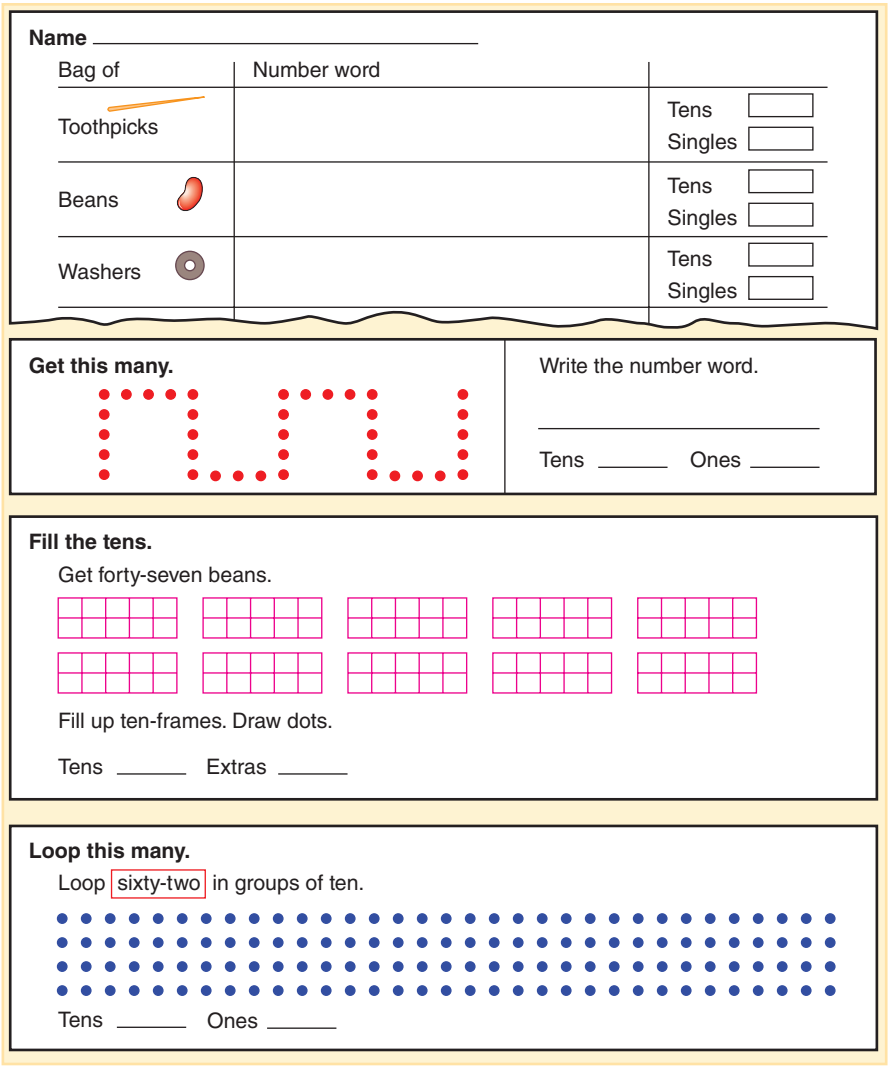


FIGURE 10.4 Activities involving groups of ten

Activity 10.3 How Long Am I?

CCSS-M: K.NBT.A.1; 1.NBT.B.2a; 1.NBT.B.2b

Give students a length to measure—for example, the length of a child lying down or the distance around a sheet of newspaper. At one end of the length, line up 10 units (e.g., 10 connecting cubes, toothpicks, rods, or blocks). On a recording sheet (see Figure 10.5 and the How Long Am I? activity page), students estimate how many groups of 10 and ones they think will match the length. Next they find the actual measure, placing units along the full length and counting them by ones and grouped in tens. Both results are recorded. Estimating the groups of ten requires students to pay attention to ten as a group or unit. Notice that all three place-value components from Figure 10.2 are included in this activity. See the Expanded Lesson: How Long Am I? for more details.

Pearson eText Activity Page: How Long Am I?

Pearson eText Expanded Lesson: How Long Am I?

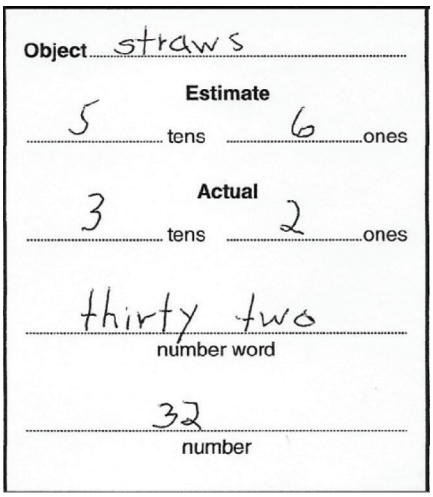


FIGURE 10.5 How Long Am I? recording sheet for estimating groups of tens and ones

FORMATIVE ASSESSMENT Notes. Use a Class Observation look-for tool to record observations about how students carry out these activities. For example, how do they count out the objects? Do they make groupings of ten? Do they count to 10 and then start again at 1? Students who count in these ways are already using base-ten structure. But what you may initially see is students counting a full set without stopping at tens and without any effort to organize the materials in groups. If you notice that behavior, conduct a diagnostic interview and ask the student to count a container of beans (with between 30 and 50 beans) and record the number. Ask the student, "If you were to place each group of 10 beans in a small cup, how many cups would you need?" If the student has no idea or makes random guesses, what would you know about the student's knowledge of place value and what activity might help? ■

Pearson eText Blackline Master: Class Observation Look-for Tool

Grouping Tens to Make 100

Here the issue is not just connecting a count-by-ones concept to a group of 100 but, rather, seeing the number in multiple ways, including as 100 single objects, as 10 tens, and as a singular thing. In curricular materials, this connection may show how 10 rods of ten can be put together to make 1 hundred piece. This quick demonstration may be lost on many students. Additionally, the word *hundred* is equally strange and can get even less attention. These word names are not as simple as they seem!

To reinforce the idea that a hundred is a group of 10 tens *and* 100 singles, consider the estimation activity Too Many to Count?

Activity 10.4 Too Many to Count?

CCSS-M: 2.NBT.A.1

Show students 250 to 1000 *groupable* items, such as a container of lima beans, a long chain of connecting links or paper clips, a box of pennies, or a grocery bag full of straws. Have students make, record, and discuss estimates of how many beans, for example, are in the container. Take a photograph of the original collection ahead of time so you have it to use later for comparison (Mix et al., 2019). Then distribute portions of the beans to pairs or triads of students to put into cups of 10 beans. Collect leftover beans and put them into groups of ten as well. Ask,

> How can we use these groups of ten to tell how many beans we have?
>
> Can we make new groups from the groups of ten?
>
> What is 10 groups of ten called?

Be prepared with larger containers or baggies into which 10 cups (or other collections of 10 tens) can be placed. When all groups are made, count the hundreds, the tens, and the ones separately. Record the total as 4 hundreds + 7 tens + 8 ones. Compare to a projected image of the original collection. What do students notice? Go back and discuss the original estimates and that the number of items is in both cases the same by labeling both amounts with 478. For students who are struggling with this concept, break apart the groupings to confirm it is the same number and then choose another amount to group into tens and hundreds. For a challenge, extend to collections of more than 1000.

Pearson eText Activity Page: Too Many Tens

You may worry that this activity will take too much time, but it helps cement the relationships among ones, tens, and hundreds that is often lost with pregrouped base-ten models. Additionally, task repetition or doing this activity again and again with different amounts helps students with special needs in math make stronger connections to the relationships in the place-value structure, so this activity is a great candidate for a learning center.

Equivalent Representations

An important variation of the grouping activities is aimed at the equivalent representations of numbers. For example, pose the following task to students who have just completed Activity 10.2 How Many in the Bag?

What is another way you can show 42 besides 4 groups of ten and 2 singles? How many ways can you find?

Interestingly, many students will go next to 42 singles. The following activities focus on creating other equivalent representations.

Activity 10.5 Can You Make the Link?

CCSS-M: 1.NBT.B.2; 1.NBT.C.5

Show a collection of materials that are only partly grouped in sets of ten. For example, you may have 5 chains of 10 links and 17 additional unconnected links. Be sure the students understand that each chain has 10 links. Have students count the number of chains and the number of singles in any way they wish. Ask, "How many in all?" Record all responses and discuss how they got their answers. Next, before their very eyes, change the groupings (make a ten from the singles or break apart one of the tens) and repeat the question. Do not change the total number of links from one time to the next. Once students begin to understand that the total does not change, ask in what other ways the items could be grouped if you use tens and ones.

If you are teaching in second grade, equivalent representations for hundreds as groups of tens can help with the concept of a hundred as 10 tens. The next activity is similar to "Can You Make the Link?" but is done using pregrouped materials and includes hundreds.

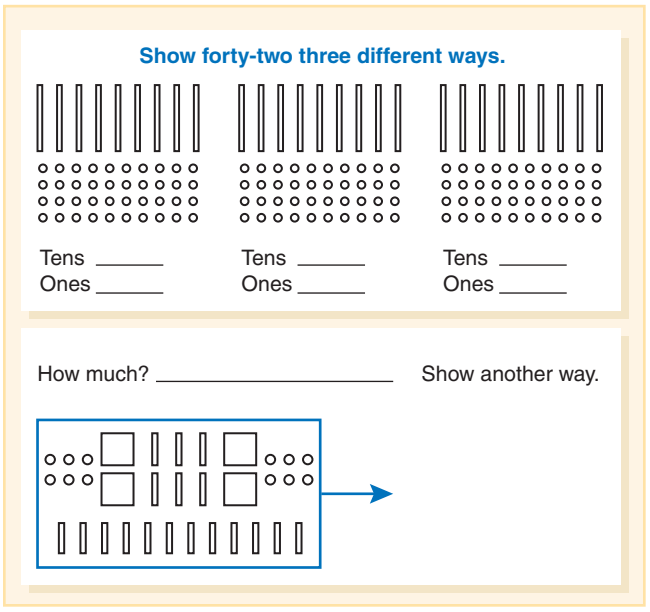


FIGURE 10.6 Equivalent representations, using square-line-dot pictures

Activity 10.6 Three Other Ways

CCSS-M: 2.NBT.A.1; 2.NBT.A.3; K.CC.B.5

Students work in groups or pairs. First, they show 463 on their desks with base-ten materials in the standard representation (4 hundreds pieces, 6 tens, and 3 ones). Next, they find and record at least three other ways of representing this quantity. A variation is to challenge students to find a way to show an amount with a specific number of pieces. "Can you show 463 with 31 pieces?" (There is more than one way to do this.)

Pearson eText Activity Page: Square Line Dot

When students have sufficient experiences with pregrouped concrete materials, a semi-concrete square-line-dot notation can be used for recording ones, tens, and hundreds (see Figure 10.6 and Square Line Dot activity page). Use the drawings to reinforce the notion of unitizing (i.e., one ten, one hundred) and as a suggestion for how students can record their thinking.

The next activity incorporates oral language with equivalent representation ideas.

Activity 10.7 Base-Ten Riddles

CCSS-M: 1.NBT.A.1; 2.NBT.A.1; 2.NBT.A.3

Base-ten riddles can be presented orally or in written form (see Base-Ten Riddle Cards). In either case, students should use base-ten materials to help solve the riddles. The examples here illustrate a variety of levels of difficulty. For a challenge, have students write new riddles when they complete these.

I have 23 ones and 4 tens. Who am I?

I have 4 hundreds, 12 tens, and 6 ones. Who am I?

I have 30 ones and 3 hundreds. Who am I?

I am 45. I have 25 ones. How many tens do I have?

I am 341. I have 22 tens. How many hundreds do I have?

I have 13 tens, 2 hundreds, and 21 ones. Who am I?

If you put 3 more tens with me, I would be 115. Who am I?

I have 17 ones. I am between 40 and 50. Who am I? How many tens do I have?

Pearson eText Activity Page: Base-Ten Riddle Cards

Reading and Writing Numbers

Learning Outcome 10.4 **Explain strategies to support students' ability to write and read numbers.**

In this section, we focus on helping students connect oral and written names for numbers (see bottom of Figure 10.2) with their emerging base-ten concepts of using groups of 10 or 100 as efficient methods of counting. Note that the ways we say and write numbers are conventions, not concepts. Students must learn these conventions by being told rather than through problem-based activities. Remember that for multilingual learners, the convention or pattern in our English number words may not be the same as it is in their native language, especially for the numbers 11 to 19.

Two-Digit Number Names

In kindergarten and first grade, students need to connect the base-ten concepts for two-digit numbers with the oral number names they repeatedly used. They know the words but may not yet think of them in terms of tens and ones. In fact, early on, they may want to write twenty-one as 201.

When teaching oral names, you will want to use base-ten models. Initially, rather than using standard language, use the more explicit base-ten language (e.g., "4 tens and 7 ones" instead of "forty-seven"). When it seems appropriate, pair base-ten language with standard language. Emphasize the teens as exceptions to the pattern. Acknowledge that the teen numbers are formed backward, and some of the decade numbers such as thirty and fifty are irregular and do not fit the pattern. The next three activities help introduce oral names for numbers. To encourage not just matching number words to models without thinking of the total quantities, always have students explain their thinking.

The next activity shows how this basic approach might be done with other base-ten models.

Activity 10.8 Tens, Ones, and Fingers

CCSS-M: 1.NBT.B.2a; 1.NBT.B.2b; 1.NBT.B.2c

Ask, "Can you show 6 fingers [or any amount less than 10]?" Then ask, "How can you show 37 fingers?" Some students will figure out that at least four students are required. Line up four students, and have three students hold up 10 fingers while the last student holds up 7 fingers. Have the class count the fingers by tens and ones. Ask other students to show different numbers. Emphasize the number of sets of ten fingers and the single fingers (base-ten language), and pair this model with standard language.

Activity 10.9 Counting with Base-Ten Models

CCSS-M: 1.NBT.B.2; 1.NBT.C.5

Show some tens pieces on a projection device, or place them on the carpet in a mixed arrangement. Ask, "How many tens?" Add or remove a ten and repeat the question. Next, add some ones. Always have students give the base-ten name and the standard name. For many students, including multilingual learners and students with special needs in math, it is helpful to post examples of base-ten names and the corresponding standard names on the math word wall. Continue to make changes in the materials displayed by adding or removing 1 or 2 tens and by adding and removing ones. For a challenge, have students create problems.

Reverse the activity by having students use base-ten materials at their desks. For example, say, "Make seventy-eight." The students make the number with the models and then give the base-ten name (7 tens and 8 ones) and standard name (78). Have students record their work (see Figure 10.7).

Pearson eText Blackline Master: Base-Ten Materials

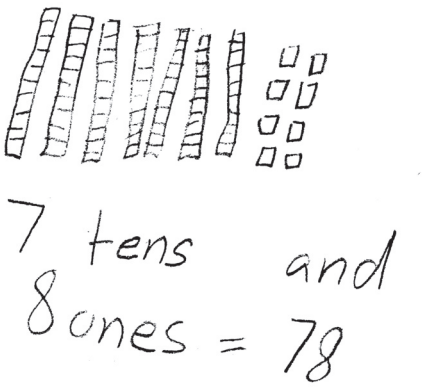


FIGURE 10.7 A student's recording of 78 with models and the base-ten name

Activity 10.10 Counting Rows of Ten

CCSS-M: 1.NBT.B.2c

Project the 10×10 Multiplication Array of dots. Cover up all but two rows, as shown in Figure 10.8(a). "How many tens? [2.] Two tens is called twenty." Have the class repeat. Show another row. Repeat the question. "Three tens is called thirty. Four tens is forty. Five tens could have been fivety but is just called fifty. How many tens does sixty have?" The names sixty, seventy, eighty, and ninety all fit the pattern. Slide the cover up and down the array, asking how many tens and the name for that many. Multilingual learners may not initially hear the difference between fifty and fifteen, sixty and sixteen, and so on, so explicitly compare these words and clearly enunciate—even overemphasize the word endings.

Use the same 10×10 array to work on names for tens and ones. Show, for example, four full lines: "forty." Next, expose one dot in the fifth row. "Four tens and one. Forty-one." Add more dots one at a time. "Four tens and two. Forty-two." "Four tens and three. Forty-three." This visual is shown in Figure 10.8(b). When that pattern is established, repeat with other decades from 20 through 90. Eventually, connect the array to a hundred chart (Top-Down or Bottom-Up) to connect the oral name and the written numeral. For example, students can locate 43 as four rows of ten down (or up on a bottom-up chart) and, over to the right, three ones.

Pearson eText Blackline Master: 10×10 Multiplication Array

Pearson eText Blackline Master: Hundred Chart

Pearson eText Blackline Master: Bottom Up Hundred Chart

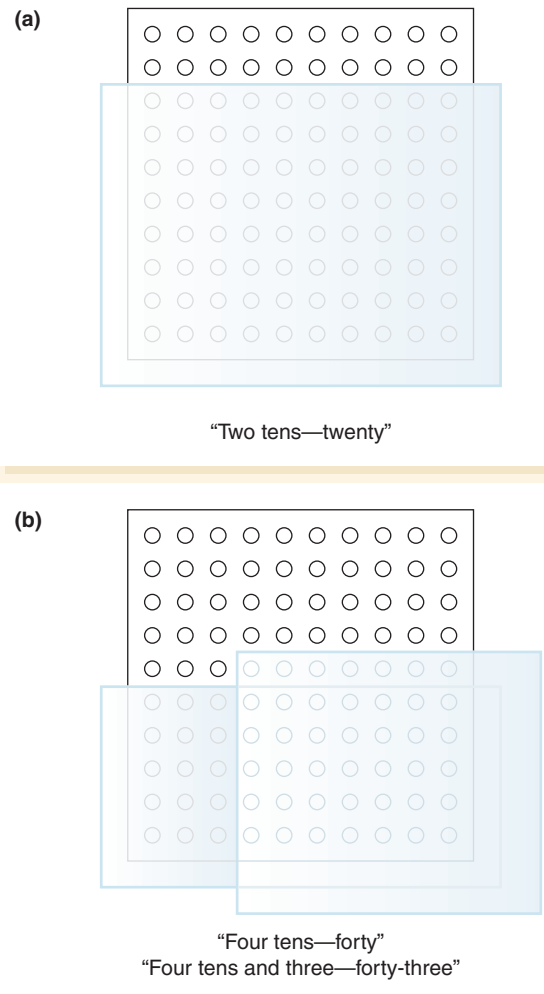


FIGURE 10.8 Arrays for modeling sets of tens and ones

Three-Digit Number Names

When students begin to study more complex numbers with more digits using fewer visual models, they are often less successful (MacDonald et al., 2018). The approach to three-digit number names starts by showing mixed arrangements of base-ten materials and have students give the base-ten name (4 hundreds, 3 tens, and 8 ones) and the standard name (438). Vary the arrangement from one example to the next by changing only one type of piece; that is, add or remove only ones or only tens or only hundreds. It is important for students with special needs in math to see counterexamples, so purposely point out that some students wrote 200803 for two hundred eighty-three, and ask them whether that is correct and to explain their reasoning. Using common errors from the research, such as this one called "additive composition" (Mix et al., 2019, p.5), is ideal for class discussion (see Table 10.4 for others). As this notation demonstrates, the connection between oral and written numbers is not straightforward, with some researchers suggesting that an early milestone on the route to full understanding is this early (incorrect) expanded form of writing numbers (Byrge, Smith, & Mix, 2013). These discussions allow students to explore their initial ideas and clear up any misunderstandings.

A challenge with three-digit numbers is with numbers involving no tens, such as 702 (or later with numbers such as 1046). Interestingly, zero was the final digit added to our Hindu–Arabic system about 1500 years ago (Ifrah, 2000) yet it is so important. As noted earlier, the use of oral base-ten language is quite helpful here. The difficulty of zero-tens (or more generally the internal zero) is more pronounced when writing numerals. For example, some students incorrectly write 7,002 for seven hundred two. At first, students do not see the importance of zero in place value and do not understand that zero helps us distinguish among such numbers as 203, 23, and 230 (Dougherty, Flores, Louis, & Sophian, 2010). Carefully avoid calling zero a placeholder, because it is a number with a value. Multilingual learners may need additional time to think about how to say and write the numerals, because they may be translating all the terms involved with the number.

Researchers note that there are significantly more errors with four-digit number names than with three-digit numbers, so do not think that students will easily generalize to larger numbers without actually exploring additional examples and tasks (Garcia-Orza et al., 2017).

Written Symbols

Place-value mats are simple mats divided into two or three sections to position ones and tens or ones, tens, and hundreds pieces, as shown in Figure 10.9. Suggest to your students that the mats are a good way to organize their materials when working with base-ten blocks. Explain that the standard way to use a place-value mat is with the space for the ones on the right and the tens and hundreds places to the left.

It is strongly recommended that two ten-frames be drawn in the ones place as shown on the Place-Value Mat. That way, the amount of ones on the ten-frames is always clearly evident, eliminating the need for repeatedly counting the ones. The ten-frame also makes it very clear how many additional ones are needed to make the next group of ten. If students are modeling two numbers at the same time (addition), one ten-frame could be used for each number.

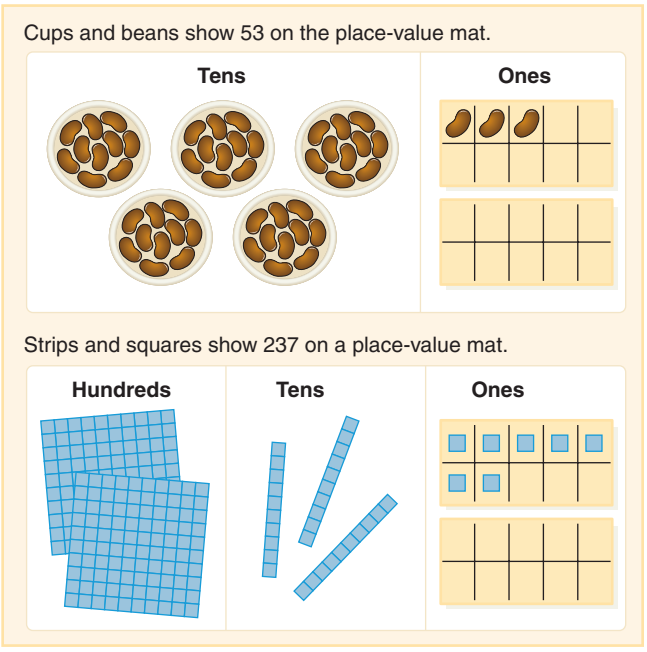


FIGURE 10.9 Place-value mats with two ten-frames

As students use their place-value mats, they make the connection to how the left-to-right order of the pieces is also the way that numbers are written.

Pearson eText Blackline Master: Place-Value Mat

Activity 10.11 Building Numbers

CCSS-M: 1.NBT.B.2; 2.NBT.A.1

To show how the numbers are built, give each pair of students a set of place-value cards—one for each of the hundreds (100–900), tens (10–90), and ones cards (1–9) (see Figure 10.10). Notice that the cards are made so that the tens card is twice as long as the ones card, and the hundreds card is three times as long as the ones card (also see https://toytheater.com/place-value-cards). Have student pairs place base-ten materials for a number (e.g., 457) on the mat and have them also place the matching cards (e.g., 400, 50, and 7) below the materials in each corresponding column. Then, the pairs start with the hundreds card and layer the others on top, right aligned. This approach will show them how the number is built while allowing them to see the individual place value parts of the number, which is especially helpful if there are zero tens.

Pearson eText Activity Page: Place-Value Cards

The place-value mat and the matching cards demonstrate the important link between the base-ten models and the written form of the numbers. There is an engaging app that counts, puts together, and unstacks any number into similar cards (https://www.ictgames.com/mobilePage/arrowCards/index.html).

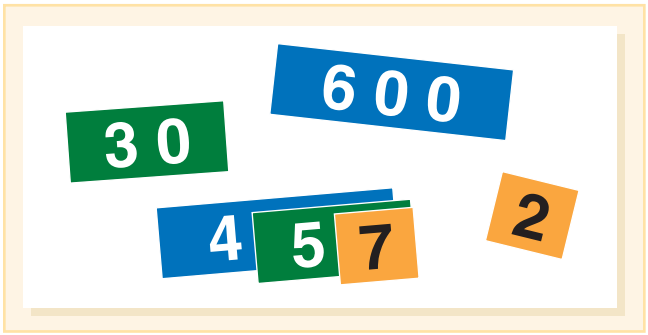


FIGURE 10.10 Building numbers with a set of place-value cards

The next two activities are designed to help students make connections between models, oral language (base-ten and standard), and written forms. The activities can be done with two- or three-digit numbers, depending on students' needs.

To support students struggling with reading a number with an internal zero, you may want to show with the place-value cards (see Figure 10.10) (or say) 7 hundreds and 4 ones. Then the class says, "Seven hundreds, zero tens, and four ones, which is—seven hundred (*slight pause*) four." The pause and the base-ten language support the correct reading of the three-digit number.

Activity 10.12 Say It/Press It

CCSS-M: 2.NBT.A.1a; 2.NBT.A.1b; 2.NBT.A.3

Display models of ones, tens, and hundreds (and thousands, if appropriate) in a mixed arrangement. Use a projection device or virtual manipulatives, or simply draw on the board, using the square-line-dot method. Students say the amount shown in base-ten language ("four hundreds, one ten, and five ones") and then in standard language ("four hundred fifteen"); next, students enter it on their calculators or use paper and pencil to respond. Have one student share their display and explain their thinking. Make a change in the materials and repeat. You can also do this activity as Show It/Press It by saying the standard name for a number and then having students use base-ten materials to show that number and enter it on their calculators (or write it). Again, pay attention to numbers with components in the teens (e.g., 317) and numbers with internal zeros (e.g., 408). Multilingual learners may need additional time to think of the words that go with the numbers, especially as the numbers get larger.

The next activity is also a good way to see whether students really understand the value of digits in two-, three- or four-digit numbers.

Activity 10.13 Digit Change

CCSS-M: 1.NBT.B.2; 1.NBT.C.5; 1.NBT.C.6; 2.NBT.A.1; 2.NBT.A.3; 2.NBT.B.5; 2.NBT.B.8

Have students enter a specific two-, three-, or four-digit number on the calculator. The task is to change one of the digits in the number without simply entering the new number. For example, change 48 to 78. Change 315 to 305 or to 295. Changes can be made by adding or subtracting an appropriate amount. Students should write or discuss explanations for their solutions. Students with special needs in math may at first need the visual support of having cards that say "add ten" or "add one." They may also need support with base-ten materials to be able to conceptualize the number and then move to more abstract work, using only the calculator.

Pearson eText Blackline Master: Base-Ten Materials

FORMATIVE ASSESSMENT Notes. Students' level of understanding of place value can be masked by their use of tens and ones pieces in prescribed ways and using the language of place value. Diagnostic interviews help you look more closely at students' understanding of the integration of the three components of place value. These classic interview tasks have been used successfully for many years (e.g., Fraivillig, 2017; Labinowicz, 1985; Kamii 1985; Ross 1986). As with all diagnostic interviews, do not give clues.

The first interview is referred to as the Digit Correspondence Task. Take out 36 blocks. Ask the student to count the blocks and then write the number that tells how many there are. Circle the 6 in 36 and ask, "Does this part of your number have anything to do with how many blocks there are?" Then circle the 3 and repeat the question. Based on responses to the task, which of Ross's (1989, 2002) five levels of place-value understanding does your student demonstrate?

1. *Single numeral.* The student writes 36 but views it as a single numeral. The individual digits 3 and 6 have no meaning by themselves.
2. *Position names.* The student correctly identifies the tens and ones positions but still makes no connections between the individual digits and the blocks.
3. *Face value.* The student matches 6 blocks with the 6 and 3 blocks with the 3.
4. *Transition to place value.* The 6 is matched with 6 blocks and the 3 with the remaining 30 blocks but not as 3 groups of 10.
5. *Full understanding.* The 3 is correlated with 3 groups of ten blocks and the 6 with 6 single blocks.

For the second interview, write the number 342 (or a two-digit number). Have the student read the number. Then have the student write the number that is 1 more. Next, ask for the number that is 10 more. You may wish to have them explore further with models. One less and 10 less can be checked the same way. Observe whether the student is counting on or counting back or whether the student immediately knows that ten more is 352.

In the third interview, ask the student to write the number that represents 5 tens, 2 ones, and 3 hundreds. Note that the task does not give the places in order. What do you think will be a common misinterpretation? If the student doesn't write 352, then ask the students to show you the number with base-ten materials, and ask what number they have with the materials. Compare to what they wrote previously, if different. What information can you obtain from the results of this interview? ■

Pearson eText
Video Example 10.3
In this video consider what the student knows about place value based on responses to the teacher's questions.

Place Value Patterns and Relationships—A Foundation for Computation

Learning Outcome 10.5 **Recognize that there are patterns in our number system that provide the structure for computational strategies.**

In this section, the focus is on the relationships of numbers to important special numbers, called *benchmark numbers*, and ten-structured thinking—that is, flexibility in using the structure of tens in our number system. These ideas begin to provide a basis for computation as students simultaneously strengthen their understanding of number relationships and place value.

Top Down:

91	92	93	94	95	96	97	98	99	100
81	82	83	84	85	86	87	88	89	90
71	72	73	74	75	76	77	78	79	80
61	62	63	64	65	66	67	68	69	70
51	52	53	54	55	56	57	58	59	60
41	42	43	44	45	46	47	48	49	50
31	32	33	34	35	36	37	38	39	40
21	22	23	24	25	26	27	28	29	30
11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10

Bottom Up:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

FIGURE 10.11 Two versions of a hundred chart.

The Hundred Chart

The hundred chart (see Figure 10.11) deserves special attention in the development of place-value concepts. K–2 classrooms should have a hundred chart displayed prominently and used often. Although you may have used the top-down chart for years, we suggest you consider the bottom-up hundred chart (Bay-Williams & Fletcher, 2017). Why? Because as it adds the number counts "up" and increases the quantity, the language makes more sense (up meaning adding on to a number); it links well to contexts (as we fill in capacity, the amount is greater); it models well with concrete materials as cubes are added or subtracted; and it aligns with coordinate graphing—up and over and so on to describe position. Therefore, in this section as we use language to describe moves, we will refer to what you say with a bottom-up chart (of course, you can align with a top-down version by reversing).

A useful version of a hundred chart can be made of transparent pockets into which each of the 100 numeral cards can be inserted. You can cover numbers, give students numbers to place on the chart, or highlight patterns such as skip counting by 2 (even numbers), 5, and 10. You can also have students skip count by threes and fours and color in each number they count. Discuss the visual pattern revealed on the chart as well as the patterns in the numbers.

In kindergarten and first grade, students count and recognize two-digit numbers on the hundred chart, including the decade numbers from 10 to 100. Additionally, first graders can use the hundred chart to develop a base-ten understanding of adding two-digit numbers with multiples of 10, noticing that moves up or down are moves of ten, while already recognizing that moves to the right or left are moves of one.

There are many patterns on the hundred chart, and during discussions, different students may describe the same pattern in several ways. Accept all ideas. Here are some of the important place-value-related patterns students may point out:

- The numbers in a column all end with the same number, which is the same as the number at the top of the chart.
- In a row, the first number (tens digit) stays the same and the "second" number (ones digit) counts 1, 2, 3,… 9, 0) as you move across.

- In a column, the first number (tens digit) counts, or goes up, by one as you move up. (Remember we are using the bottom-up chart for our examples.)
- You can skip count by tens (decade numbers) going up the far right-hand column.
- Starting at 11 and moving up on the diagonal, you can find that numbers increase by 11, with the same digit repeating in the tens and ones places (e.g., 11, 22, 33, 44, and so on).

For students, these patterns are not always obvious. For example, one student may notice the pattern in the column under the 4—every number ends in a 4. Two minutes later, another student will discover the parallel pattern in the column headed by 7. That this pattern is in every column may not be completely apparent.

Once you've discussed some of these patterns, try the next activity.

Activity 10.14 Missing Numbers

CCSS-M: K.CC.A.1; K.NBT.A.1; 1.NBT.A.1; 1.NBT.B.2; 1.NBT.C.5

Provide students with a hundred chart on which some of the number cards have been removed. The students' task is to replace the missing numbers. To begin, have only a random selection of individual numbers removed. Later, remove sequences of several numbers from three or four rows. Finally, remove all but one or two rows or columns. Eventually, challenge students to replace all of the numbers on the blank hundred chart. For students with special needs in math, model the placement of a missing number using a think-aloud to describe how you make your decision and what key features of the number you think about as you place the number properly on the chart.

Pearson eText Blackline Master: Hundred Chart

Pearson eText Blackline Master: Bottom-Up Hundred Chart

Pearson eText Blackline Master: Blank Hundred Chart

FORMATIVE ASSESSMENT Notes. Replacing the number cards on a blank chart is a good station activity for two students to try. By listening to how students determine the correct places for numbers, you can assess how well they have constructed an understanding of the 1-to-100 sequence and whether they recognize and purposefully use patterns in our number system to re-create the hundred chart. ■

Activity 10.15 Finding Neighbors on the Hundred Chart

CCSS-M: K.CC.A.1; K.NBT.A.1; 1.NBT.A.1; 1.NBT.B.2; 1.NBT.C.5

Begin with a blank or nearly blank hundred chart (projecting it or giving copies to individual students). Then circle a particular missing number. Students should fill in the designated number and its neighbors, the numbers to the left, right, above, and below. After students become comfortable naming the neighbors of a number, ask what they notice about the neighboring numbers. The numbers to the left and right are one less and one more than the given number. Those above and below are ten less and ten more, respectively. How do numbers differ on the diagonal? By discussing these relationships on the chart, students begin to see how the sequence of numbers is related to numerical relationships.

Pearson eText Blackline Master: Blank Hundred Chart

Pearson eText Blackline Master: Bottom-Up Hundred Chart

In the following activity, number relationships are made more explicit by modeling the numbers with base-ten materials and associating those models with the hundred chart.

Activity 10.16 Models with the Hundred Chart

CCSS-M: K.CC.A.1; K.NBT.A.1; 1.NBT.A.1; 1.NBT.B.2; 1.NBT.C.5

Use either base-ten materials or the little ten-frame cards to model two-digit numbers with which the students are familiar.

- Give students one or more numbers to first make with the models and then find on the hundred chart. Use groups of two or three numbers in either the same row or the same column. Ask students how the numbers are alike and how they are different.
- Indicate a number on the chart. What would you have to change about that number to equal each of its neighbors (the numbers to the left, to the right, above, and below)?

As a first step in moving to larger numbers, continue your 100 chart to 200 or 300 and extend the same questions to the larger numbers. Then for a challenge, extend the hundred chart to a thousand chart.

Pearson eText Blackline Master: Base-Ten Materials

Pearson eText Blackline Master: Little Ten-Frame Cards

Activity 10.17 Arrow around the Bottom-Up Hundred Chart

CCSS-M: 1.NBT.C.4; 1.NBT.C.5; 1.NBT.C.6; 2.NBT.B.5; 2.NBT.B.6

Start on a bottom-up hundred chart by giving students a number like 41. Then suggest a series of arrows such as →↑↑↑ to navigate from 41. First move to the right one space to 42 and "up three times (or 30)" to 72. This movement shows the sum of 41 + 31. Ask students whether the arrows can be used in a different order. Try these:

14 →→←←	63 →→↑↑→→	45 →→→↑→→←

Also ask students, "What arrows do we need to get from 72 to 94? From 5 to 45? From 34 to 79?" Students should begin to connect these place-value concepts to the operations of addition and subtraction. Other students may recognize that a downward arrow undoes an upward arrow—an indication that they are moving toward generalizations (Blanton, 2008).

Pearson eText Blackline Master: Bottom-up Hundred Chart

***TECHNOLOGY* Notes.** Several web-based resources include a hundred chart that allows students to explore patterns. Investigate http://hundredgrid.classworks.com, which has a calculator and a hundred chart and can switch into a 0–99 chart or extend to a thousand chart. Students can skip-count by any number as well as begin their counts at any number. ■

Activity 10.18 A Thousand Chart

CCSS-M: 2.NBT.A.1; 2.NBT.A.2; 2.NBT.A.3; 2.NBT.B.8

Provide students with several blank hundred charts. Assign groups of three or four students the task of creating a 1-to-1000 chart. The chart is made by taping 10 hundred charts together in a long vertical strip. Students should decide how they will divide up the task, each taking different parts of the chart. The thousand chart should be discussed as a class to examine how numbers change as you count from one hundred to the next hundred, what patterns they find, and so on. All of the earlier activities can be extended to a thousand chart.

Pearson eText Blackline Master: Blank Hundred Chart

Pearson eText Blackline Master: Bottom-Up Hundred Chart

Relative Magnitude, Using Benchmark Numbers

Number sense also includes having a solid grasp of the size of numbers and the order of numbers, given two or more. Relative numerical magnitude refers to the size relationship one number has with another—is it much larger, much smaller, close, or about the same? How

students think about these comparisons is supported by the use of models and the development of benchmark numbers that can be used as signposts or referents for a number's location (as on a number line).

A valuable feature of both the hundred chart and the little ten-frame cards is how clearly they illustrate the distance to the next multiple of 10—the number of spaces to the end of the row on the chart or the blank spaces on the ten-frame. Multiples of 10, 100, and occasionally other numbers, such as multiples of 25, are referred to as *benchmark* numbers. Students learn to use this idea when they work with informal methods of computation. When finding the difference between 74 and 112, a student might say, "First, I added 6 onto 74 to equal 80, which is a benchmark number. Then I added 2 tens onto 80 to get to 100 because that's another benchmark number." Whatever terminology is used, understanding how numbers are related to these referent numbers is an important step in students' development of number sense and place-value understanding.

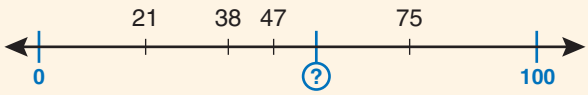
The number line is also an excellent way to explore how one number is related to another, and students' ability to find numbers' relative magnitude accurately on a number line is a predictor of their future mathematics performance (Bull, Lee, & Muñez, 2020; Dietrick et al., 2016; Schneider et al., 2018).

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Activity 10.19 Who Am I?

CCSS-M: 1.NBT.A.1; 1.NBT.B.2; 1.NBT.B.3; 2.NBT.A.1; 2.NBT.A.2; 2.NBT.A.4

Draw a long line with arrows on the ends (or use cash register tape) and label 0 and 100 at opposite ends. Mark a point with a "?" (on a sticky note) that corresponds to your secret number and have students try to determine it. For each estimate, place and label a mark at that number on the line until your secret number is discovered. Have students explain how they are making their estimations, including highlighting any use of benchmark numbers. As a variation, use the Who Am I? activity page where the end points are different, such as 0 and 1000, 200 and 300, or 500 and 800. For students with special needs in math, mark the guesses that already occurred and where they are located. Labeling those numbers at their actual locations will support students' reasoning in the process of identifying the secret number. Also consider using https://apps.mathlearningcenter.org/number-line/for a virtual number line.



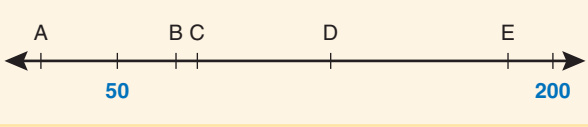
Pearson eText Blackline Master: Number Lines

Pearson eText Activity Page: Who Am I?

Activity 10.20 Who Could They Be?

CCSS-M: 1.NBT.A.1; 1.NBT.B.2; 1.NBT.B.3; 2.NBT.A.1; 2.NBT.A.2; 2.NBT.A.4

Label two points on a number line (not necessarily the ends) with benchmark numbers. Show students different points labeled with letters and ask what numbers they might be and why they think that. In the example shown here, B and C are less than 100 but probably more than 60. E could be about 180. You can also ask where 75 might be or where 400 is located. About how far apart are A and D? Why do you think D is more than 100? For multilingual learners and students with special needs in math, say as well as write the numbers on a note card, or ask students to both write and say the numbers. Use the Who Could They Be? activity page for more examples.



Pearson eText Blackline Master: Number Lines

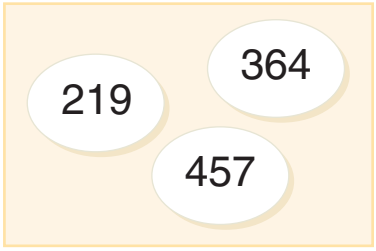
Pearson eText Activity Page: Who Could They Be?

The next activity encourages students to apply some of the same ideas about benchmark numbers that we have been exploring.

Activity 10.21 Close, Far, and In Between

CCSS-M: 1.NBT.A.1; 1.NBT.B.2; 1.NBT.B.3; 2.NBT.A.1; 2.NBT.A.2; 2.NBT.A.4

Put any three numbers on the board that are appropriate for your students. With these three numbers as referents, ask questions such as the following, encouraging discussion of all responses:



Which two numbers are closest? How do you know?

Which is closest to 300? To 250?

Name a number between 457 and 364.

Name a multiple of 10 between 219 and 364.

Name an even number that is greater than all of these numbers.

About how far apart are 219 and 500? 219 and 5000?

If these are big numbers, what are some small numbers? Numbers that are about the same? Numbers that make these seem small?

For multilingual learners, this activity can be modified by using prompts that are similar (rather than changing the prompts each time, which increases the linguistic demand). Also, multilingual learners and students with special needs in math may benefit from using a visual, such as a number line, and from writing the numbers rather than just hearing/saying them.

Look at the corresponding Expanded Lesson: Close, Far, and In Between, in which students estimate the relative size of a number between 0 and 100 and strengthen their conceptual understanding of number size and place value.

Pearson eText Expanded Lesson: Close, Far, and In Between

Van Garderen, Lannin, and Kamuru (2020, pp. 10–12) researched a progression of approaches students use as they develop place-value structure to compare number magnitude and justify their thinking, as shown in Table 10.3.

By considering patterns of their justifications, students can begin to formulate generalizations that will support their continuing growth in number sense.

TABLE 10.3 PROGRESSION OF JUSTIFICATIONS FOR COMPARING NUMBER MAGNITUDE

Justification	Sophistication Level	Sample Student Rationales
Larger or Smaller Number	Developing	36 is more than 29.
Counting or Math Fact	Developing	62 is greater than 55 because if you start at 55, five more equals 60, and then I can count 61 and 62. So 62 is seven more.
Superficial Place Value	Developing	50 is bigger than 20 because 5 is bigger than 2.
Benchmarking	Target	125 is more than 89 because 125 is bigger than 100 and 89 is less than 100.
Position on the Number Line and Robust Place Value	Target	87 is more than 52 because it is on the right of 52 on the number line. 54 is greater than 45 because it has 5 tens and 4 ones and the other has only 4 tens and 5 ones.

Approximate Numbers and Rounding

The most familiar form of computational estimation is rounding, which changes numbers in a problem to other numbers that are easier to compute mentally. Students in third grade use place-value understanding to round numbers to the nearest 10 or 100, and students in fourth grade should be able to round any multi-digit whole number to any place value. At all grades, students can use rounding to assess the reasonableness of answers. Fifth graders use place-value understanding to round decimals to any place.

To be useful in estimation, rounding should be flexible and conceptually well-understood. To round a number simply means to select a compatible number. (Note that the term *compatible* is not a mathematical term. It refers to numbers that would make the problem easier to compute mentally.) The compatible number can be any close number and need not be a multiple of 10 or 100, but in many cases, students are asked to round to one of these places.

A number line with benchmark numbers highlighted can be useful in helping students select compatible numbers. An open number line like the one shown in Figure 10.12 can be made using strips of poster board taped end to end, cash register tape, or a digital version. Labels are written above the line. The ends can be labeled 0 and 100, 100 and 200, . . . 900 and 1000. Indicate the location of a number above the line that you want to round. Discuss the locations of numbers that are close. Present the convention that if a number that is being rounded has a 5 in the place being considered, although it is halfway between two numbers, they round up. The number line is a powerful tool for these discussions.

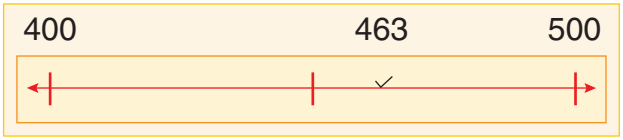


FIGURE 10.12 A number line supports rounding

Connections to Real-World Ideas

As students study place-value concepts, encourage them to notice numbers in the world around them. Students K–1 should be thinking of numbers to 100 and 120, respectively, while students in the second grade should be thinking about numbers up to 1000. Where are these numbers found around the school? You might use the number of students in the third grade, minutes devoted to mathematics each week, or days since school has started. There are measurements, numbers discovered on a field trip, numbers in books in other subject areas, and so on. What do you do with these numbers? Turn them into interesting graphs, write stories using them, and make up problems. For example, how many cartons of chocolate and plain milk are served in the cafeteria each month? Can students estimate how many cartons will be served in a year? Collecting data and then grouping into tens and hundreds (or thousands) will help reinforce the value of grouping in situations when you need to count and compare quantities. Using real-world connections increases understanding of place-value concepts (Bray & Blais, 2017).

Numbers Beyond 1000

Learning Outcome 10.6 **Describe how the place-value system extends to large numbers.**

Explicitly understanding the difference between a million and a billion matters! To interpret our world, working with very large numbers has never been more important (Howe, 2019). For students to have good concepts of numbers beyond 1000, the place-value ideas you've carefully developed must be extended. This development is sometimes difficult because physical models for thousands are not readily available, or you may just have one large cube to show. At the same time, number-sense ideas must also be developed. In many ways, connecting very large numbers to real amounts is just as important as connecting smaller numbers to real quantities.

Extending the Place-Value System

Two important ideas developed for three-digit numbers should be extended to larger numbers as students move to thinking about 1,000,000 in fourth grade. First, students should develop the multiplicative structure of the number system as a generalization. That is, ten in any position makes a single thing (group or unit) in the next position to the left, and one position to the right is one tenth of that unit (or it takes ten of those units). Second, the oral and written patterns for numbers in three digits are duplicated in a pattern for every three digits to the left. At first, these two related ideas may not be as easy for students to grasp immediately. Because availability of models for large numbers can't be assumed or are easy to visualize, many curricular materials deal with these ideas in a predominantly symbolic manner. That is not sufficient!

Activity 10.22 What Comes Next? (Concrete)

CCSS-M: 2.NBT.A.1; 2.NBT.A.3

To help students generalize the place-value pattern, this activity uses three-dimensional base-ten materials. The first three shapes are distinct: a *cube* (1), a *long* (10), and a *flat* (100). What comes next? Have students explore. They can stack 10 flats to make a larger cube—the same shape as the little one, only 1000 times larger. What comes next? (See Figure 10.13.) Ten large cubes make another long. What comes next? Ten big longs make a big flat. The first three shapes have now repeated! Ten big flats will make an even bigger cube, and the trio of shapes begins again. Although we are using the terms *cube, long*, and *flat* to describe the shape of the materials, it is still critical to call these representations "ones, tens, and hundreds," particularly for students with special needs in math. We need to name them consistently by the number they represent rather than by their shape. This language reinforces conceptual understanding.

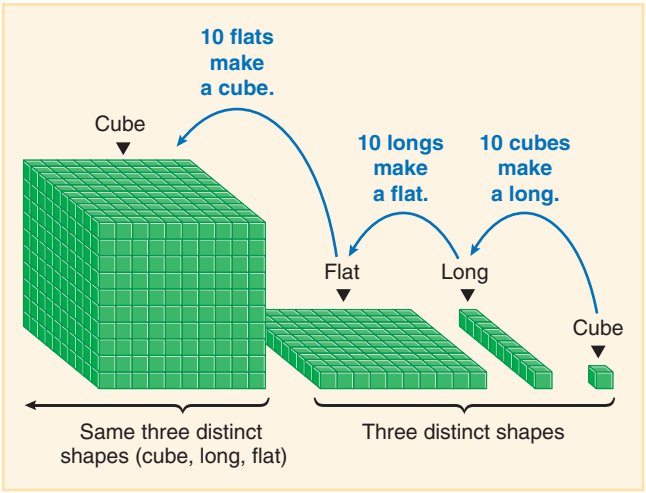


FIGURE 10.13 With every three places, the shapes of the base-ten materials repeat

How far you want to extend this generalization of the square-strip–square-strip sequence depends on your class. The idea that 10 in one place makes 1 in the next can be brought home dramatically and memorably. It is quite possible with older students to make the next 10-m × 10-m square, using chalk lines on the playground. The next strip is 100 m × 10 m. This model can be measured out on a large playground with students marking the corners. By this point, the payoff includes an appreciation of both the exponential increase in size of each successive amount and the 10-makes-1 progression (powers of ten). The 10-m × 10-m square models 1 million, and the 100 m × 10 m strip is the model for 10 million. The difference between 1 million and 10 million is dramatic.

Each cube has a name. The first one is the *unit* cube, the next is a *thousand*, the next a *million*, then a *billion*, and so on. Each long is 10 cubes: 10 units, 10 thousands, and 10 millions. Similarly, each flat shape is 100 cubes and so on.

Activity 10.23 What Comes Next? (Semi-Concrete)

CCSS-M: 2.NBT.A.1; 2.NBT.A.3

Use the paper base-ten materials where the unit or ones piece is a 1-cm square, the tens piece is a 10-cm × 1-cm strip, and the hundreds piece is a 10 cm × 10 cm square. What is next? What shape would a thousand be? Students can tape together a long strip made of 10 paper hundreds squares. What comes next? Reinforce the idea of "10 makes 1" that has progressed to this point. Ten one-thousand strips would make a square measuring 1 meter (m) on each side, making a paper 10,000 model. Once the class has figured out the shape of each piece, the problem posed to them is, "What comes next?" Let small groups work on the dimensions of a 100,000 piece (they will likely need space in the hallway to do this!).

Pearson eText Blackline Master: Base-Ten Materials

The reading of large numbers requires students to understand the logical structure. To read a number, first mark it off in triples from the right. The triples are then read, stopping at the end of each to name the unit for that triple (see Figure 10.14). Leading zeros in each triple are ignored. If students learn to read numbers like 059 (fifty-nine) or 009 (nine), they should be able to read any number. To write a number, use the same scheme. If first mastered orally, the system is quite straightforward. Remind students *not* to use the word *and* when reading a whole number. For example, 106 should be read as "one hundred six," not "one hundred *and* six." The word *and* will be needed to signify a decimal point. Please make sure you and your students read numbers accurately.

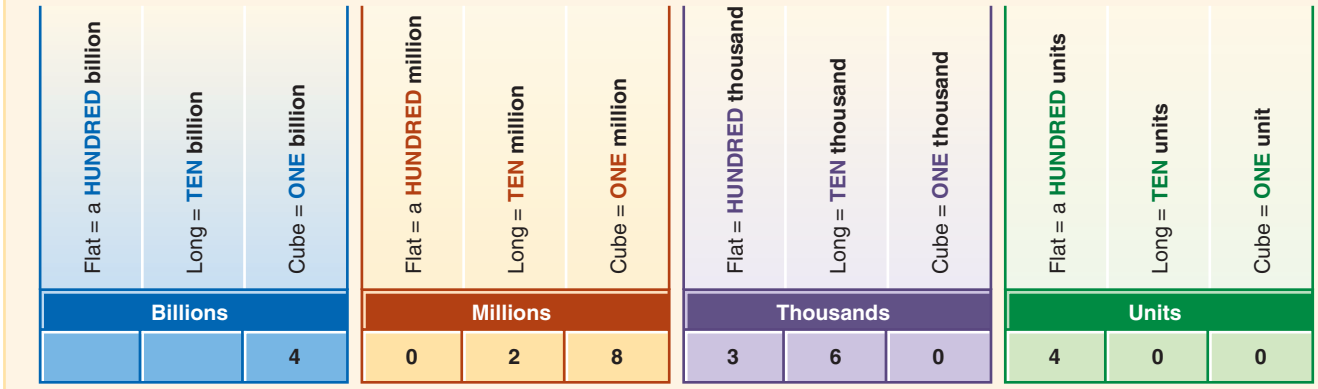


FIGURE 10.14 The triples system for naming large numbers

Conceptualizing Large Numbers

The ideas just discussed are only partially helpful in thinking about the actual quantities involved in very large numbers. For example, in extending the paper square-strip-square-strip sequence, some appreciation for the quantities of 1000 or of 100,000 is acquired. But it may be challenging for students to translate quantities of small squares into quantities of other items, distances, or time.

These ideas are important to discuss as a class, because students need to explore these relationships.

Pause & Reflect

How do you think about 1000 or 100,000? Do you have any real concept of a million? ●

In the following activities, numbers like 1000; 10,000 (see 10,000 Grid Paper); or even 1,000,000 are translated literally or imaginatively into something that is more meaningful or engaging to think about. Interesting quantities become memorable reference points or benchmarks for large numbers and thereby add meaning to numbers encountered in real life.

Pearson eText Blackline Master: 10,000 Grid Paper

Pearson eText
Video Example 10.4
In this video consider what Donny knows about place value based on his reasoning about 192,000 as 19,200 tens or 1920 hundreds.

Activity 10.24 Collecting 10,000

CCSS-M: 2.NBT.A.2; 2.NBT.A.3

As a class or grade-level project, collect some type of object with the objective of reaching some specific quantity—for example, 1,000 or 10,000 bread tabs or soda can pop tops. If you begin aiming for 100,000 or 1,000,000, be sure to think it through. One teacher spent nearly 10 years with her classes before amassing a million bottle caps. It takes a small dump truck to hold that many!

Activity 10.25 Showing 10,000

CCSS-M: 2.NBT.A.2; 2.NBT.A.3

Sometimes it is easier to create large amounts than to collect them. For example, start a project in which students draw 100 or 200 or even 500 dots on a sheet of paper. Each week, different students contribute a specified number. Another idea is to cut up newspapers into pieces the same size as dollar bills to see what a large quantity of money would look like. Paper chain links can be constructed over time and hung down the hallways with benchmark numbers marked. Let the school be aware of the ultimate goal.

Activity 10.26 How Long?/How Far?

CCSS-M: 2.NBT.A.1a; 2.NBT.A.3

Talk about real or imagined distances with students by posing investigations for them to consider, such as, "How long is a million baby steps?" Other ideas that address length include a line of toothpicks, dollar bills, or energy bars end to end; students holding hands in a line; blocks or bricks stacked up; or students lying down head to toe. Standard measures—feet, centimeters, meters—can also be used with students noting that larger numbers in the measurements emerge when the smallest units are used.

Activity 10.27 A Long Time

CCSS-M: 3.MD.A.1

How long is 1000 seconds? How long is a million seconds? A billion? How long would it take to count to 10,000 or 1,000,000? (To make the counts all the same, use your calculator to do the counting. Just press [=].) How long would it take to do some task like buttoning a button 1000 times?

Activity 10.28 Really Large Quantities

CCSS-M: 2.MD.A.3; 3.MD.A.1; 3.MD.C.5

Ask how many:

- Energy bars would cover the floor of your classroom.
- Steps an ant would take to walk around the school building.
- Grains of rice would fill a cup or a gallon jug.
- Quarters could be placed in one stack from floor to ceiling.
- Pennies can be laid side by side down the entire hallway.
- Pieces of notebook paper would cover the gym floor.
- Books on shelves in the library.
- Bricks in a wall (consider using *Billions of Bricks: A Counting Book about Building* (Cyrus, 2016)).
- Seconds you have lived.

Common Challenges related to Place Value

Big-number projects can be explored over several weeks as a group task, take-home, or, perhaps best of all, translated into great schoolwide estimation events. In Chapter 22, the discussion of large numbers extends to using scientific notation to record quantities (which is based on place value).

Now that you've explored many place-value ideas in this chapter, look at Table 10.4 for possible common challenges identified in the research. Suggestions on what you might notice and how to help are included.

TABLE 10.4 COMMON CHALLENGES RELATED TO PLACE VALUE

Common Challenge	What It Looks Like	How to Help
1. Recognizing that each digit in a multi-digit numeral carries a value dependent on its position in the number.	When students are asked to compare the value of numbers, as shown bolded in the following two amounts, they will say they are the same. **2**357 and 49,99**2**.	• Have students show with base-ten materials the value of these two numbers. Support students to read numbers in addition or subtraction problems as their value rather than as the digits (saying 5 tens or fifty rather than just 5). • Use the place-value cards (see Figure 10.10) to reinforce how each digit is the product of that digit by the value assigned to its position. • Students hear numbers like 2357 read as two, three, five, seven—when they should always be read two thousand, three hundred, fifty-seven. • Use the digit correspondence task described in this chapter to identify which of the five levels of understanding matches your student's performance and work from that point.
2. Reversing the digits when writing two-digit numbers.	Writes "53" when should write "35."	• Have students use virtual base-ten materials that display the corresponding number to check their answer. • Have the student model both 53 and 35 with base-ten materials and describe how the numbers are similar and different.
3. Representing a number with base-ten materials using the face value of the digits.	When asked to represent 13 with base-ten materials, the student uses one piece for the "1" and three pieces for the "3" as shown 1 3 	• Have the student use base-ten materials to count out 13 single units. Then ask them to compare that amount to what they previously showed. • Have the student build the number with the place value cards and then use base-ten materials to represent the corresponding amounts. Again, compare to the amount originally shown.
4. Inserting the word "and" in a number when they read it aloud.	When reading 1016 students will say "one thousand and sixteen."	• Practice reading numbers without using the word "and." The only time the word "and" is used is to represent a decimal point.
5. Using a form of expanded number writing called additive composition.	Students write "three hundred eighty-five" as something like 300805, 310085, or 3085.	• Provide examples of base-ten materials on a place-value mat and use the place-value cards to show how the matching number is built.
6. Ignoring or misunderstanding an internal zero.	Given 5 hundreds and 8 units in base-ten materials, students will write that number as 85. The student believes that 802 and 8002 represent the same amount.	• Focus on the meaning of a zero in any number by starting with 408 and asking how that would be shown with materials. Explicitly discuss the role of an internal 0 in the number. • Never refer to 0 as a placeholder. This terminology gives the impression that it is not a numerical value and is there just to fill a space. • Never read or refer to 0 as oh or zip. Say "zero" as it is a number.
7. Writing a number as given left to right, not considering place value.	When students are asked to write the number that represents: 7 ones, 4 tens, 1 thousand, and 3 hundreds. They write 7413.	• Use base-ten materials and the place value mat to take out the same amount of base-ten materials as in the number. Then have the student write the number of base-ten materials. Then they compare the two answers to consider which one is accurate.
8. Interpreting the value of the base-ten materials incorrectly.	Students think the value of the 1000 place value block is 600 by just calculating the number of squares on each face of the cube.	• Show students how the blocks are built (grouping into a unit). So explicitly show the building of the 1000 cube by taking ten hundreds and forming a cube (grouping them with elastic bands).

RESOURCES FOR CHAPTER 10

WRITING TO LEARN

Assess your understanding and application of chapter content by answering these questions.

1. Name the three ways one can count a set of objects. Explain how these methods of counting can be used to coordinate concepts and oral and written names for numbers.
2. Describe the three types of physical models for base-ten concepts. What is the significance of the differences between these models?
3. How can students learn to write two- and three-digit numbers in a way that is connected to the base-ten meanings of ones and tens or ones, tens, and hundreds?
4. What are some of the ways the hundred chart (or bottom-up hundred chart) can be used to identify and use place-value concepts?
5. What are benchmark numbers? Describe the relationships you want students to develop concerning benchmark numbers. Describe an activity that addresses these relationships.

LITERATURE CONNECTIONS

Books that emphasize groups of things, even simple counting books, are a good beginning to the notion of ten things in a single group. Many books have wonderful explorations of large quantities and how the numbers can be composed and decomposed. Here are some suggestions by the numbers.

One Hundred:

Miss Mingo and the 100th Day of School

Harper (2020)

The 100th Day of School from the Black Lagoon

Thaler (2014)

One Thousand:

A Thousand Glass Flowers: Marietta Barovier and the Invention of the Rosetta Bead

Turk (2020)

A Thousand No's: A Growth Mindset Story of Grit, Resilience, and Creativity

Corchin & Dougherty (2020)

Thousand Star Hotel

The Okee Dokee Brothers (2017)

One Million:

How Much Is a Million?

Schwartz (2004)

On Beyond a Million: An Amazing Math Journey

Schwartz (2001)

If I Had a Million Bucks

Johnson (2012)

RECOMMENDED READINGS

Articles

Burris, J. T. (2013). Virtual place value. *Teaching Children Mathematics*, *20*(4), 228–236.

This article explores teaching place value using both concrete base-ten materials and virtual versions to help reinforce conceptual structures, particularly for showing equivalent representations. Use any of the virtual materials suggested in this chapter.

Books

Richardson, K. (2003). *Assessing math concepts: Grouping tens.* Mathematical Perspectives.

This book is one of a nine-part series on using diagnostic interviews and other assessment tools to understand students' grasp of a concept—in this case, grouping by tens.

Vennebush, G. P. (2020). *One-hundred problems involving the number 100*. NCTM.

Here is a collection of challenging problems revolving around 100 that are aligned to a range of mathematics standards for Pre-K–12.

CHAPTER

11

Developing Strategies for Addition and Subtraction Computation

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **11.1** Identify the interplay between place-value understanding and addition and subtraction reasoning strategies.
- **11.2** Explain multiple reasoning strategies students can use for carrying out addition and subtraction with multi-digit numbers.
- **11.3** Explain the development of the standard algorithms for addition and subtraction, including ways to record students' thinking.
- **11.4** Identify ways to teach computational estimation to develop students' flexibility and ability to recognize reasonable answers.
- **11.5** Identify a variety of computational estimation strategies.

Much of the public sees computational skills as the hallmark of what it means to know mathematics at the elementary school level. Although that is far from the whole story, learning computational skills with whole numbers is, in fact, a critical component of the curriculum. Expectations for competency in today's workforce as well as in daily life mean that changes are warranted in how computation is taught, focusing more on reasoning strategies than on computations that are otherwise done on a phone or calculator.

Chen and Siegler (2000) suggest a process that young children use when solving problems, and this process also reflects how students develop new computational strategies over time. This progression is presented in Table 11.1 and is mapped to a strategy you learned in Chapter 9 (basic facts).

Rather than presenting a single method of adding or subtracting, the most appropriate method can and should change flexibly as the numbers and the context change. This is procedural fluency as described in Chapter 2. The goal is no longer just a matter of knowing how to subtract three-digit numbers; rather, it is the development over time of an assortment of flexible skills and procedures that are meaningfully linked to conceptual understanding. *Adding It Up* (National Research Council, 2001) describes it this way:

More than just a means to produce answers, computation is increasingly seen as a window on the deep structure of the number system. Fortunately, research is demonstrating that both skilled performance and conceptual understanding are generated by the same kinds of activities (NRC, p. 182).

TABLE 11.1 PROGRESSION OF NEW STRATEGY DEVELOPMENT

Process	What a Student Might Think or Say
1. New strategy acquisition	I see how to use the Making Tens strategy (i.e., that for $9 + 6$, I can move one over to rethink the problem as $10 + 5$).
2. Mapping the strategy to new problems	Hmm, that strategy can work for $6 + 8$! I can rethink it as $4 + 10$.
3. Strengthening the strategy for use with original and new problem types	This strategy is easy; look, it works for any fact combination that adds up to more than 10.
4. Refining the choice of a strategy for a particular problem, based on known strategies	I don't want to use Making Tens strategy for $6 + 7$, so I am going to use a different method, like $6 + 6 + 1$. I will use the Making Tens strategy when I see addends of 8 or 9.
5. Building the effective implementation of the strategy	I can use this strategy correctly, and it makes sense to me.

Adapted from Chen & Siegler, 2000, p. 8–9

To reach these expectations, the following big ideas are essential.

BIG IDEAS

- Flexible methods of addition and subtraction computation involve taking apart (decomposing) and combining (composing) numbers in a wide variety of ways. Most of the decomposition of numbers is based on place-value parts or compatible numbers—which are number pairs that work easily together, such as 25 and 75.
- Reasoning strategies provide flexible methods of computing that vary with the numbers and the situation. Successful use of the strategies requires that they be understood by the one who is using them.
- Flexible methods for computation require a deep understanding of the operations and properties of the operations (the commutative property and the associative property), including the inverse relationship between addition and subtraction.
- Standard algorithms are step-by-step processes based on place value.
- Multi-digit numbers can be built up or taken apart in a variety of ways to make the numbers easier to work with. These parts can be used to estimate answers in calculations rather than using the exact numbers involved. For example, 36 is the same as 30 and 6 or 25 and 10 and 1. Also, 483 can be thought of as $500 - 20 + 3$.
- Computational estimation involves using easier-to-handle parts of numbers or substituting difficult-to-handle numbers with close compatible numbers so that the resulting computations can be done mentally.

Addition and subtraction strategies that build on decomposing and composing numbers in flexible ways contribute to students' overall number sense. In most everyday instances, these strategies for computing are easier and faster than standard algorithms and can often be done mentally. Therefore, it is best for students to learn a variety of methods that they can select from as needed.

Consider the following problem:

Mary has 114 spaces in her photo album. So far, she has put 89 photos in the album. How many more photos can she put in the album to fill it?

Pause & Reflect

Try solving the photo album problem, using a method other than the standard algorithm you were taught in school. If you are tempted to begin subtracting the 9 from the 4, try a different approach. Can you solve it mentally using number sense? Can you solve it in more than one way? Work on this problem before reading further. ●

Here are five of many possible ways primary school students used reasoning to solve the photo album problem (114 – 89):

89 + 11 equals 100; 11 + 14 equals 25.

90 + 10 equals 100, and 14 more is 24, plus 1 (because we should have started at 89, not 90) equals 25.

From 114, subtract 14 and then subtract 10 more, which equals 90, and then subtract 1 more to get to 89, which is 25 in all.

89, 99, 109 (that's 20), 110, 111, 112, 113, 114 (keeping track on fingers) equals 25.

89 + 11 equals 100. Adding the same amount of 11 to 114 equals 125. The difference between 125 and 100 equals 25.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Reasoning strategies such as these can often be done more efficiently than standard algorithms, and make more sense to the person using them. Every day, too many students (and adults) resort to standard algorithms when a reasoning strategy would be more efficient and less prone to error (Fuson & Beckmann, 2012/2013).

Flexibility is an important component of fluency (as illustrated in Chapter 2 and Figure 2.4) and as such is necessary for success in mathematics and in daily life. If you haven't developed your own repertoire of reasoning strategies, you will learn them as you read this chapter and as you teach your students to use them! Let's resume where we just left off—place value.

Connecting Addition and Subtraction to Place Value

Learning Outcome 11.1 **Identify the interplay between place-value understanding and addition and subtraction reasoning strategies.**

Concepts of numeration and place value can and should be developed in tandem with teaching the operations. Research suggests that problems involving addition and subtraction are a good context for learning place-value concepts (Wright, Martland, Stafford, & Stanger, 2008). If students only understand computation as a digit-by-digit exercise and not through the actual value of the numbers involved, they are more likely to make errors and are unable to judge the reasonableness of their answers.

Consider the work of two first graders (Figure 11.1). Jerrika solves a word problem for 10 + 13 + 22, using connecting cubes. Her written work shows she is still at a pre–place value stage. We can observe that she is beginning to use the idea of "1 ten" but likely counted on the remaining 35 cubes by ones. Her classmate, Monica, solved the same problem but used more base-ten understanding. Place-value concepts such as these continue to grow with additional problem solving and sharing of ideas in class discussions.

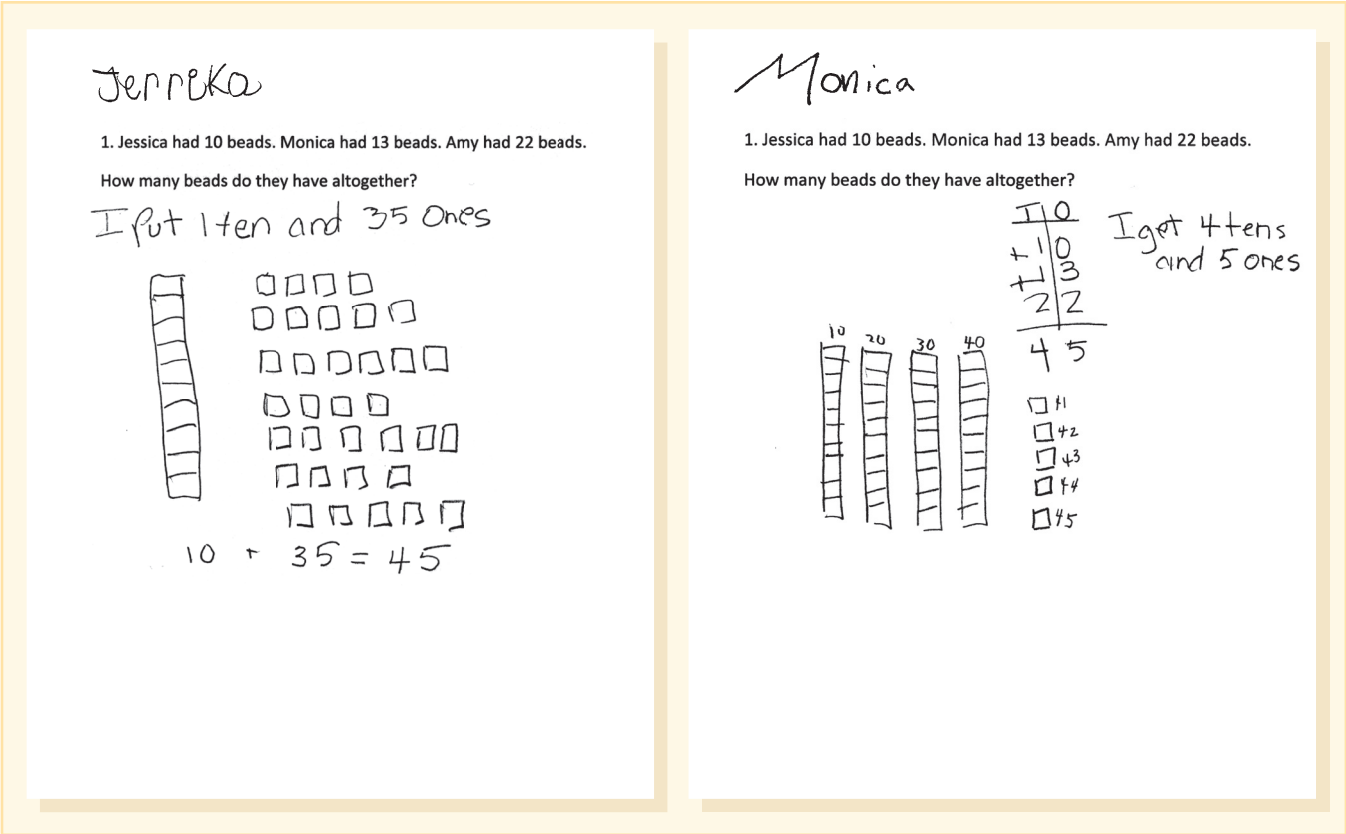


FIGURE 11.1 First-grade students solve 10 + 13 + 22

The activities in this chapter are designed to further students' understanding of place-value concepts while engaging them in addition and subtraction. The first of these bridging activities involves adding or subtracting 10 and noticing patterns (then adding and subtracting other values).

Activity 11.1 Calculator Challenge Counting

CCSS-M: 1.NBT.C.5

This activity begins with students adding 10 on a calculator to notice patterns. Students press any two-digit number on the calculator (e.g., 17) and then [+] 10. They say the sum before they press [=]. Then they continue to add 10 mentally, challenging themselves to say the sum before they press [=]. Students should see how far they can go without making a mistake.

Differentiate this activity for students by adjusting the numbers. You may want to begin with a starting number less than 10 for students with special needs in math or with a three-digit number such as 327 for students who need a challenge. Also, the constant addend ([+] 10 in the preceding example) can be any number with one, two, or three digits. Some students will find jumps of 5 can be challenging if the starting number is not a multiple of 5. Skip counting by 20 or 25 will be easier than counting by 7 or 12 and will help develop important patterns and relationships.

Then students try the reverse, subtracting by 10, entering a number such as 53 (or 123) in the calculator and press [−] 10 (or for a challenge, try [−] 5). As before, students state the expected result before pressing [=]. Each successive press subtracts whatever constant was entered. Have students share strategies for determining the sum or difference and discuss patterns that they notice.

A second important connection is knowing combinations that equal 100, for example, knowing the missing value in equations such as this: $78 + \_\_\_\_\_ = 100$. Benchmark numbers such as 100 and 50 are used in computational strategies to make the calculations easier. Activities 11.2 and 11.3 focus on these important benchmarks.

Activity 11.2 The Other Part of 100

CCSS-M: 1.NBT.B.4; 2.NBT.B.5

Two students work together with a set of little ten-frame cards. One student makes a two-digit number by using the ten-frame cards. Then both students work mentally to determine what should be added to the ten-frame amount to equal 100. They write their solutions on paper and then check by making the other part with the cards to see whether the total equals 100. Students take turns making the original number (see Other Part of 100 Activity Page). Figure 11.2 shows three thought processes that students might use. Have students look at Maurice's chart, decide whether Maurice is right, and, if needed, describe how to correct the mistake. For a challenge, change the whole from 100 to other multiples of 10 such as 70 or 80 or extend the whole to any number less than 100.

Pearson eText Blackline Master: Little Ten-Frame Cards
Pearson eText Activity Page: Other Part of 100
Pearson eText Activity Page: Maurice's Chart

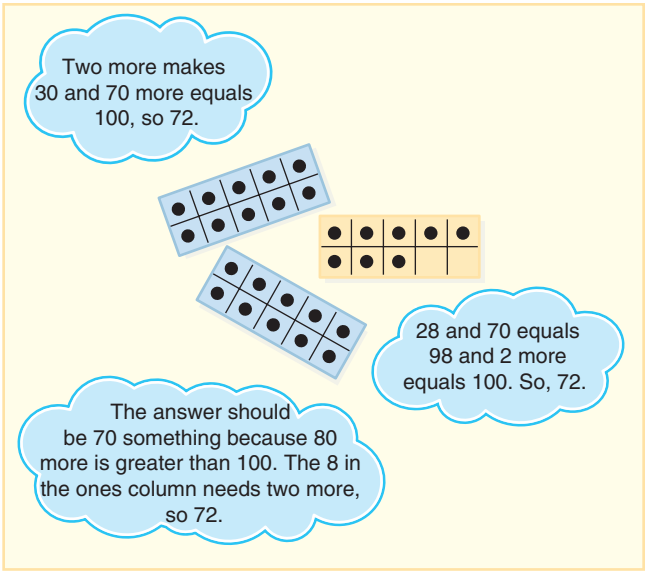

Two more makes 30 and 70 more equals 100, so 72.
28 and 70 equals 98 and 2 more equals 100. So, 72.
The answer should be 70 something because 80 more is greater than 100. The 8 in the ones column needs two more, so 72.

FIGURE 11.2 Thinking about the "other part of 100" when shown 28 on a card

Pause & Reflect

What if we wanted to extend the reasoning to numbers other than benchmarks? Suppose the whole in Activity 11.2 is 83. Use or imagine little ten-frame cards showing 36. What number added to 36 equals 83? How did you think about solving it? How do the ten-frames support your reasoning? ●

Maybe you added up from 36 to 83. Or you might have subtracted 33 from 83, which equals 50 and then subtracted 3 more. Either way, notice that you did not regroup. Most likely you did this calculation in your head, possibly using benchmark numbers and place value. With lots of practice, students as early as second grade can do this computation without the aid of the ten-frame cards.

Activity 11.3 50 and Some More

CCSS-M: 1.NBT.B.4

Say or write a number between 50 and 100. Students respond with "50 and _____." For 63, the response is "50 and 13." Any benchmark number can be used instead of 50. For example, you could use any number that ends in 50 for the first part, such as "450 and some more." Or you can use numbers such as 70 or 230 as starting points.

Compatible numbers for addition and subtraction easily combine to equal benchmark numbers. Numbers that equal tens or hundreds are the most common examples. Activity 11.4 focuses on looking for combinations that work together to equal various benchmarks.

Activity 11.4 Compatible Pairs Search

CCSS-M: 2.NBT.B.5

This activity can be done with the full class or as a learning center. If it is full class, project one of the searches in Figure 11.3 (see Compatible Pairs activity pages). Pair students up to find and record as many pairs as they can. Debrief by asking what helped students find the pairs. Ask, "What patterns did you notice?" As a learning center, students record their pairs and respond to the prompt in writing.

Pearson eText Activity Page: Compatible Pairs

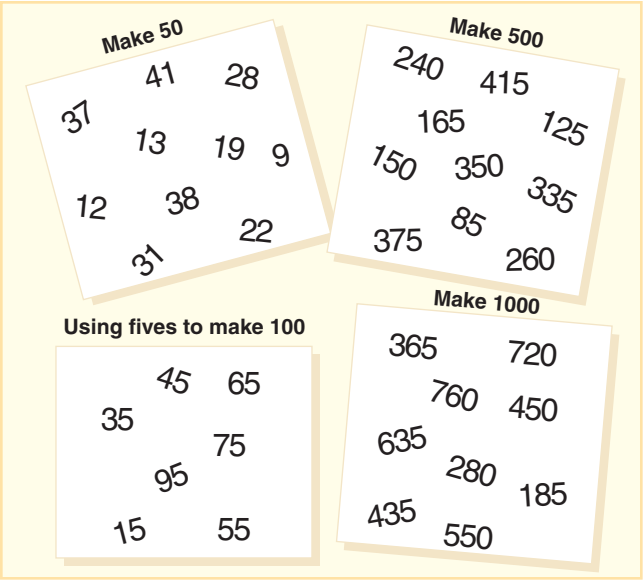


FIGURE 11.3 Compatible-pair searches

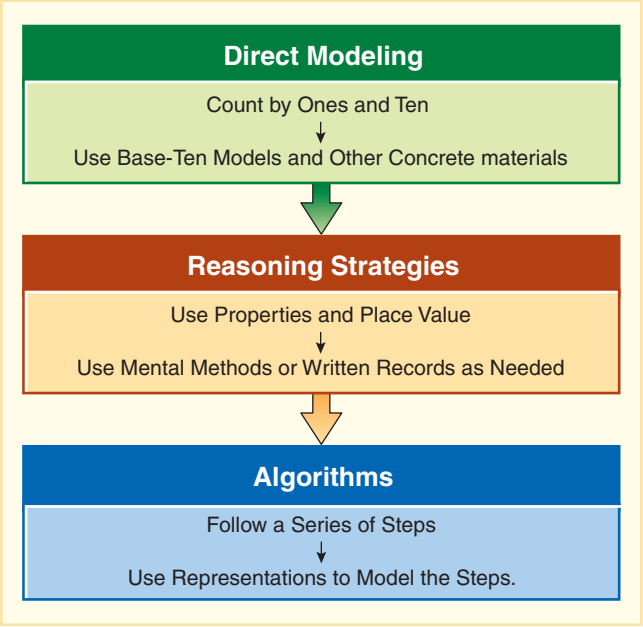

Direct Modeling
Count by Ones and Ten
Use Base-Ten Models and Other Concrete materials
Reasoning Strategies
Use Properties and Place Value
Use Mental Methods or Written Records as Needed
Algorithms
Follow a Series of Steps
Use Representations to Model the Steps.

FIGURE 11.4 Three ways to compute

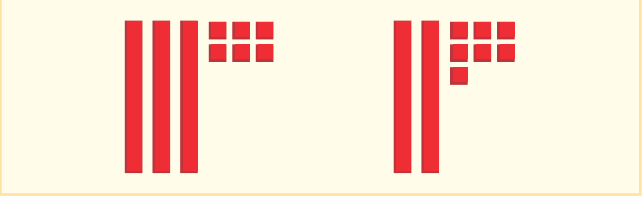


FIGURE 11.5 A direct modeling of 36 + 27, using base-ten models

Once students have an understanding and skill at finding combinations that equal benchmarks, they are ready to begin using their place-value knowledge to add and subtract.

Progression for Computational Fluency

How do students go from place-value understandings to being fluent with addition and subtraction? Figure 11.4 illustrates a general progression from more concrete experiences toward the abstract use of algorithms. The arrows show how students move from acting out situations with direct modeling and counting to the use of reasoning strategies to the development of standard algorithms. The *direct modeling* methods can, with guidance, develop into an assortment of more flexible and useful *reasoning strategies*, many of which can be carried out mentally. The *algorithms* remain an important part of what students need to learn. Remember, a strategy is a general method to solve a problem flexibly, whereas an algorithm is an established step-by-step procedure, and both are needed to reach the goal of computational fluency.

Direct Modeling. The developmental step that is linked to what students already know about counting strategies and place values is called *direct modeling*. This strategy involves the use of counting with manipulatives or drawings to represent the meaning of the addition or subtraction operation or related story problems directly (see Figure 11.5).

Students who consistently count by ones in additive situations may need to have more experience developing base-ten grouping concepts. That does not mean you should avoid giving them problems involving two-digit numbers. Rather, as you work with students who are still challenged to see ten as a unit, suggest that they use thinking tools, such as placing counters in a ten-frame or making groups of ten from connecting cubes. Some may initially use the base-ten rod of 10 as a counting device to keep track of counts of ten, even though they are counting each 10 rod by ones. As students complete each intermediate count, have them write down the corresponding numbers for memory support.

Eventually, students need to move on from relying on counting by ones and transition to other approaches (Hopkins, Russo, & Siegler, 2020). When students have constructed the idea of ten as a unit, they should begin to use this idea to bridge from direct modeling to reasoning strategies derived from number sense and the properties of operations and tailored to particular problems. Direct modeling is a necessary phase for students to work through. This progression is important because students who need more experiences before using more efficient methods are able to explore the same problems as classmates who have progressed beyond this stage. Avoid pushing students to prematurely abandon

concrete approaches using materials; however, some students may need encouragement to move away from the prolonged use of direct modeling. Here are some ideas to promote the fading of direct modeling:

- Record students' verbal explanations on the board in ways that they and others can follow.
- Ask students who have just solved a problem with direct modeling if they can do the calculations mentally.
- Ask students to make a written numeric record of how they solved the problem with models. Then have them try the same written method on a new problem.
- Introduce opportunities for students to explore and compare reasoning strategies; some researchers suggest there is too heavy a focus on counting strategies and not enough on other approaches (Cheng, 2012; Gaidoschik, 2019).

Reasoning Strategies. *Reasoning strategies* are flexible methods to compute that are based on a person's understanding of place value and the properties. We use the language *reasoning* here in that the development of these mathematical strategies emerge from a constant process of decision making as students select strategies based on the characteristics of the problem. Reasoning strategies often become mental methods after the ideas have been explored, understood, and practiced. For example, after some experiences with ten-frames or number lines, students will be able to solve $75 + 19$ mentally, using a strategy such as compensation: $75 + 20$ is 95; less 1 is 94.

A number of research studies have focused attention on how students of a variety of ages respond to computational situations when they have been given options for multiple strategies (Csikos, 2016; Keiser, 2010; Rittle-Johnson, Star, & Durkin, 2020; Verschaffel, Greer, & De Corte, 2007). One of the best ways for students to grow their repertoire is to listen to classmates' strategies as they are shared and explored and then try them out. However, students should not use any strategy without understanding it.

Benefits of Reasoning Strategies. The development and use of reasoning strategies generate procedural proficiency and number sense. The positive benefits are as follows:

- *Students make fewer errors.* Research reveals that students using methods they understand make many fewer errors because they select their own methods (Gravemeijer & van Galen, 2003). If students do not understand the underlying concepts of standard algorithms, not only do they make errors, but the errors are often systematic and difficult to remediate. Errors with reasoning strategies are less frequent and rarely systematic.
- *Less reteaching is required.* Teachers often are concerned when students' early efforts with reasoning strategies are slow and time-consuming. But the productive struggle in these early stages builds a meaningful and well-integrated network of ideas that is robust and long-lasting, and decreases the time required for reteaching.
- *Students develop number sense.* Students' development and use of number-oriented, flexible approaches offers them a rich understanding of the number system through strategies they understand.
- *Supports mental math and computational estimation.* When reasoning strategies are used, mental computation is part of the process and not a separate skill. As students become more proficient with these flexible methods, they find they are able to use them mentally or sometimes only need to jot down intermediate steps.
- *Frequently more efficient than standard algorithms.* Consider $38 + 44$. A reasoning strategy might involve making a ten, rethinking the problem as $40 + 42$, which equals 84. This process is much quicker than carrying out the steps of the standard algorithm.
- *Strategy development involves generalization, which is part of "doing mathematics."* Students who choose and use a variety of strategies are making sense of the quantities.
- *Supports students on standardized tests.* Evidence suggests that students using reasoning strategies do as well or better than students using standard algorithms in problems involving computation on standardized tests (Fleischman, Hopstock, Pelczar, & Shelley, 2010; Fuson, 2003).

Pearson eText
Video Example 11.1
In this video consider the two different reasoning strategies this student uses to mentally solve a two-digit addition problem.

- *Develops positive mathematics identities.* As students use strategies that make sense to them, and are invited and expected to choose strategies, they develop confidence in their ability to do mathematics. Thus, a focus on reasoning strategies encourages positive mathematics identity and a sense of mathematical agency.

Pause & Reflect

Try mental computation with this example:

$$396 + 157 + 489$$

Could you carry out this task mentally without recording any written record? Did you add by place value and hold the numbers in your working memory? At some point, maintaining many numbers in one's head is difficult and calls for a way to record thoughts. Did it take you a while to do it? When problems do not lend to mental processes, we rely on efficient algorithms. Posing a problem like this for students offers a teachable moment to help students recognize the role of formalized procedures. ●

Algorithms. An algorithm is an established set of steps that can be used to solve a mathematical computation. A standard algorithm refers to the algorithm that has been adopted as the best algorithm (which may vary from country to country, which is why you will sometimes see the designation "U.S. standard algorithm"). Without strong foundational experiences, standard algorithms can tend to cause students to think in terms of digits rather than the composite number that the digits make up, so without connections to the concepts, students may lose track of the actual place value of a digit.

There are three main components to know about the standard algorithms:

(1) Knowing the step-by-step procedure and how it is performed
(2) Knowing why the algorithm works and how it can be applied
(3) Knowing when an algorithm should be used and comparing the value and usefulness of different algorithms (Fan & Bokhove, 2014)

The focus in teaching standard algorithms should not be as merely a memorized series of steps, but as making sense of the procedure as a process. Algorithms should have the characteristics of certainty (precise procedures), reliability (always a correct answer if carried out properly), transparency (the process is understood), efficiency (effective approach), and generalizability (solves a collection of similar problems) (Fan & Bokhove, 2014).

The standards in every state require that students eventually have knowledge of the standard algorithms (see Table 11.2). Notice that the grades in which this knowledge of standard algorithms is required occurs after the topic is introduced, sometimes years later. This timeline points to the need for conceptual development and the learning of reasoning strategies to be developed first.

TABLE 11.2 WHAT STATE STANDARDS DOCUMENTS COMMONLY SUGGEST REGARDING THE DELAY OF STANDARD ALGORITHMS

Standard Algorithm	When Topic Is Introduced and Developed	When Standard Algorithm Is Required
Add and Subtract Multi-digit Whole Numbers	Grades 1–3	Grade 3 (end) or 4
Multiply Multi-digit Whole Numbers	Grades 3 and 4	Grade 5
Divide Multi-digit Whole Number	Grades 3 and 4	Grade 6

Standard Algorithms Must Be Understood. Students may learn the standard algorithms from siblings and other family members while you are still trying to teach a variety of other strategies. Some of these students may resist learning more flexible strategies, thinking that they already know the right approach. What do you do then?

First and foremost, apply the same rule to standard algorithms as to all strategies: *If you use it, you must understand why it works and be able to explain how it works.* In an atmosphere that says, "Let's figure out why this procedure works," students profit from making sense of

standard algorithms in the same way they should be able to reason about other approaches. But the responsibility for the explanations should be theirs, not yours. Remember, "Never say anything a kid can say!" (Reinhart, 2000).

The standard algorithm, once understood, is one more strategy to put in the students' toolbox of methods. Algorithms are a significant part of the development of a deep understanding of mathematics (Fan & Bokhove, 2014). Regularly reinforce the idea that just like the other strategies, it may be more useful in some instances than in others. For example, pose problems in which a mental strategy is more useful than an algorithm, such as for 504 − 498, or a problem such as 4568 + 12,813, where the standard algorithm has distinct advantages. Discuss which strategy seems best in a variety of situations.

Standards for Mathematical Practice
MP7. Look for and make use of structure

Delay! Delay! Delay! You will likely introduce and explain each algorithm to students and help them understand how and why it works. The question is when you should do that, because no matter how carefully you introduce these algorithms into your classroom as simply another alternative, students may conclude it is the one right way or the best way. So spend a significant amount of time with other strategies—months, not weeks. The understandings students gain from working with reasoning strategies makes it easier for you to teach standard algorithms meaningfully. If you think you are wasting precious time by delaying, just be reminded of how many years standard algorithms are taught over and over to students who still make errors and are unable to explain them. (In one study, only 33% of the pre-service teachers could accurately explain the addition and subtraction algorithms) (Thanheiser, 2009).

Pearson eText
Video Example 11.2
In this video consider how the student compares a reasoning strategy with the standard algorithm.

Cultural Differences in Algorithms. Some people falsely assume that mathematics is universal. In fact, there are many international differences in notation, conventions, and algorithms. Knowing more about the diverse algorithms students might bring to the classroom and their ways of recording symbols for doing mathematics will assist you in supporting students and responding to families. What the United States calls the standard algorithm may not be customary in other countries, so encouraging a variety of algorithms is important in valuing the experiences of each and every student.

For example, *equal addition* is a subtraction algorithm used in many Latin American countries (Lopez, n.d.). It is based on the knowledge that adding the same amount to both the minuend and the subtrahend will not change the difference (answer). Therefore, if the expression to be solved is 15 − 5, there is no change to the answer (or the difference) if you add 10 to the minuend and subtrahend and solve 25 − 15. There is still a difference of 10. Let's consider 62 − 27. Using the algorithm that you may think of as standard, you would likely regroup by crossing out the 6 tens, adding the 10 with a small 1 to the 2 in the ones column (making 12 ones), and then subtract 7 from the 12 and so forth. In the *equal addition* approach (see Figure 11.6), you add 10 to the ones place (2) in 62 by just mentally adding a small 1 in the ones column (to represent the 10) resulting in 12, and then counteract that addition of 10 to the minuend by mentally adding 10 to the 27 (subtrahend) by increasing the tens column by one and subtracting 37.

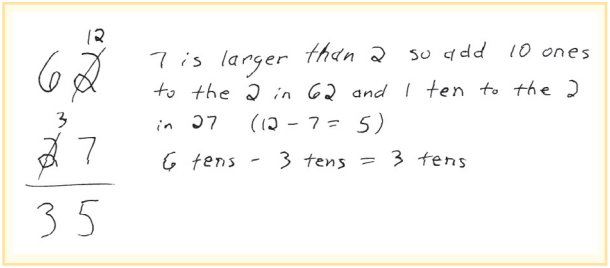
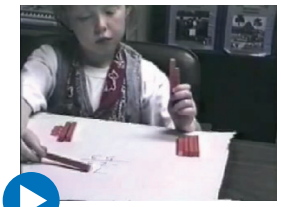


FIGURE 11.6 Example of the equal-addition subtraction algorithm



Pearson eText
Video Example 11.3
In this video consider the different methods used, the solutions the student gets, and the reaction to these solutions. If you were this student's teacher, what would you consider doing next?

Another key component to understanding cultural differences in algorithms is the emphasis on mental mathematics in other countries (Perkins & Flores, 2002). In fact, some students pride themselves on their ability to do math mentally. Don't be surprised if students from other countries can produce answers without showing work.

Reasoning Strategies in Addition and Subtraction

Learning Outcome 11.2 **Explain multiple reasoning strategies students can use for carrying out addition and subtraction with multi-digit numbers.**

There are three groups of reasoning strategies to solve addition and subtraction situations that can be extended to higher numbers (Ebby, Hulbert, & Broadhead, 2020; Torbeyns, De Smedt, Ghesquiere, & Verschaffel, 2009). These groups are:

- ***Jump strategies*** (counting on [addition], Think-Addition [subtraction as difference], or Counting-Back [subtraction as take away]).
- ***Decomposition strategies*** (partitioning or breaking apart one or both numbers).
- ***Compensation strategies*** (changing the problem in some way and then compensating for that change).

Usually, students do not spontaneously invent computational methods while you watch; instead, teachers' methods and instructional tasks have an effect on the approaches students develop (Verschaffel et al., 2007). Here we discuss pedagogical methods that support students' development of reasoning strategies for multi-digit addition and subtraction.

Creating a Supportive Environment for Reasoning Strategies

Reasoning strategies are developed out of a strong understanding of numbers, which is supported when learners publicly share emerging ideas. Students need a classroom environment where they can act like mathematicians and investigate ways to solve problems, knowing that mistakes may happen along the way, and these are opportunities for learning, not points of embarrassment. As students consider new reasoning strategies, they should find your classroom a safe and nurturing place for expressing their ideas. These factors will help to create such an environment:

- Use familiar contexts in story problems to build background and connect to individual students' perspectives. Avoid using naked-number problems as a starting point, because they do not encourage strategy development.
- Avoid immediately identifying the right answer when a student states it. Give other students a chance to consider whether they think the answer and approach are correct.
- Expect and encourage student-to-student interactions, questions, discussions, and conjectures about which computational strategies are efficient and effective. Allow plenty of time for discussions.
- Encourage students to clarify previous knowledge and make attempts to construct new ideas from their prior learning.
- Promote curiosity and openness to trying new strategies.
- Talk about ideas in a non-evaluative and nonthreatening way.
- Move less-efficient strategies to more-efficient strategies through strategic questioning, such as asking, "Which of these strategies would you choose to use the next time you see a problem like this one?" and "When is the [new, efficient reasoning strategy] a good strategy for us to use?"
- Show worked examples of computational approaches (both correct and incorrect) and invite students to critique the reasoning used (Barlow, Gerstenschlager, & Harmon, 2016; McGinn, Lange, & Booth, 2015). (Worked examples are discussed in Chapter 4.)

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

Jump Strategies

The *jump strategies* focus on partitioning one addend and often use the representation of an *open number line* (also known as an *empty number line*). The open number line has no prewritten numbers or tick marks. Students can use this tool to incorporate a sequential jump strategy that is very effective for thinking about addition and subtraction situations (Caldwell, Kobett, & Karp, 2014; Gravemeijer & van Galen, 2003; Verschaffel et al., 2007). The open number line is more flexible than the traditional number line because it can be jotted down anywhere,

works with any numbers, and eliminates the need for formalized tick marks and precisely proportional spaces between them. It is also important in building a mental number line. Also, students are less prone to making computational errors when using an open number line (Verschaffel et al., 2007).

Counting on (also called adding up) is an important addition strategy and a good place to begin whole-number addition because it uses place-value concepts. Similarly, counting back is a good first strategy for teaching subtraction. In both cases, the idea is that students are skip-counting by 10s and 1s to find the sum or difference.

Activity 11.5 uses semi-concrete base-ten representations to engage students in exploring the counting on and counting back strategies.

Activity 11.5 Numbers, Squares, Lines, and Dots

CCSS-M: 2.NBT.B.7

Use the Numbers, Squares, Lines, and Dots activity pages to introduce students to the use of hand-drawn squares (hundreds), lines (tens), and dots (ones) to represent concrete base-ten materials. Then, as illustrated in Figure 11.7 (addition) or Figure 11.8 (subtraction), display problems by using a number along with a quantity represented by squares, lines, and dots. Students then mentally compute the sums or differences. A good starting place is the subtraction problems with the removed amount represented by the numeral. To scaffold the activity, have students share their thinking for each task before moving to the next. You can do this by displaying individual problems or cutting the activity page into single problems and have students record their thinking. It is always important to discuss these ideas as a class.

Pearson eText Activity Page: Numbers, Squares, Lines, and Dots – Addition
Pearson eText Activity Page: Numbers, Squares, Lines, and Dots – Subtraction

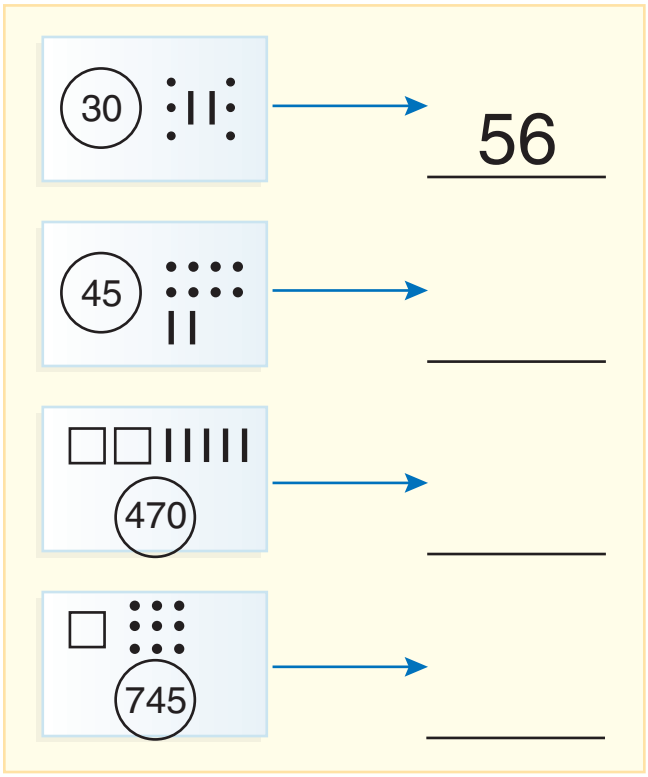


FIGURE 11.7 Tasks to explore counting on that link semi-concrete and abstract representations

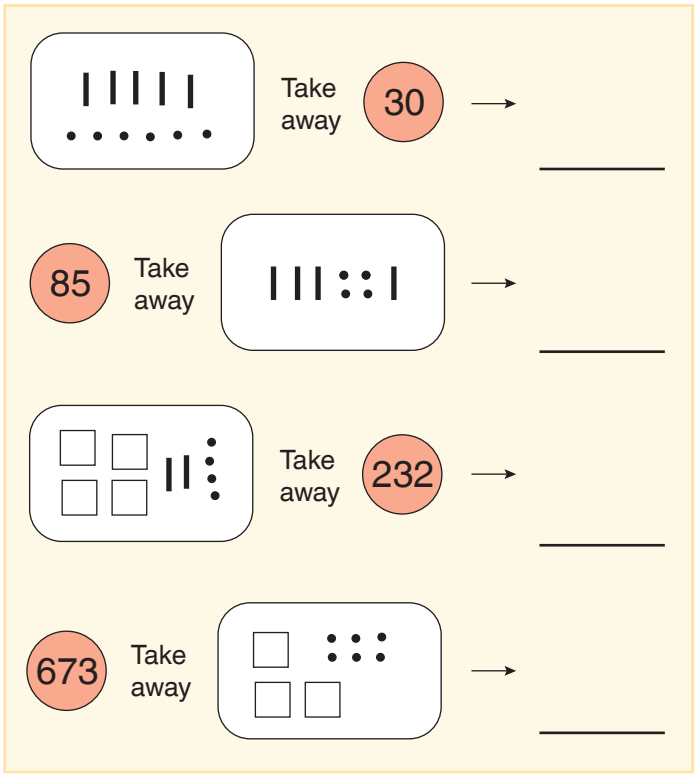


FIGURE 11.8 Tasks to explore counting back to subtract that link semi-concrete and abstract representations

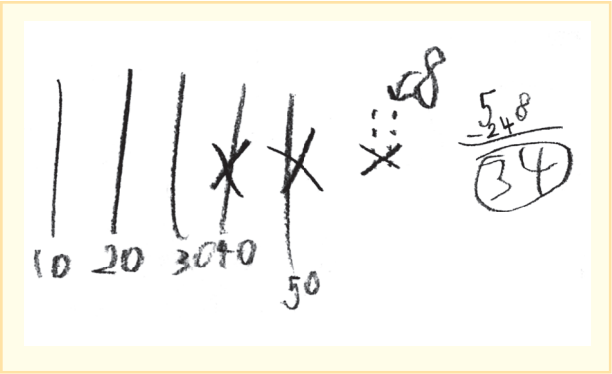


FIGURE 11.9 A student subtracts using a line-and-dot model

Students can also use these square, line, and dot representations as a way to calculate or check answers, as shown in the work of a second grader in Figure 11.9.

FORMATIVE ASSESSMENT Notes. Students who exhibit difficulty with any of these activities may be challenged with adapting computation strategies to the different numbers in a problem. Therefore, conduct a diagnostic interview to find how students strategize the addition and subtraction exercises in Activity 11.5, "Numbers, Squares, Lines, and Dots." Students are required to have sufficient understanding of base-ten concepts to use those ideas in meaningful counts. If students are still counting by ones, before moving on, back up and consider additional counting activities in which students have opportunities to see the value of grouping by ten. ■

Students can also use other semi-concrete place-value models that include ten-frames and the hundred chart for addition computation, as shown in the next activities.

Activity 11.6 Little Ten-Frame Sums

CCSS-M: 2.NBT.B.5

Provide pairs of students with two sets of little ten-frames. Each student chooses a number. An example (47 + 36) is shown in Figure 11.10. Partners work to find the total number of dots, recording the pair of numbers and the sum. The activity can also be done by projecting the two ten-frames for 10–15 seconds and asking the students to write the total. Give students a second look at the cards if they need one!

Pearson eText Blackline Master: Little Ten-Frames

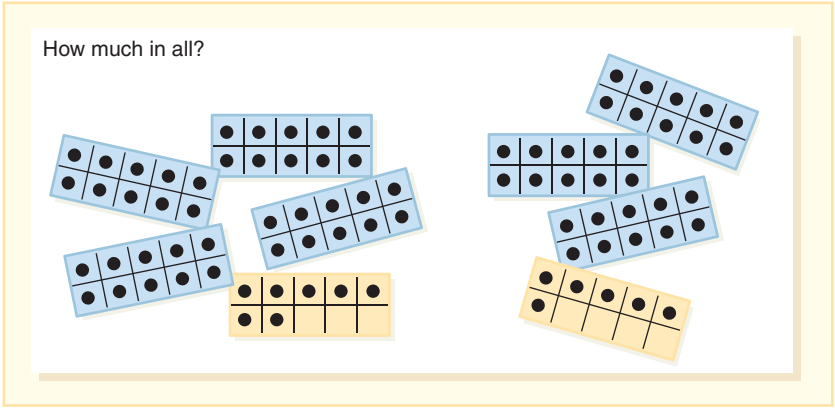


FIGURE 11.10 Using little ten-frame cards to add

Activity 11.7 Hundred Chart Addition

CCSS-M: 2.NBT.B.5

Display a hundred chart (or a thousand chart) or, alternatively, give students their own individual hundred charts, using the four-small-hundred charts. Students use the hundred chart to add two numbers (e.g., 38 and 14). There are many ways students can use the hundred chart for addition, so the value is in having students work on one sum at a time and then discussing and comparing the different methods used.

Pearson eText Blackline Master: Four Small Hundred Charts
Pearson eText Blackline Master: Four Small Bottom-Up Hundred Charts

The hundred chart can be thought of as a stacked number line—one that highlights the distance from any number to the next multiple of 10. Movement down a row is the same as adding 10 (notice they are using a top-down chart), and a move up a row is 10 less. As illustrated in Figure 11.11(a), one student added 13 + 12 by just counting by ones. If students simply count on 12 individual squares from 13, that may indicate that they do not yet grasp the important place-value concept of counting by groups of tens from any starting value. Consider how a student might use the hundred chart to help think about the sum of 27 and 12. This next student's approach is to begin at 27, move down one row, and count over 2 to 39; see Figure 11.11(b). Figure 11.11(c) shows a subtraction problem with a drawing of part of a hundred chart, beginning at 39, moving up one row, and moving backward 2. Notice how the student checked her answer by using an equation.

Think-Addition

The *Think-Addition* strategy, first learned for basic facts, is an important strategy for solving multi-digit subtraction problems. This strategy is particularly successful for students with special needs in math (Peltenburg, van den Heuvel-Panhuizen, & Robitzsch, 2012). For example, for 38 – 19, the idea is to think, "How much do I add to 19 to get to 38?" A counting-up strategy is employed, such as jumping from 19 to 29 (+10) to 39 (+10) and back 1 (–1) for a total difference of 19. Notice that Think-Addition is not as efficient for 42 – 6. This problem is more efficiently solved with counting back. As students explore subtraction, it is important for them to notice features of the problem that lend to one strategy or the other.

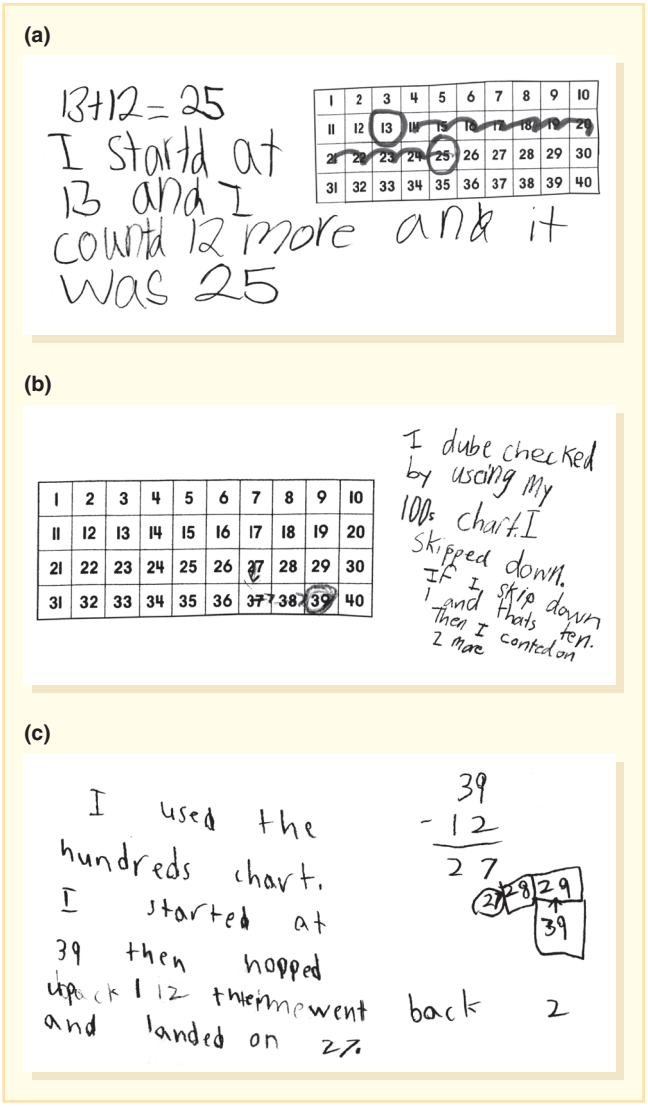

(a)
13+12=25
I startd at 13 and I countd 12 more and it was 25
(b)
I dube checked by using my 100s chart. I Skipped down. If I skip down 1 and that's ten. Then I 2 mae
(c)
I used the hundreds chart. I started at 39 then hopped up back 1 12 then I went and landed on 27.
39
-12
27
back 2

FIGURE 11.11 Three students use a top down hundred chart to add or subtract

Pause & Reflect

For what types of subtraction problems might you use count back? For what types might you use Think-Addition? ●

When the minuend and subtrahend are close together, using Think-Addition makes sense, because it is efficient to find the difference between the two numbers. But when the subtrahend is relatively small, it is likely more efficient just to count back.

Story situations can help students understand Think-Addition, using *join with change-unknown* problems, *missing-part* problems, or *compare* problems (discussed in Chapter 8). Here is an example of each.

Juanita had 46 baseball cards. She went to a card show and bought more cards for her collection. Now she has 73 cards. How many cards did Juanita buy at the card show?

Sam counted all of the teacher's pencils. Some were sharpened and some not. He counted 73 pencils in all; 46 pencils were not sharpened. How many pencils were sharpened?

Erik raised $73 and Mira raised $46. How much more did Erik raise?

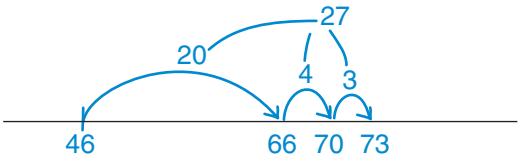
Figure 11.12 shows Think-Addition strategies to solve 73 – 46. As you can see, using tens is also an important part of this thinking. Simply asking for the difference between two numbers may also prompt these strategies.

Add Tens to Get Close, Then Ones

73 – 46

46 and 20 equals 66. (30 more equals too much.) Then 4 more equals 70 and 3 equals 73. That's 20 and 7 or 27.

$46 + 20 = 66$
$66 + 4 = 70$
$70 + 3 = 73$
$20 + 4 + 3 = 27$

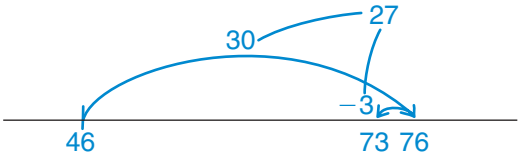


Add Tens to Overshoot, Then Come Back

73 – 46

46 and 30 equals 76. That's 3 too much, so it's 27.

$46 + 30 \longrightarrow 76 - 3 \longrightarrow 73$
$30 - 3 = 27$

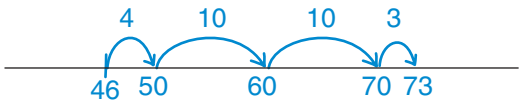


Add Ones to Make a Ten, Then Tens and Ones

73 – 46

46 and 4 equals 50. 50 and 20 equals 70 and 3 more equals 73. The 4 and 3 equals 7 and 20 equals 27.

$46 + 4 \longrightarrow 50$
$50 + 20 \longrightarrow 70$
$70 + 3 \longrightarrow 73$
$4 + 20 + 3 = 27$



Similarly, 46 and 4 equals 50.
50 and 23 equals 73.
23 and 4 equals 27.

$46 + 4 \longrightarrow 50$
$50 + 23 \longrightarrow 73$
$23 + 4 = 27$

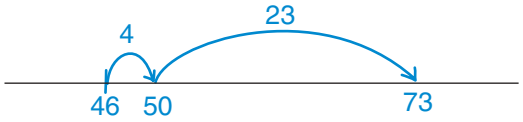


FIGURE 11.12 Three ways students might apply the Think-Addition strategy

Activity 11.8 How Far to My Number?

CCSS-M: 2.NBT.B.5; 2.NBT.B.9

Students work in pairs with a single set of little ten-frames. One student uses the cards to make a number less than 50 while the other student writes a number greater than 50 on a piece of paper, as shown in Figure 11.13. For students with special needs in math or students who would like a challenge, you may choose to suggest the size of the second number (e.g., less than 100, less than 500) by writing it on a sticky note and putting it near the partners. The task is for the pairs to find out how much more must be added to the number shown with the ten-frames to get to the larger number and write the corresponding equation. Once an answer is determined, they should demonstrate how their answer, combined with the smaller number, matches the larger number. Over time, students can decide they do not need the little ten-frame cards.

Pearson eText Blackline Master: Little Ten-Frame Cards

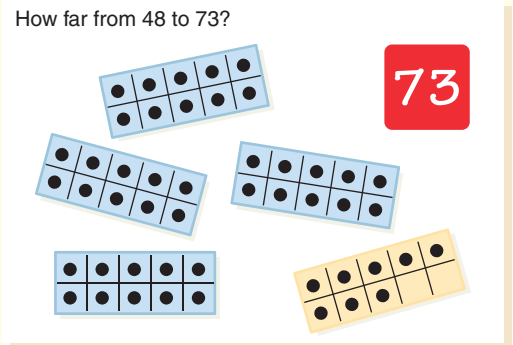


FIGURE 11.13 Use Think-Addition to solve How Far? problems

The following activity uses a blank hundred chart to explore the idea of Think-Addition as a reasoning strategy for subtracting.

Activity 11.10 engages students in Think-Addition, using compare situations.

Bar diagrams (sometimes called tape diagrams) are length models that can also be effectively used to support students' thinking of comparison subtraction situations and help them explain their ideas to others. Bar diagrams work particularly well for contexts that fit a comparison subtraction situation and a part-part-whole model as discussed in Chapter 8. See Figure 11.14 for a sample of each, using the Part-Part-Whole Graphic Organizer.

Pearson eText Activity Page: Part-Part-Whole Graphic Organizer

Activity 11.9 How Much Between?

CCSS-M: 2.NBT.B.5

Provide students with a blank hundred chart. Give the students two numbers. Their task is to determine how much from one number to the next. Your choice of the two numbers will have an impact on the strategies students will use. A good place to begin is with a pair of numbers in the same column on the hundred chart, such as 24 and 64. When the larger number is in a different column from the smaller number (e.g., 24 and 56), students can add on tens to get to the target number's row and then add or subtract ones. Of course, this strategy is also reasonable for any two numbers, but consider 17 and 45, where the column that 45 is in on the chart is to the left of the column that 17 is in. A student might count by tens and move 3 rows (+30) to 47 and then move 2 (−2) to 45. The difference between is now 30 − 2 or 28. Celebrate all possible approaches.

Pearson eText Blackline Master: Blank Hundred Chart

Ty read 29 books since the first day of school. Audra read 43. How many more books does Ty have to read to match Audra?

Whole	
43	
Part	Part
29	?

Lizzy had 117 pistachios. She ate 98 of then. How many pistachios are left?

Whole	
117	
Part	Part
98	?

FIGURE 11.14 Using bar diagrams to support the Think-Addition strategy

Activity 11.10 Celebrating Book Birthdays

CCSS-M: 2.NBT.B.5; 2.NBT.B.8

Students work in pairs with a place-value T chart (T for tens and O for ones). The teacher selects a book such as *Tales of a Fourth Grade Nothing* (Blume, 1972). Students are asked to use the T chart and Think-Addition, skip-counting up from 1972 in this case by first tens and then ones from the book's birthday (copyright date) to see how old the book is. Keep a tally of the counts on the T chart. Let students select books and find the difference between the birthday year and the current year. Students can organize books in order of age from oldest to youngest or group books by their decade (books in their teens or twenties, etc.).

Pearson eText Activity Page: T Chart

Thinking about the distances between numbers, particularly on a number line, is helpful (Whitacre et al., 2016/2017). A student, using a strategy based on equal addition called *same difference* (Humphreys & Parker, 2015) that was described earlier in this chapter, might add 3 to both 62 and 27, which results in 65 − 30, a much easier problem to solve (Figure 11.15). The student is basically moving the same difference to a new location on the number line. This approach may sound confusing to you—but try it. Especially when there are zeros in the minuend (e.g., 302 − 178), you may find this method a productive option.

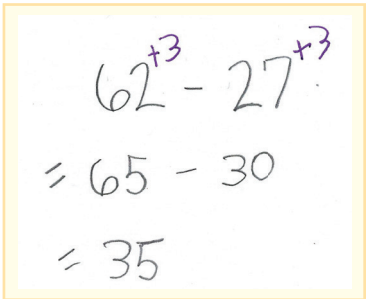


FIGURE 11.15 A strategy, using equal addition or same difference

Count Back – Take-Away

Students learn subtraction as take-away before they learn subtraction as find-the-difference (compare). But, when it comes to using strategies, a take-away action is considerably more difficult to do mentally than a counting-up action. Four student solutions are shown in Figure 11.16 for the following story problem. Notice that take-away actions fall across the groups of strategies.

There were 73 students on the playground. The 46 second-grade students came back into the building. How many students were still on the playground?

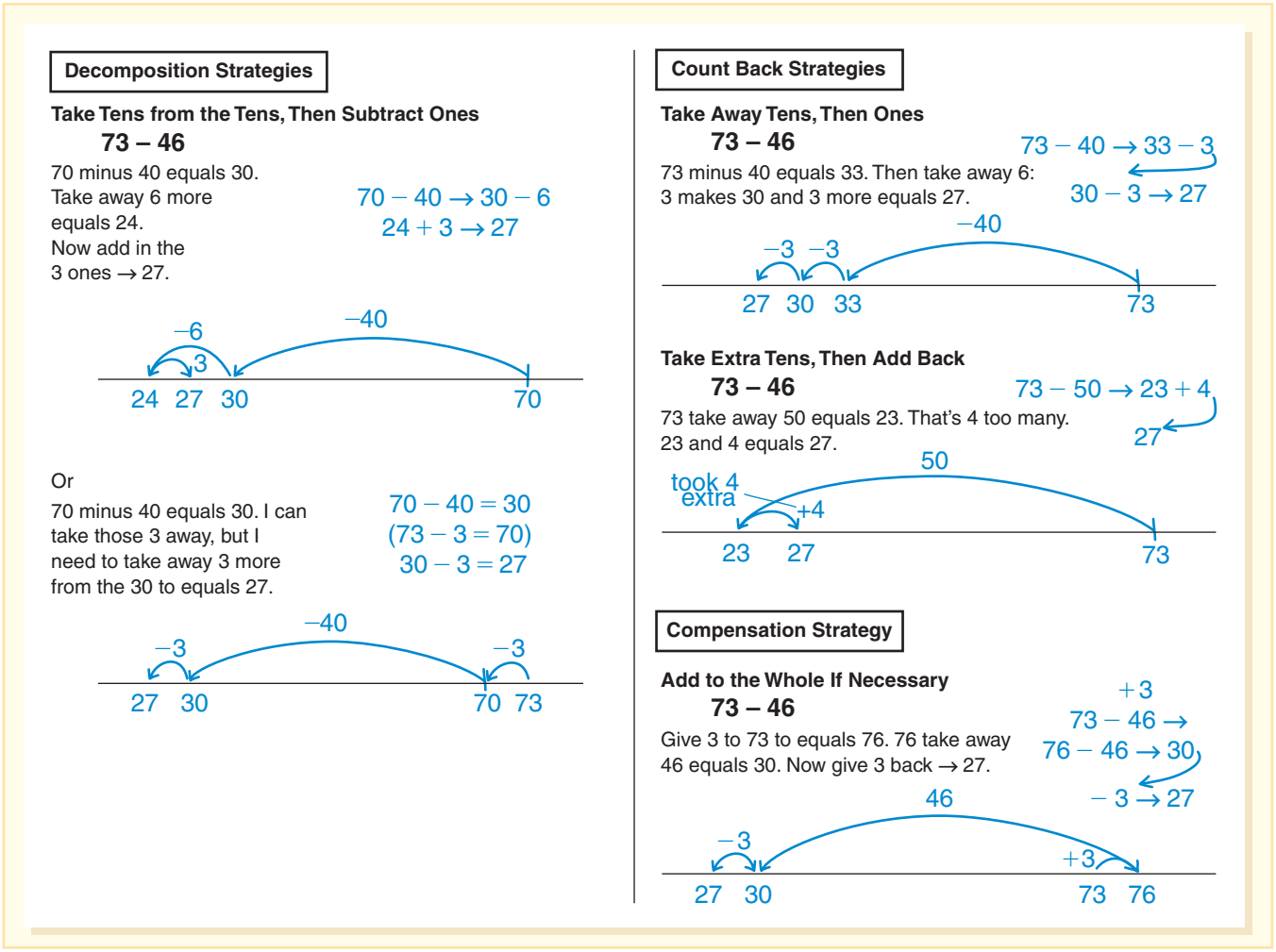

Decomposition Strategies
Take Tens from the Tens, Then Subtract Ones
73 – 46
70 minus 40 equals 30.
Take away 6 more
equals 24.
Now add in the
3 ones → 27.
70 – 40 → 30 – 6
24 + 3 → 27
–6
–40
–3
24 27 30
70
Or
70 minus 40 equals 30. I can take those 3 away, but I need to take away 3 more from the 30 to equals 27.
70 – 40 = 30
(73 – 3 = 70)
30 – 3 = 27
–40
–3
–3
27 30
70 73
Count Back Strategies
Take Away Tens, Then Ones
73 – 46
73 minus 40 equals 33. Then take away 6:
3 makes 30 and 3 more equals 27.
73 – 40 → 33 – 3
30 – 3 → 27
–40
–3 –3
27 30 33
73
Take Extra Tens, Then Add Back
73 – 46
73 take away 50 equals 23. That's 4 too many.
23 and 4 equals 27.
73 – 50 → 23 + 4
27
50
took 4 extra
+4
23 27
73
Compensation Strategy
Add to the Whole If Necessary
73 – 46
Give 3 to 73 to equals 76. 76 take away 46 equals 30. Now give 3 back → 27.
+3
73 – 46 →
76 – 46 → 30
– 3 → 27
46
–3
+3
27 30
73 76

FIGURE 11.16 Solutions to 73 – 46 that reflect take-away subtraction

The two methods shown that begin by taking tens from tens are reflective of what most students do with their base-ten materials. The other two methods leave one of the numbers intact to do the subtraction. When the subtracted number is a multiple of 10 or close to a multiple of 10, take-away can be carried out mentally, using compensation. Try 83 – 29 in your head by first taking away 30 and adding 1 back. Some students may initially become confused when they hear a classmate describe this strategy for 83 – 29. In particular, they might not understand why you add 1 back. They think because you added 1 to 29 to equal 30, you should subtract 1 from the answer. Let them act it out with base-ten materials so they can see that when they take away 30, they took 1 too many away and that is why they need to add 1 back.

Pause & Reflect

Try computing 82 – 57. Use both Take-Away and Think-Addition methods. Can you identify all of the strategies in Figures 11.12 and 11.16 without looking? ●

Keep in mind that for many subtraction problems, especially those with three digits, the Think-Addition approach is significantly easier than a take-away approach. For students who could benefit from a Think-Addition strategy but are not using it, revisit missing-part activities to refresh and encourage that thinking.

Decomposition Strategies

The *decomposition strategy* partitions numbers parts, frequently place-value parts. For example, a problem such as $146 + 321$ uses students' understanding of place value, and they can say $100 + 300 + 40 + 20 + 6 + 1$. They can also start with the first addend (146) and say $146 + 300 + 20 + 1$. The notion of decomposing or breaking a number into parts is a useful strategy for all operations and a good structure by which to compute mentally (see Figure 11.17).

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

(a) How much is 86 and 47?

S: I know that 80 and 20 more is 100.

T: Where do the 80 and the 20 come from?

S: I split the 47 into 20 and 20 and 7 and the 86 into 80 and 6.

T: (illustrates the splitting with lines) So then you added one of the 20s to 80?

S: Yes, 80 and 20 is 100. Then I added the other 20 and got 120.

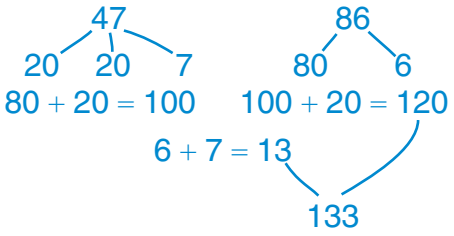
T: (writes the equations on the board)

S: Then I added the 6 and the 7 and got 13.

T: (writes this equation)

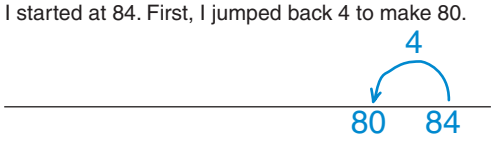
S: Then I added the 120 to the 13 and got 133.

T: Indicates with joining lines.



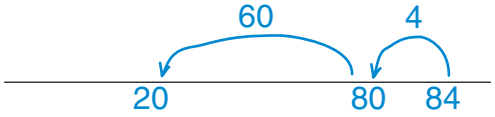
(b) What is 84 minus 68?

S: I started at 84. First, I jumped back 4 to make 80.

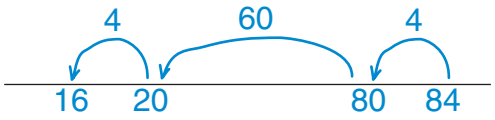


T: Why did you subtract 4 first? Why not 8?

S: It was easier to think about 80 than 84. I will save the other part of 8 until later. Then I jumped back 60 to get 20.



S: Then I jumped back 4.



T: Why 4?

S: That was how much I still had left over from 68.

FIGURE 11.17 Two methods of recording students' thought processes so that the class can follow the strategy

Introduce the open number line by using it to model a student's thinking for the class, explicitly linking it to their understanding of a unit. Remind students that the number line, like a ruler, is a length marked off into particular units—and in the case of the open number line, they are creating their own units. Students determine the start and result numbers based on the problem they are solving. Then they often use friendly numbers to make their jumps and then calculate the total of the lengths (Barker, 2009). The jumps on the line can be recorded as the students share or explain each step of their solution, counting up or down from an initial number. With time and practice, students will find the jump strategy paired with the open number line to be effective in supporting and explaining their thinking.

***TECHNOLOGY* Note.** You can find interactive number lines on the Internet where children can practice using a jump strategy. For example, try https://apps.mathlearningcenter.org/number-line. ■

Pearson eText
Video Example 11.4
In this video consider how the student is challenged when solving a two-digit subtration problem with a zero in the minuend.

https://www.youtube.com/watch?v=8N4N2EE1uYg

Compensation Strategies

Compensation strategies involve the flexible adjustment of numbers by overshooting or undershooting and then equalizing the change. Recall the basic fact strategy called Pretend-a-10, wherein students add $9 + 7$ by pretending the 9 is a 10, adding $10 + 7$ (17) and then adjusting their answer by 1 to equal 16. This is compensation. The following activity supports students' thinking about adjusting numbers by using 10 as an anchor.

Activity 11.11 Just Adjust It

CCSSM-M: 2.NBT.B.5; 2.NBT.B.9

Create a string of problems, using numbers that will help students gravitate toward the idea of using 10, or use the Just Adjust It activity page. The following problems help students use reasoning to become more aware of different ways to adjust numbers:

$50 + 30$ $48 + 30$ $50 + 32$ $51 + 28$ $53 + 29$

Project one problem at a time and give students time to solve it before you ask for explanations of their approaches with their answer. Record students' strategies so that you can refer back to them. If no student suggests the idea of using the first problem of $50 + 30$ to help solve the subsequent problems, challenge them to determine how they could use the answer to that problem to help them solve all the others. By listening to classmates, other students—particularly students with special needs in math—can become more aware of different ways to adjust numbers.

Pearson eText Activity Page: Just Adjust It

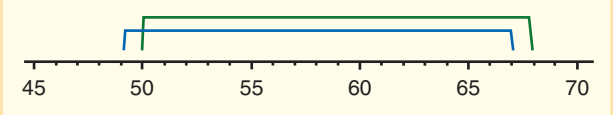
There are multiple ways to add the numbers from the preceding activity by using 10 as an anchor. For example, for $51 + 28$, a student may change 51 to 50 and 28 to 30, add $50 + 30$ to equal 80, and then subtract 2 to adjust for adding 30 (instead of 28) and add 1 for adding 50 (instead of 51) to get the answer 79. Other students may think that 51 is really close to 50, so they add the 1 from 51 to the 28 and add $50 + 29$ to equal 79. Still other students may add the tens $(50 + 20)$ and then the ones $(1 + 8)$ to equal 79. Remember, your goal is to help students develop multiple strategies that are efficient and that make sense to them. By having students listen to how their classmates reasoned to find the sums, you expose students to a variety of strategies.

Compensation is a very useful strategy for subtraction, but it is more complicated than addition (Bay-Williams & SanGiovanni, 2021). Let's consider three cases: changing the minuend (first number), changing the subtrahend (the second number), and changing both.

Emphasizing reasoning strategies before moving to standard algorithms can enhance students' number sense and place-value understanding. Try your hand at doing these computations without using the standard algorithms: $487 + 235$ and $623 - 587$. For subtraction, a counting-up strategy is sometimes the easiest, especially when the numbers are relatively close together, as in $623 - 587$. Occasionally, other strategies are more advantageous. For example, chunking multiples of 50 or 25 is often a useful method. For $462 + 257$, pull out 450 and 250 to equal 700. That leaves 12 and 7 more, for 719. These are just a few of the many reasoning strategies students can use.

TABLE 11.3 THREE WAYS TO USE COMPENSATION STRATEGIES TO SUBTRACT

Idea	The Related Reasoning	Example: 67 − 49 Changes in Red
Change the minuend (first number)	Change the total and then apply the inverse to the answer after subtracting (if you *add* 2 to the minuend, *subtract* 2 from the answer).	Change: 69 − 49 = 20 Adjust: 20 − 2 = 18
Change the subtrahend (second number)	Change how much is subtracted and then do the same to the answer (if you *add* 1 to the subtrahend, you also *add* 1 to the answer because you took away 1 too many).	Change: 67 − 50 = 17 Adjust: 17 + 1 = 18
Change both	Change both the same amount, keeping the difference the same. It is like a shift on the number line (blue is original, green is adjusted).	Change: add 1 to both Adjust: 68 − 50 = 18



As students expand their understanding through many experiences, they can use these strategies efficiently and effectively. Your goal might be that each of your students has at least one or two methods that are reasonably efficient, mathematically correct, and useful with lots of numbers. Expect different students to settle on different strategies that play to their strengths. In other words, all students do not have to use the same strategy for the same problem. Here we explore a variety of reasoning strategies that students often use.

Adding or Subtracting a Single-Digit Number

When adding or subtracting small amounts or finding the difference between two reasonably close numbers, many students will use a **jump strategy** to solve the problems. This is a good time to introduce a **decomposition strategy**, specifically a Make Tens strategy (building on students' knowledge of making tens with basic facts). When the calculation crosses a ten (e.g., $58 + 6$), using the decade number (60) and thinking $58 + 2 + 4$, extends the use of the Making Tens strategy. The **compensation strategy** is also a connection here, pretending the sum is $60 + 6$ and then adjusting by subtracting 2 (building on the Pretend-a-10 strategy). Similarly, for subtraction, students can extend the Down Under 10 strategy. For instance, for $53 - 7$, subtract 3 (of the 7) to equal 50 and then subtract 4 more equals 46.

Activity 11.12 Crossing a Decade

CCSS-M: 1.NBT.C.4; 2.OA.A.2

Quickly review the Make Tens and Down Under 10 strategies for basic facts (from Chapter 9), using little ten-frames. Then have little ten-frames, base-ten materials, a place-value mat with double ten-frames, or a hundred chart available for students' use. Pose an addition or subtraction story problem that crosses a decade number and involves a change or difference of less than 10. Here are some example problems:

- Tommy was on page 47 of his book. Then he read 8 more pages. What page did he end up on?
- How far is it from 68 to 75?
- Ruth had 52 cents. She bought a small toy for 8 cents. How much does she have left?

Pearson eText Blackline Master: Little Ten-Frames
Pearson eText Blackline Master: Base-Ten Materials
Pearson eText Blackline Master: Place-Value Mat with Double Ten-Frames
Pearson eText Blackline Master: Hundred Chart
Pearson eText Blackline Master: Bottom-Up Hundred Chart

Listen for students who are counting on or counting back without paying attention to the ten and suggest they use either the hundred chart or the little ten-frames as shown in Figure 11.18 to support their thinking. Also, find out how they solve fact combinations such as $8 + 6$ and $13 - 5$. The use of tens for these facts is essentially the same as for the higher-decade problems. Have students who use the strategy share their ideas with others. They might say, for $47 + 6$, "I added 3 from the 6 to the 47 to equal 50. Then I added the remaining 3 to equal 53."

Standards for Mathematical Practice
MP7. Look for and make sense of structure

Adding Tens and Hundreds

As you transition students from single-digit addends and subtrahends to multi-digit ones, they must be proficient at adding and subtracting multiples of tens and hundreds. Sums and differences involving multiples of 10 or 100 are easily computed mentally. Write a problem such as the following:

$$300 + 500 + 20$$

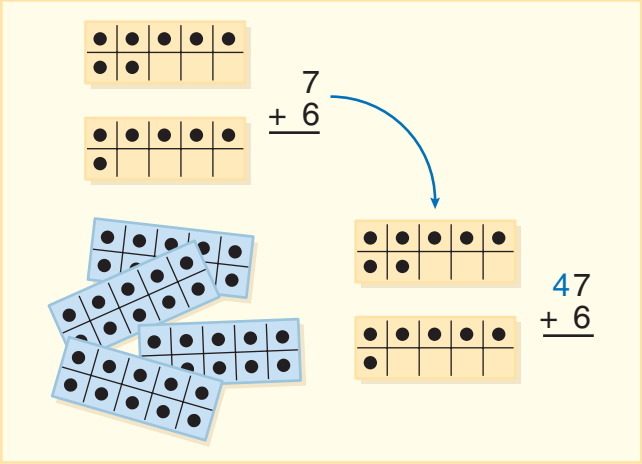


FIGURE 11.18 Students extend the Making Tens basic fact strategy to larger numbers

Challenge students to solve it mentally. Ask students to share their thinking. Listen for the use of place-value words: "3 *hundred* and 5 *hundred* is 8 *hundred*, and 20 more equals 820." Start with problems that do not require regrouping and then evolve to more difficult problems such as 70 + 80. Continue to use base-ten materials to help students think in terms of units of tens and hundreds.

Adding Multi-digit Numbers

The development of fluency with multi-digit addition and subtraction begins in grade 1 with the addition of multiples of 10 and continues through grade 4, when students are expected to add and subtract multi-digit whole numbers with fluency. This fluency must be built with years of explorations, using concrete models or drawings and strategies based on the inverse relationship between addition and subtraction, place value, and the properties of operations. Problems involving the sum of 2 two-digit numbers will usually elicit a wide variety of reasoning strategies across these three categories, and these strategies extend to three-digit numbers (and beyond). Student performance can decline as the number of digits increases (Erdem, 2017), so it is important to help students transfer the strategies they have learned with one- and two-digit numbers to larger numbers.

Story situations provide contexts that help students reason. In each case, the numbers in the problem and the type of problem will influence the strategies students use. Therefore, it is important to think carefully about the type of story problem you pose as well as the numbers you select!

Let's start by looking at the following story problem.

Two grades went on a field trip to the museum. There were 46 fourth graders and 38 fifth graders. How many students went on the trip?

Figure 11.19 illustrates four solutions to this problem. The ways that the solutions are recorded support the students' explanations; particularly note the frequent use of the open number line.

Activity 11.13 I Am, but Who Is?

CCSS-M: 1.NBT.C.4; 1.NBT.C.5; 1.NBT.C.6

Have students practice their mental calculations and prepare for multi-digit computation by adding or subtracting multiples of tens (or other familiar combinations) with two-digit numbers. Use the I Am/Who Is? game cards for a class of up to 30 students. If you have fewer than 30 students, give students who would like a challenge two cards. There are also some easier combinations (e.g., Who is 20 + 10? Who is 37 + 10?), so consider giving those cards to students with special needs in math to ensure that they understand the game and are using units of ten. Start the game by calling out any number you see on a nearby card, such as "Who is 22?" The student should respond with "I am 22, who is…" and then read the rest of the card. The game will loop all the way around through the 30 problems.

Pearson eText Activity Page: I Am/Who Is? Game Cards

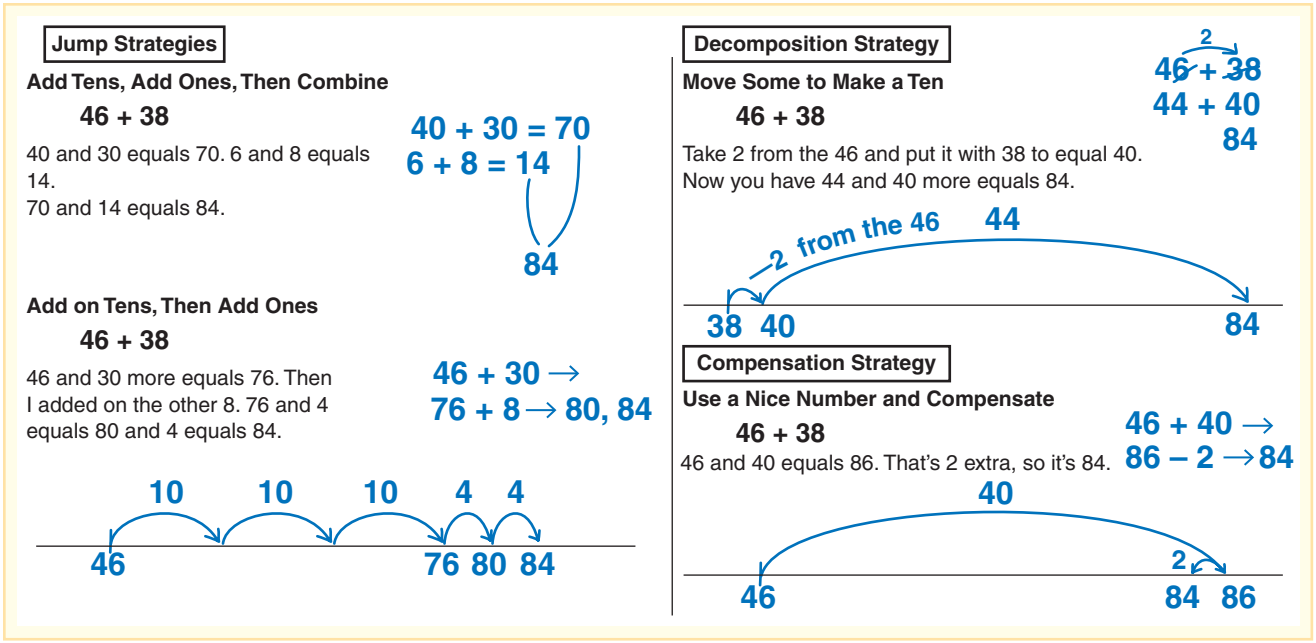

Jump Strategies
Add Tens, Add Ones, Then Combine
46 + 38
40 and 30 equals 70. 6 and 8 equals 14.
70 and 14 equals 84.
40 + 30 = 70
6 + 8 = 14
84
Add on Tens, Then Add Ones
46 + 38
46 and 30 more equals 76. Then I added on the other 8. 76 and 4 equals 80 and 4 equals 84.
46 + 30 →
76 + 8 → 80, 84
10 10 10 4 4
46 76 80 84
Decomposition Strategy
Move Some to Make a Ten
46 + 38
Take 2 from the 46 and put it with 38 to equal 40.
Now you have 44 and 40 more equals 84.
2
46 + 38
44 + 40
84
2 from the 46
44
38 40
84
Compensation Strategy
Use a Nice Number and Compensate
46 + 38
46 and 40 equals 86. That's 2 extra, so it's 84.
46 + 40 →
86 – 2 → 84
40
2
46
84 86

FIGURE 11.19 Four ways to solve 46 + 38

Pause & Reflect

Try adding 367 + 155 in as many ways as you can. How many of your ways are like those in Figure 11.19? ●

Stories, such as the ones in Activity 11.14, are also important for subtraction.

Activity 11.14 Exploring Subtraction Strategies

CCSS-M: 2.OA.A.1

Use the set of problems on the Looking at Collections activity page. Ask students, "How can you use bar diagrams, open number lines, or break-apart strategies to show how you are solving these problems?" See the Expanded Lesson: Exploring Subtraction Strategies for more details on how to adapt this activity for students with special needs in math and multilingual learners.

Pearson eText Activity Page: Looking at Collections
Pearson eText Expanded Lesson: Exploring Subtraction Strategies

FORMATIVE ASSESSMENT Notes. Periodically focus on one student to determine their flexible use of strategies. Pose the following problem in a diagnostic interview: 46 + 35. See whether the student begins by breaking the numbers apart by place value or to make the next ten. That is, for 46 + 35, a student may add on 4 to the 46 to equal 50 and then add the remaining 31. Or first add 30 to 46 to get to 76, add 4 to equal 80, and then add the remaining 1. In either case, be mindful of how flexibly the student uses ten as a unit. Students may use an open number line and count up with lengths of 10, using a technique to count by tens and then add the ones. They would count up, saying "46, 56, 66, 76" as they draw corresponding lengths marked with their jump of +10 and so on. Another approach may involve splitting the numbers into other parts such as 50 or 25 as an easier compatible number to work with. If a student is not consistently using ten as a unit, they may need more experiences with developing place-value ideas. ■

Selecting Problems

Each of the examples in the preceding sections involved sums less than 100, and all involved *crossing a ten;* that is, if done with a standard algorithm, they required regrouping. As you plan instruction, consider whether a problem requires crossing a ten, the size of the numbers, and the potential for students doing problems mentally.

Crossing a Ten (or More). For most strategies, it is easier to add or subtract when crossing a ten is not required. Try the Think-Addition and Take-Away strategies with $34 + 52$ or $68 - 24$ to see how they work. Easier problems have their purpose, one of which is to build confidence. They also permit you to offer challenges to your students by asking, "Would you like a harder one?" There is also the issue of crossing 100 or 1000. Try $58 + 67$ with these different strategies. Crossing across hundreds is also an issue for subtraction. Problems such as $128 - 50$ or $128 - 45$ are more difficult than problems that do not cross a hundred as in $298 - 187$.

Larger Numbers. Second-grade students should add and subtract three-digit numbers, using a variety of strategies. How would you do these problems without using the standard algorithms: $487 + 235$ and $623 - 247$? Again, for subtraction, a Think-Addition strategy is usually better understood and more effective. Again, students can also chunk multiples of 50 or 25, such as in $835 + 488$, as they chunk $825 + 475$ to equal 1300. Then they add 10 and 13 more, which equals 1323.

Activity 11.15 engages students in computation with larger numbers.

Activity 11.15 City Populations

CCSS-M: 2.NBT.B.5; 2.NBT.B.6; 2.NBT.B.7; 3.NBT.B.2; 4.NBT.B.2; 4.NBT.B.4

Have students combine or compare the populations of classrooms, cities, states, or counties in your school or region. Place the names of the cities, for example, on cards and either write the population directly on the card or have a corresponding data chart for students to refer to. Then give students two cards and use the City Population Word Problem activity page, or have students create their own word problems. Students wanting a challenge can work with densely populated locations, while students needing a gentler ramp up to the content can start with two sparsely populated areas.

Pearson eText Activity Page: City Population Word Problem

Algorithms for Addition and Subtraction

Learning Outcome 11.3 **Explain the development of the standard algorithms for addition and subtraction, including ways to record students' thinking.**

It is your role to help students understand how and why the standard algorithms work through the strategies they've already developed. Teaching in a conceptual manner means helping students attend to the units of hundreds, tens, and ones as they work.

Consider the following significant differences between the reasoning strategies just discussed and standard algorithms.

1. *Reasoning strategies are number-oriented rather than digit-oriented.* Using the standard algorithm for $45 + 32$, students think of $4 + 3$ instead of 40 and 30. Kamii and Dominick (1998) suggest that standard algorithms "unteach" place value. By contrast, a reasoning strategy for $618 - 254$ might begin with $600 - 200$ is 400. Another approach might begin with 254 and add 46, which equals 300. Then 300 more, equaling 600, and so on. In these cases, the computation is number-oriented.
2. *Reasoning strategies are left-handed rather than right-handed.* Reasoning strategies often begin with the largest units (leftmost digits) because they focus on the entire number.

For 263 + 126, you would possibly begin with "200 + 100 equals 300," providing some sense of the size of the eventual answer in just one step. In contrast, the standard algorithm begins with the ones units, with 3 + 6 equals 9, which is also counter to how students write. By beginning on the right with a digit orientation, the solution is hidden until the end. The exception is the standard algorithm for division.

3. *Reasoning strategies are a range of flexible options rather than one right way.* Reasoning strategies depend on the numbers involved, so students can be flexible and make the computation easier. Try each of these problems mentally: 465 + 230 and 526 + 98. Did you use the same method? The standard algorithm suggests using the same tool on all problems. The standard algorithm for 7000 – 25 typically leads to student errors, yet a mental strategy is relatively simple.

Importantly, students who used just the standard algorithm for either addition or subtraction had lower math achievement than students who used a mixture of reasoning strategies (Sahin, Dixon, & Schoen, 2020).

Partial Sums and Differences

A bridge to the standard algorithms for addition and subtraction is to use partial sums or differences. The partial-sum method begins with the largest place value, records that partial sum below the problem, and then continues toward the smaller place values. Figure 11.20 shows an example. Partial sums highlight the actual value of the numbers and are relatively efficient, at least for up to three digits. Because it is more conceptual, it is a particularly effective method for students with special needs in math.

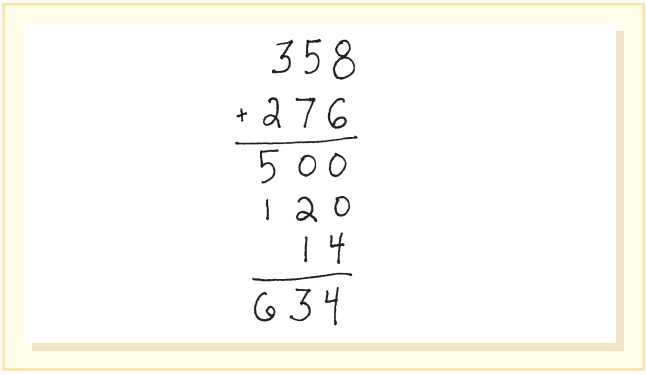


FIGURE 11.20 The partial-sums approach can be used from left to right as well as from right to left

The partial differences method also begins with the largest place value. If all the digits of the minuend (first number) are larger than the digits in the subtrahend (second number), this is straightforward. If, however, that is not the case, the subtraction must be notated to illustrate the negative value. Two examples are provided in Figure 11.21.

Partial sums and differences set the stage for making sense of the standard algorithms, which differ in the order in which the place values are considered (the algorithms start with the ones place and work to larger place values) and the use of regrouping. *Regrouping* is exchanging 10 in one place-value position for 1 in the position to the left—or the reverse, exchanging 1 for 10 in the position to the right. The terms *carrying* and *borrowing* are conceptually misleading and therefore obsolete. The word *regroup* may initially have little meaning for young students, so start with the term *trade*. Ten ones are *traded* for a ten. A hundred is *traded* for 10 tens. Notice that none of the reasoning strategies we've discussed involve regrouping, but this process is critical to using algorithms.

Pearson eText
Video Example 11.5
In this video author Jennifer Bay-Williams discusses partial sums and partial differences algorithms.

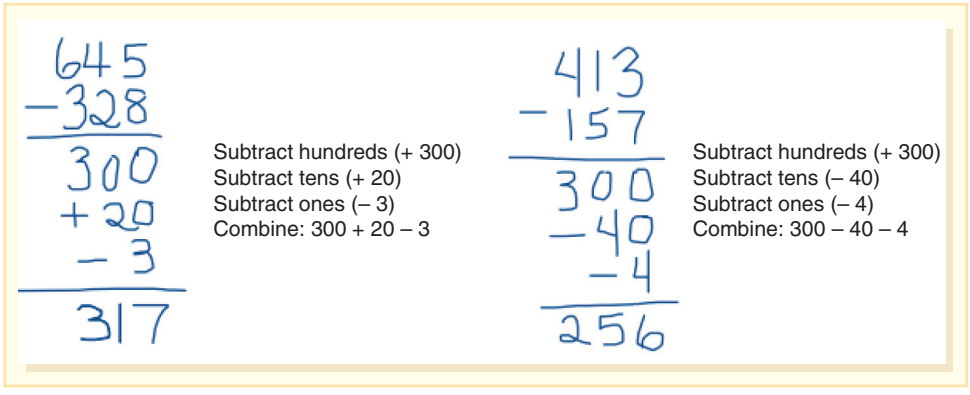


FIGURE 11.21 Examples of partial differences

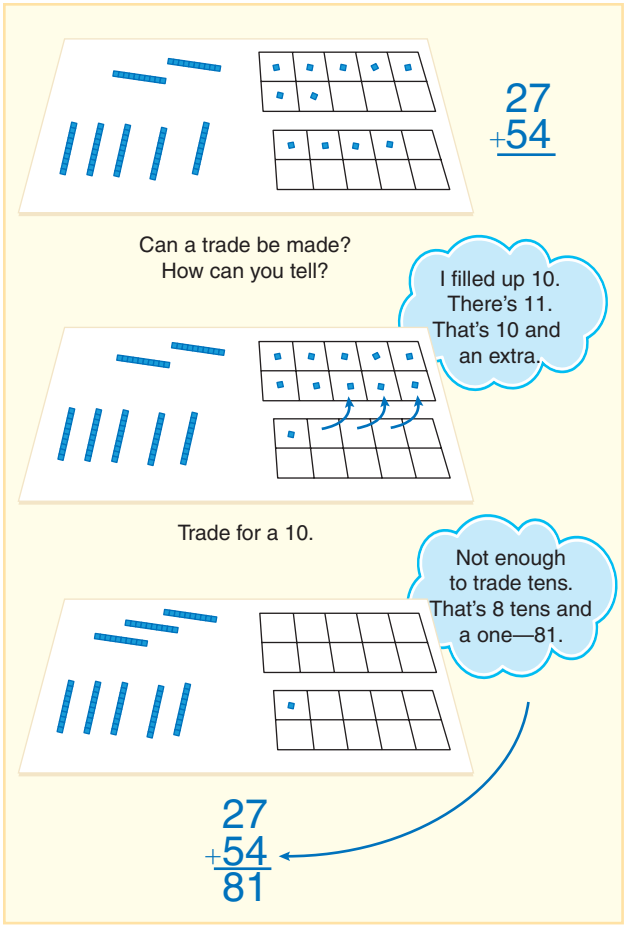


FIGURE 11.22 Working from right to left in addition

Standard Algorithm for Addition

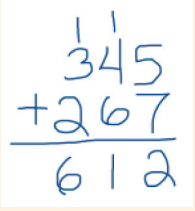
As you teach the standard algorithm for addition, be sure students view it as one possibility, a good choice in some situations (while their other strategies may be better choices in other situations). Begin instruction with concrete models and then make explicit connections to the symbolic process of combining and regrouping, as described here. The algorithm is a record of acting out the operation with the materials.

Begin with Models Only. In the beginning, focus on regrouping without recording the numerical process. Provide students with place-value mats and base-ten materials. Have students use the base-ten materials to represent one number at the top of the mat and a second beneath it, as shown in the top portion of Figure 11.22. Point out this procedure: *Begin in the ones column.* Then let students solve the problem on their own. Allow plenty of time and then have students explain what they did and why. Let students project their work or use interactive whiteboard models to support their explanations. One or two problems in a lesson with lots of discussion are more productive than many problems based on rules students don't understand or can't explain.

Pearson eText Blackline Master: Place-Value Mats
Pearson eText Blackline Master: Base-Ten Materials

Develop the Written Record. Use the Addition and Subtraction recording pages to help students record numerals in columns as they model each step of the procedure they carry out as they combine (add) the base-ten materials. The first few times you do this, guide each step carefully through questioning, as shown in Figure 11.23. A similar approach would be used for three-digit or four-digit problems. Have students work in pairs with one responsible for the models and the other recording the steps. They should reverse roles with each problem.

With more experience, the language transitions from "trading" to "regrouping." For example, to solve 345 + 267, a student talk-aloud might sound like this:

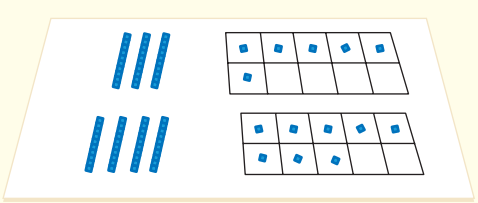


I added 5 and 7, regrouped 1 ten and recorded the 2. Then I added 1 ten plus 4 tens plus 6 tens, and regrouped 1 hundred and recorded 1 ten. Then, I added 1 hundred plus 3 hundreds plus 2 hundreds. The answer is 612.

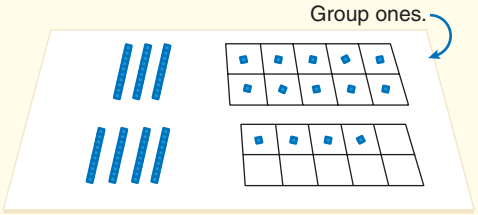
A common error students make is to record their answers in this way:

$$\begin{array}{r} 57 \\ +\ 69 \\ \hline 1116 \end{array}$$

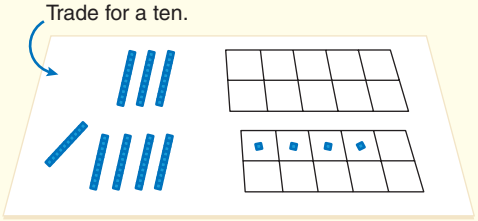
As you can see, this student has lost the connection to place value and the need to regroup. This student would benefit from solving using partial sums.



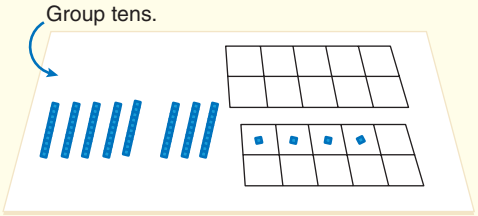
Group ones.



Trade for a ten.



Group tens.



Tens	Ones
	□
3 + 4	6 8

How much is in the ones column? (14)

Will you need to make a trade? (yes)

How many tens will you make? (1)
How many ones will be left? (4)

Tens	Ones
	□
1 3 + 4	6 8
	4

Good! Make the trade now.

Let's stop now and record exactly what we have done. You had 14 ones, and you made 1 ten and 4. Write a "1" in the tens column to show the ten you put there and a "4" in the answer space of the ones column for the 4 ones left.

Tens	Ones
	□
1 3 + 4	6 8
8	4

Look at the tens column on your mat. You have 1 ten on top, 3 from the 36, and 4 more from the 48. See how your paper shows the same thing?

Now add all the tens together. Write how many tens that is in the answer space for the tens column.

FIGURE 11.23 Students record what they do on their place-value mats

Importantly, this answer is not reasonable. Because the standard algorithm is less intuitive than other strategies, it is all the more important to check for reasonableness—for example, asking, "Do you think your answer will be between 100 and 200 or 200 and 300?" or more generally, "What's the largest this sum might be? What's the least it might be?" More estimation ideas are discussed in the final section of this chapter.

Standard Algorithm for Subtraction

The general approach to developing the subtraction algorithm is the same as for addition. When the procedure is completely understood with models, a do-and-record approach connects it with the written form.

Begin with Models Only. Start by having students treat the subtraction problem as a take-away situation. With this meaning of subtraction, they model only the top number (minuend) in a subtraction problem on the top half of their place-value mats, using the top double ten-frame for any ones. For the amount to be subtracted, have students write each digit on a small piece of paper and place these numerals near the bottom of their mats in the respective columns, as in Figure 11.24. To reduce errors, suggest carrying out all regrouping first. That way, the full amount on the paper slip can be taken off immediately. Also, explain to

Pearson eText
Video Example 11.6
In this video consider how Talecia solves a multidigit addition problem. If you were this student's teacher, what is your next instructional step?

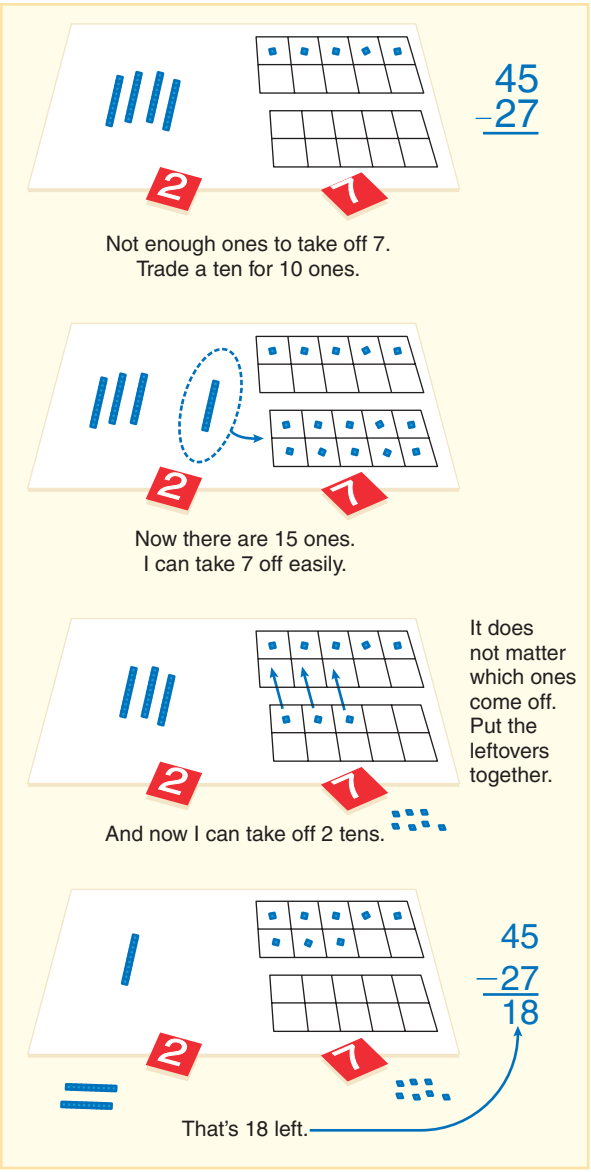


FIGURE 11.24 Two-digit subtraction with models

students that they are to begin working with the ones column first, as they did with addition.

Anticipate Difficulties with Zeros. Problems in which internal zeros are involved cause special difficulties.

The common errors that emerge when students regroup across zero are best addressed at the modeling stage. For example, in 403 – 138, students must make a double trade: regrouping a hundreds piece for 10 tens and then a tens piece for 10 ones. After students have experience with making these double regroupings, using base-ten materials, use the following activity before giving students any hints about how they might approach regrouping across an internal zero.

Activity 11.16 Trading across Zero

CCSS-M: 2.NBT.B.7; 2.NBT.B.9

Pose a problem that requires regrouping across a zero, such as 203 – 78. Students work in pairs, using base-ten materials and place-value mats. Once they have identified an answer, they now check their answer, using the algorithm. If they did not get the same answer with the base-ten materials and the algorithm, encourage them to determine why. Follow up with a discussion that starts with students sharing their ideas.

Pearson eText Blackline Master: Place-Value Mats
Pearson eText Blackline Master: Base-Ten Materials

Develop the Written Record. The process of recording each step as it is done is the same as was suggested for addition using the Addition and Subtraction recording pages. When students can explain the use of symbols involved in the recording process, they can decide to transition away from the use of the concrete materials and move toward the use of symbols. Again, be attentive to problems with internal zeros.

Pearson eText
Video Example 11.7
In this video consider how the student can respond doing mental math but is challenged when given the written problem. What did the teacher do to push the child's thinking? How would you follow up with this student?

Pearson eText Activity Page: Addition and Subtraction Recording Pages

Pause & Reflect

Contrast teaching students to regroup in subtraction, especially regrouping across zero, with the Think-Addition strategy. For example, try solving 203 – 78 (counting up from 78 to 203). Now think about solving the same problem with the standard algorithm. Which is easier? Why? ●

Procedural fluency means choosing appropriate strategies. An appropriate strategy is one that can be done in about as many steps or about as much time as other appropriate methods (Bay-Williams & SanGiovanni, 2021). For example, for 405 + 397, a Make-a-Hundred strategy (moving 3 over to make 400) or compensation (pretending it is 100 and then subtracting 3 from the answer) are both appropriate choices, but the standard algorithm is not, because it is not as efficient. The next activity asks students to determine which strategy from a variety of strategies might be most useful.

Activity 11.17 Pick Your Strategy

CCSS-M: 1.NBT.C.4; 2.NBT.B.5; 2.NBT.B.6; 2.NBT.B.7; 2.NBT.B.9

Tell students you are going to show them a problem, but they are *not* to solve it—instead, they should decide which method from their reasoning strategies and standard algorithms they would choose and why. Projecting a list of possible strategies and thinking tools for addition and subtraction problems will support students with special needs in math and multilingual learners. After students make their selection, call out each method and have them raise their hand to indicate which strategy or thinking tool they chose. Then they use their selected method to solve the problem. When finished, they raise their thumb and hold it to their chest. Share students' solutions for each strategy used. Then ask the class which method seemed to work best for each problem and why. Use a variety of problems whose numbers lend themselves to different strategies.

Pearson eText Activity Page: List of Addition and Subtraction Strategies and Thinking Tools

Computational Estimation

Learning Outcome 11.4 Identify ways to teach computational estimation to develop students' flexibility and ability to recognize reasonable answers.

Computational estimation is a higher-level thinking skill that involves determining an efficient way to estimate a sum or difference. It should be blended into the overall addition and subtraction strategy instruction to support students' checking for reasonableness of answers. Students' successful computational estimation is correlated with higher levels of problem-solving ability (Desli & Lioliou, 2020).

Everyone needs to solve numerical situations that need an about answer. You have likely been in a circumstance where a friend has quickly done some approximate calculations in their head and you realize how useful and important that mental math skill is. Computational estimation is the abstract thinking and reasoning that provide a foundation for future mathematical topics, because we want students to estimate either before they compute to reinforce their number sense or after they compute to assess reasonableness of their answer or both!

Whenever we are faced with a computation in real life, we have a variety of choices to make concerning how we will find a reasonable answer. A first decision is: "Do we need an exact answer, or will an approximation be okay?" If precision is called for, we can use a reasoning strategy, a standard algorithm, or even a calculator. If we do not need a precise answer, then we select a way to estimate the sum or difference mentally. How close an estimate must be to the actual computation is a matter of context, as was the original decision to use an estimate.

Understanding Computational Estimation

The goal of computational estimation is to be able to produce a reasonable *estimate* flexibly and quickly that is a suitable approximation for an exact number, given the particular context. In the K–8 mathematics curriculum, *estimation* refers to three quite different ideas:

- *Measurement estimation:* Determining an approximate measure without making a precise measurement. For example, we can estimate the length of a room or the weight of a watermelon.
- *Quantity estimation:* Approximating the number of items in a collection. For example, we might estimate the number of students in the auditorium or jelly beans in the estimation jar.
- *Computational estimation:* Determining a number that is an approximation of a computation that we cannot or do not need to determine precisely. For example, we might want to know the approximate amount we are spending at a store and need to add the cost of several items to see whether $20.00 will cover the amount.

Students are often not as good at computational estimation as they are at producing exact answers and sometimes find computational estimation uncomfortable (Siegler & Booth, 2005). Many students confuse the idea of estimation with guessing. None of the three types of estimation involves guessing. Each involves reasoning using strategies. Therefore, it's important to (1) avoid using the words *guess* and *guessing* when working on estimation activities and (2) explicitly help students see the difference between a guess and a reasonable and well-informed estimate.

Computational estimation may be underemphasized in situations where the focus is on the standard algorithm, but in the problem-solving process, the last of the four steps is to look back. If students practice estimating when they are computing and look back after completing a computation, they should be able to see whether the answer is in the ballpark. Take 402−138. At a glance, this answer must be more than 200 and less than 300, so an incorrect answer of 175 (a common error) would be recognized as not reasonable.

Suggestions for Teaching Computational Estimation

Here are some general principles that are worth keeping in mind as you help your students develop computational estimation skills.

Use Real Examples. Discuss real-life situations in which computational estimations are used. Some examples include comparative shopping (which store has the item for less); adding distances in trip planning; determining monthly totals (school supplies, lawn-mowing income, time playing games on a tablet); and figuring the cost of going to a sporting event or movie, including transportation, tickets, and snacks. Look at news headlines to identify numbers that are the result of estimation and numbers that are the result of precision such as comparing "Hundreds of Students Leave School Ill" versus "Fourteen Students Injured in Bus Accident." Students are often motivated with real examples—for example, asking older students, "Are you a million seconds old? How can you find out?"

Use Estimation Language. Words and phrases such as *about*, *close*, *just about*, *a little more* (or *less*) *than*, and *between* are part of the language of estimation. Students should understand that they are trying to get as close as possible by using efficient methods, but there is no one correct or winning estimate. Language can help convey that idea.

Focus on Reasonableness (Not Accuracy). Do not reward or overemphasize an estimate that is the closest. It is already very difficult for students to handle approximate answers; worrying about accuracy and acknowledging the one closest answer exacerbates this problem. Instead, focus on whether the answers given are *reasonable* for the situation or problem at hand.

Use a Context. Situations play a role in estimation. For example, it is important to know whether the cost of a car would be $950 or $9500. Could attendance at the school play be 30 or 300 or 3000? A simple computation can provide the important digits, with knowledge of the context providing the rest.

Focus on Flexible Methods. Remember that having students reflect on the strategies used by classmates will lead to additional strategy development. Class discussions are just as important as they were for the development of reasoning strategies. For any given estimation, there are often several very good but different methods of estimation. Here is an activity in which a specific number is not required to answer the questions.

Offer a Range as an Option. What is a reasonable answer for $267 + 295$? Well, it has to be greater than $200 + 200$ (400) and it has to be less than $300 + 300$ (600). The answer must be between 400 and 600. Sometimes setting a range for a reasonable answer is easier than implementing an estimation strategy. A related option is to use a benchmark and ask if the answer is over or under that value. Activity 11.18 focuses on this idea.

Activity 11.18 Over or Under?

CCSS-M: 3.OA.C.8

Use the Over or Under activity page on a document projector (as in Figure 11.25). In this case, each answer is either over or under $1.50, but the "over or under" number can be changed for each task. You can add an interesting context to make the activity accessible to more learners, but remember that using multiple contexts can be difficult for multilingual learners who must learn the language around each new context. Consider playing Over or Under? by picking one context (e.g., the price of energy bars or fruit) and then varying the values (5 at 43 cents each, then 6 at 37 cents each, and so on).

Pearson eText Activity Page: Over or Under

Here are some addition examples for the Over or Under? activity:

37 + 25 (over/under 100)
712 − 458 (over/under 300)

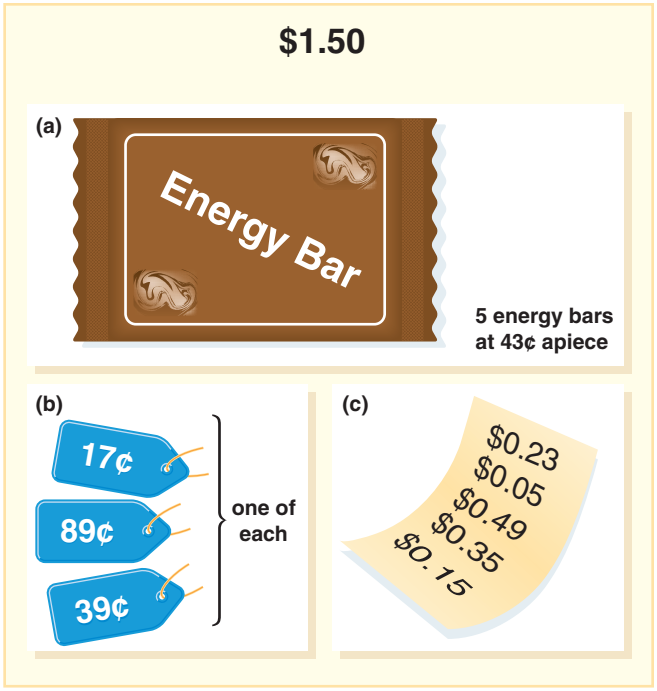


FIGURE 11.25 Over or Under? is a good beginning estimation activity

Computational Estimation Strategies

Learning Outcome 11.5 **Identify a variety of computational estimation strategies.**

As with exact computation, there are strategies for computing estimates in addition and subtraction. Here are a few to teach to students explicitly.

Front End

A front-end computational estimation simply means add the front end—the largest place value. It is a reasonable option for addition or subtraction when all or most of the numbers have the same number of digits. For initial experiences, practice first just using the front digits. That means that when a number has fewer digits than the rest, that number is ignored.

With experience doing front-end only, students are ready to learn to adjust to get a more accurate estimate. Adjusting means looking at the numbers that were ignored and deciding how much to raise or lower the original estimate. Adjusting can lead to students worrying too much about accuracy, so to assist, ask, "Is there enough to go up one more [hundred]? If so, adjust, if not, then keep your estimate." Figure 11.26 illustrates this idea.

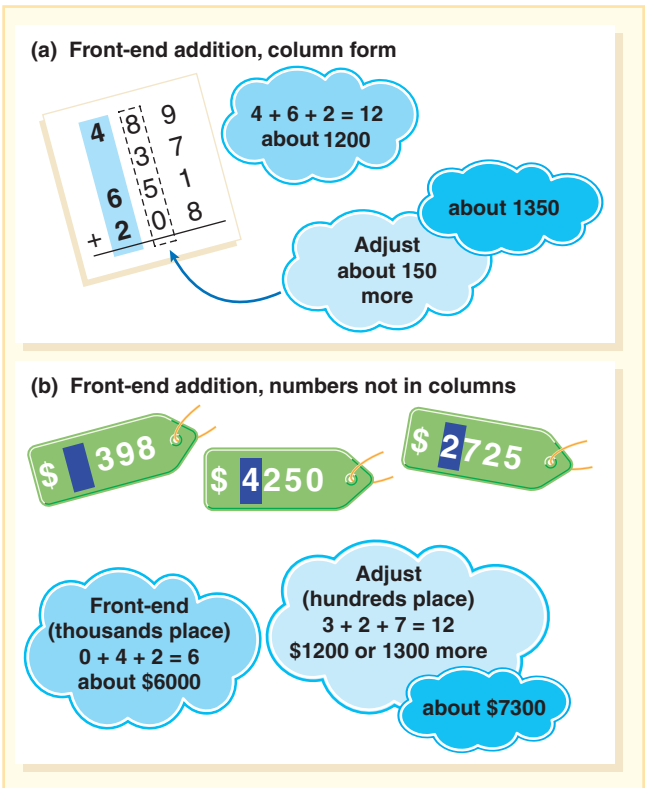


FIGURE 11.26 Front-end estimation in addition

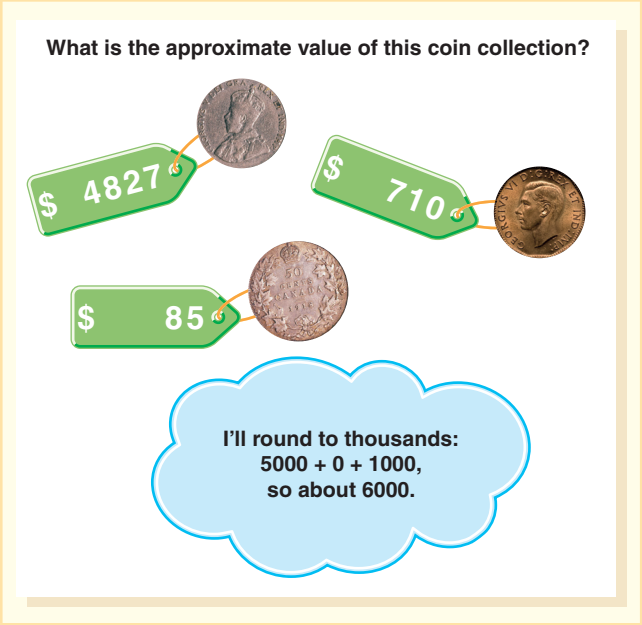


FIGURE 11.27 Rounding using place value.
Image credit: Yaroslaff/Shutterstock

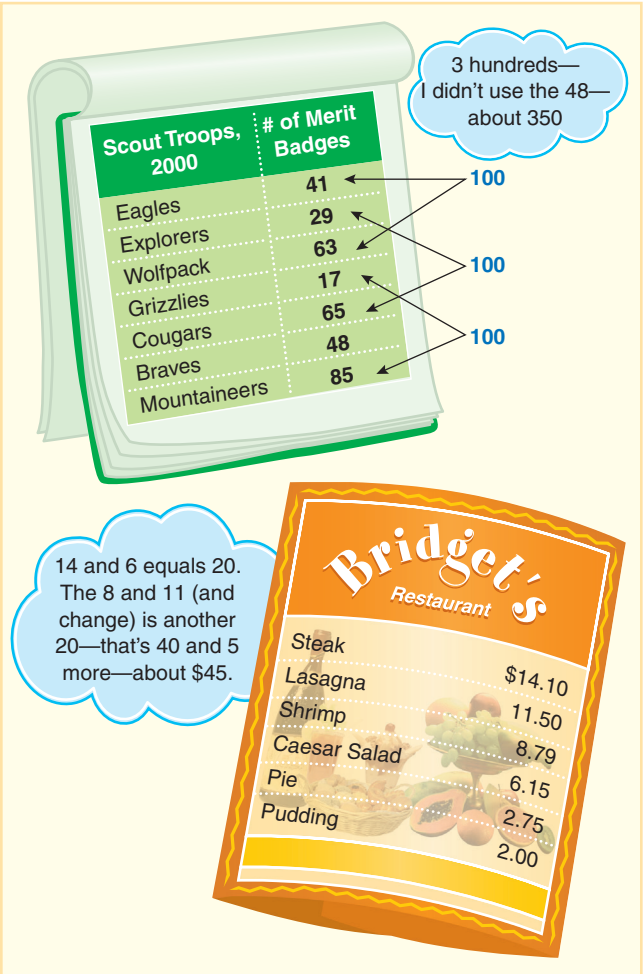


FIGURE 11.28 Compatible numbers used in addition

The front-end strategy is efficient because it does not require rounding or changing numbers. Students, however, must pay close attention to place value and only consider digits in the largest place, especially when the numbers vary in the number of digits.

Rounding

Rounding is an expectation in standards, often at grade 3 or 4. The focus of such rounding is often learning to round to the closest ten or hundred. Rounding to estimate is more flexible and does not have to abide by rounding rules. Remember the goal is to get a reasonable estimate. For example, to estimate $347 + 439$, rounding both down gives an underestimate. Rounding one up and one down will result in an estimate closer to the actual answer. Like with front end, rounding is more challenging when the numbers are different numbers of digits. Figure 11.27 shows such an example.

Using rounding to estimate begins with understanding the process of rounding, the focus of Activity 11.19.

Activity 11.19 Round Up?

CCSS-M: 3.NBT.A.1

Create a number line on the floor with painter's tape, clothesline, or cash register tape. Use sticky notes to label the benchmark numbers of the tens (10, 20, 30,…), hundreds (100, 200, 300,…) or whatever range of numbers you are considering as a class. Have the numbers far apart so that 3 to 4 students can stand facing forward between each number. Then distribute numbers for students to round. For example, give 53. The student stands on the line where 53 should be, then rounds to the nearest ten (50). Discuss the case of a 5 in the ones position (or in other halfway positions for larger numbers) as a convention—we all agree that we round up when we are midway between numbers.

For addition and subtraction problems involving only two numbers, one option is to round only one of the two numbers. For example, you can round only the subtracted number, so $6724 - 1863$ becomes $6724 - 2000$, resulting in 4724. You can stop here, or you can adjust. Adjusting might go like this: Because you subtracted a larger number, the result must be too small. Adjust to about 4800.

Compatible Numbers

Sometimes it is useful to look for two or three compatible numbers that can be grouped to equal benchmark values (e.g., 10, 100, 500). If numbers can be adjusted slightly to equal these amounts, that will make finding an estimate more straightforward. This approach is illustrated for addition in Figure 11.28. In subtraction, it is often possible to adjust only one number to produce a readily observed difference, as illustrated in Figure 11.29.

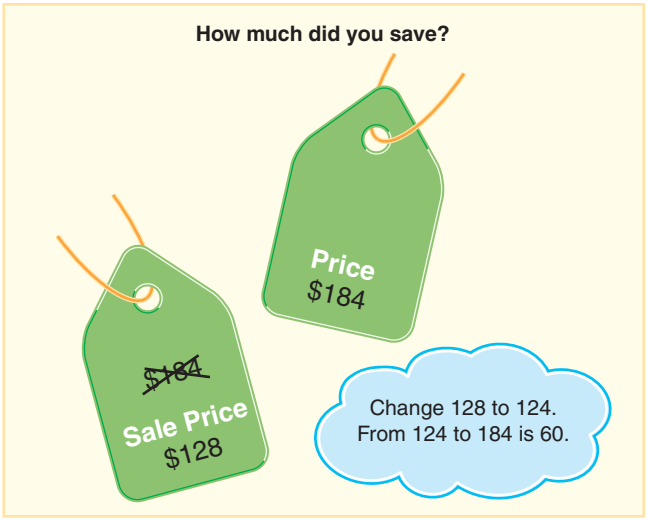


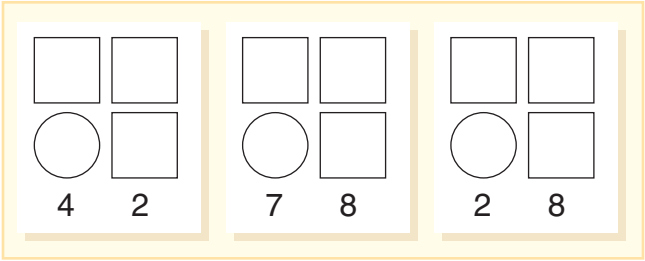
FIGURE 11.29 Compatible numbers can help find the difference

The following activity (adapted from Coates & Thompson, 2003) blends mental computation and estimation. Figuring out where the numbers go to create the exact solution involves estimation.

Activity 11.20 Box Math

CCSS-M: 2.NBT.B.5

Give students three number cards (e.g., 3, 5, 7) and two operations cards (+ and −), cut out from card stock so they can manipulate them easily on the Box Math Template. Show students a set of equations with answers only and ask them to use their number cards (in the squares) and operations (in the circle) to get to the answers shown.



For students with special needs in math, you may want to have the operation signs built into the templates of answers (such as pages with an addition box or a subtraction box). These fixed operation signs will reduce the amount of decision making needed and allow for a focus on the numbers.

Pearson eText Blackline Master: Number Cards
Pearson eText Activity Page: Operations Cards
Pearson eText Activity Page: Box Math Template

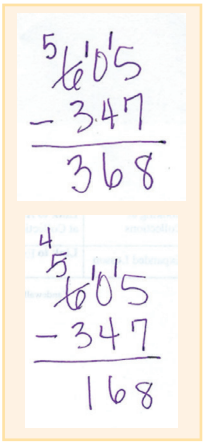
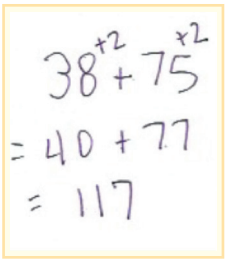
FORMATIVE ASSESSMENT Notes. In a diagnostic interview, ask a student to solve the following: "Charlie wants to estimate how much he needs to save to buy two video games. One game is $99 and the other is $118. About how much should he save to purchase the two games?" Ask, "How did you come up with your estimate?" If the student is trying to carry out the standard algorithm mentally in the air or on the table with a finger, they likely need more

experiences with estimation strategies. Because \$99 is so close to \$100, it would be important for the student to use that relationship in combining the two prices. If the student is unclear whether to change 99 to 100 or to 98 (as it is equally close), that pinpoints a lack of understanding of the purpose of estimation. Estimation cannot be easily assessed with paper-and-pencil tasks, so interviews that require explanations give more substantial evidence of student performance. ■

Common Challenges related to Addition and Subtraction Computation

TABLE 11.4 COMMON CHALLENGES RELATED TO ADDITION AND SUBTRACTION COMPUTATION

Common Challenge	What It Looks Like	How to Help
1. When using a decomposition strategy, knowing what remains when a part is taken from an addend.	When adding $68 + 17$, the student gives 2 from the 17 to make $68 + 2 = 70$, then they add on 17.	• Have students model the value of the addends with base-ten materials and then make the move from one quantity to the other and record.
2. When using compensation with subtraction, knowing when the adjusting involves addition and when it involves subtraction.	When students subtract $83 - 19$, they decide to subtract 20 but are unsure how to counteract the compensation so they subtract another 1 from their answer.	• Model the value of the minuend (83) with base-ten materials and then subtract the 20. Discuss whether they subtracted too much or too little.
3. Using the standards algorithm, place value and regrouping are not accurately integrated into the process (Nelson & Powell, 2018). There is a disconnect between the digit and the reference unit.	When students add: $\begin{array}{r} 56 \\ +\ 97 \\ \hline 1413 \end{array}$ Students record the entire sum of the digits in each column. This challenge also may result in adding each digit collectively, such as: $\begin{array}{r} 23 \\ +\ 41 \\ \hline 10 \end{array}$ Here they don't see the 2 as twenty, for example, indicating they lost the reference unit.	• Have students model the value of the addends with base-ten materials on a place-value mat and then combine the quantities. • Avoid letting students read numbers in addition or subtraction problems as digits (for example, don't say 5; instead, they should say 5 tens or fifty). • Have students estimate the answer first. Will the answer be in the hundreds? Thousands? Then ask, "Is this answer reasonable?" • Have students use partial sums to record and find the total. • Have students use another strategy to check the answer.
4. The recording of regrouping in the standard algorithm is reversed.	When students add: $\begin{array}{r} 63 \\ +\ 52 \\ \hline 16 \end{array}$ they start adding in the tens place and add backward, disregarding place value. So, in this case when they add $6 + 5$ and get the answer of 11, they put down the one and regroup the one back to the ones column: $1 + 3 + 2 = 6$.	• Ask students to check their answer for reasonableness—if we start with 63 items and add more will we end up with only 16 items? • Have students model using base-ten materials with a place-value mat to carry out the computation. Show how the actions are recorded so that they match the standard algorithm.
5. The smaller number (digit) is subtracted from the larger number (digit) rather than regrouping.	For the problem $\begin{array}{r} 70 \\ -\ 23 \\ \hline 53 \end{array}$ students say, "0 from 3 equals 3 and 2 from 7 equals 5. The answer is 53."	• Remind students about subtracting from 0 or other smaller numbers by using a concrete model. Show 3 cubes in your hand and ask, "Can you take away 5?" Reinforce the result of a situation where you are asked for more than you have. • Put materials representing the minuend (70) on a place-value mat. Then subtract the subtrahend (23) and highlight the need for regrouping when there are not enough of the units needed.

Common Challenge	What It Looks Like	How to Help
6. When regrouping across zeroes in a subtraction problem, the proper exchange units aren't understood.		• Have students use estimation to check for reasonableness of answer. • Have students use base-ten materials and a place-value mat to represent the minuend and act out the subtraction of the other number. Compare to original answer and discuss the differences. • Ask the student to use another strategy to check answer.
7. There is an overgeneralization of the same-difference or equal-addition strategy to addition situations.	 For problems like 38 + 75, the student reasons that because 38 is 2 away from 40, they will add 2 to 38 (40) and then thinks they need to add 2 to 75 (77) because they do not understand the equal additions strategy and why it is only applicable to subtraction situations.	• Have students use base-ten materials to represent the quantities and solve the original addition problem and the problem they created. Then ask the student to compare. • Have students use a number line to represent and solve the original addition problem and the problem they created. Ask the student to try to determine why the answers are different. • Have students solve a subtraction problem, using equal additions. Then show the original and the modified subtraction problems on a number line. Have students determine why the equal-additions strategy results in equivalent answers.
8. When asked to estimate, there is a focus on finding an exact answer.	When asked to estimate, say, 624 − 83, the student changes 83 to 100 to find 624 − 100 or about 524. But then immediately calculates to find the exact answer.	• When students are asked to find estimates, avoid having them check their estimates by finding the exact answer. • Use real situations as examples for when estimations (and not exact answers) are useful. • Discuss why an exact answer is unnecessary when finding estimates.

RESOURCES FOR CHAPTER 11

WRITING TO LEARN

Assess your understanding and application of chapter content.

1. What are the benefits of reasoning strategies as compared to standard algorithms?
2. List two reasoning strategies for adding 46 + 38.
3. Use two strategies for solving 545 − 267.
4. When teaching standard algorithms, why is an assessment that explores student understanding so essential?
5. What are some important considerations for teaching computational estimation?

LITERATURE CONNECTIONS

Children's literature plays a valuable role in helping you develop problems that lead to multiple strategies and mental computation.

The Great Math Tattle Battle

Bowen (2006)

This entertaining book is largely about correcting errors in double-digit addition calculations. You can share a student worksheet that contains lots of worked examples and see

whether students can find errors and correct the paper. There's also an important message about tattling!

National Geographic Kids Almanac 2023

National Geographic (2022)

These nonfiction books include interesting facts that can be used to create a variety of addition and subtraction problems.

RECOMMENDED READINGS

Article

O'Loughlin, T. A. (2007). Using research to develop computational fluency in young mathematicians. *Teaching Children Mathematics*, *14*(3), 132–138.

Here a second-grade teacher describes the journey to improve students' computational fluency through the use of reasoning strategies. An interesting collection of student work and related thought-provoking debriefing by the teacher is provided.

Whitacre, I., Schoen, R. C., Champagne, Z., & Goddard, A. (2016/2017). Relational thinking: What's the difference? *Teaching Children Mathematics*, *23*(5), 302–308.

Students' subtraction computations are the focus of this article. Explore the role of reasoning and flexibility in students' development of their ideas.

Books

Caldwell, J., Kobett, B., & Karp, K. (2014). *Essential understanding of addition and subtraction in practice, grades K–2*. Reston, VA: NCTM.

This book emphasizes the pedagogical content knowledge needed to teach the big ideas of addition and subtraction. There are tasks to try and samples of students' work to discuss!

CHAPTER

12

Developing Strategies for Multiplication and Division Computation

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **12.1** Recognize how understanding place value and the properties of operations support the learning of reasoning strategies in multiplication.
- **12.2** Identify a variety of models and recording approaches for developing the multiplication algorithm.
- **12.3** Recognize how understanding place value and the properties of operations support the learning of reasoning strategies in division.
- **12.4** Explain the development of the standard algorithm for division and ways to record students' thinking.
- **12.5** Identify ways to teach computational estimation for multi-digit multiplication and division as a way to develop students' flexibility and ability to recognize reasonable answers.

The strategies students use to multiply and divide depend on the instruction you provide and their developing experiences with these ideas. Researchers suggest that reasoning strategies for multiplication and division are less well documented than for addition and subtraction (Verschaffel et al., 2007) and the relationship between the operations more difficult to grasp (Robinson & LeFevre, 2012). So let's look at ways to counter these challenges by creating an instructional environment that rewards flexibility and the exploration and testing of new ideas. As with addition and subtraction, students who only have knowledge of the standard multiplication and division algorithms have difficulty following steps they do not fully understand (Burns et al., 2014; Fuson & Beckmann, 2012/2013). When students can compute multi-digit multiplication and division problems in a variety of ways, complete written records of their work, explain their thinking, and discuss the merits of one strategy over another, they are developing as independent learners.

BIG IDEAS

- Multiplication and division reasoning strategies involve decomposing and composing numbers in a variety of ways.
- Reasoning strategies require a deep understanding of the commutative and associative properties of multiplication, the distributive property of multiplication over addition, and the inverse relationship of multiplication and division.
- Reasoning strategies provide flexible methods of computing that vary with the numbers and the situation. Successful use of the strategies requires that they be understood by those who are using them—hence, they must grasp that the process and the outcome of that process are related.
- Standard algorithms are generalizable steps for computing that have been developed over time. Each is based on performing the operation on one place value at a time.
- Computational estimation is an important skill. Computational estimation strategies involve using easier-to-handle parts of numbers or close compatible numbers so that the adapted expression can be computed mentally.

Reasoning Strategies for Multiplication

Learning Outcome 12.1 **Recognize how understanding place value and the properties of operations support the learning of reasoning strategies in multiplication.**

A reasoning strategy is a flexible method for computing (unlike an algorithm, which is a step-by-step procedure implemented the same way in all cases). Reasoning strategies are often done mentally, though they do not have to be. To be clear, mental mathematics is not doing computations in your head, where you imagine the algorithm on a mental chalkboard and carry out the calculations; instead, it is reasoning about numbers and making decisions about which strategy to use (Erdem, 2017; Varol & Farran, 2007). When students use reasoning strategies, they begin to understand why mathematical ideas are true as they extend their prior knowledge to new contexts. Reasoning with multiplication is more challenging than addition or subtraction, because students move from acting on units that are the same, such as adding 23 *fruit* to 28 *fruit*, to the binary operation of multiplication, where they act on two quantities of differing units, such as 23 *fruit* at 45 *cents* each (Calabrese et al., 2020). In other words, in addition and subtraction, quantities are linked via part-part-whole relationships, and multiplication involves one-to-many relationships (Nunes et al., 2016). But similar to addition and subtraction, these strategies are number-based rather than digit-based and based on reasoning rather than on a strict step-by-step routine.

You will help students, using the CSA approach as you move from basic arrays for single-digit multiplication problems to larger arrays that emphasize the distributive property, to mapping simultaneously, to recording partial products, and then over time to the standard algorithm. Let's see how these ideas and representations all fit together! Students' ability to compose and decompose numbers flexibly is important for multiplication. This skill hinges in part on the full understanding of the distributive property of multiplication over addition. For example, to multiply 5×43, one might think about breaking 43 into 40 and 3, multiplying each by 5, and then adding the results. Students require ample opportunities to develop these concepts by making sense of their own ideas and those of their classmates.

As with addition and subtraction, it is helpful to place multiplication tasks in context and let students model the problems in ways that make sense to them. The strategies described here are grounded in student reasoning, as described in research on multiplicative reasoning in Table 12.1 (Baek, 2006; Confrey, 2008; Hulbert et al., 2017).

Pearson eText
Video Example 12.1
In this video consider how the student uses the distributive property to solve 7×12.

TABLE 12.1 PROGRESSION OF MULTIPLICATION STRATEGIES AND ALGORITHMS

Strategy	Characterized by:
Direct Modeling	Repeatedly grouping the same number of items through a variety of representations.
Decomposition	Breaking at least one of the factors into components, making a chain of multiplication calculations with simpler numbers.
Compensation	The problem is changed to an easier one, and then an adjustment or compensation is made such as halving one factor and then doubling the other.
Partial Products	Breaking apart the numbers by tens and ones and finding the product of each part (the distributive property). Partial products are then added to find the total product.
Standard Algorithm	A sequenced step-by-step process used to calculate a multiplication situation.

All of these strategies and algorithms should be part of students' repertoire for multi-digit multiplication computation. Introduce different representations as ways to explore multiplication until you are comfortable that students have a collection of useful ideas. Although teachers may worry that presenting multiple strategies to solve problems will overwhelm and confuse students, researchers found that comparing a variety of strategies from the start helped students gain flexibility and enhanced learning (Rittle-Johnson et al., 2020). In fact, not teaching these strategies to students is an equity issue because they are being denied access to reasoning and effective methods for solving multiplication problems (Bay-Williams & SanGiovanni, 2021).

Direct Modeling

The problem 6×34 may be represented in a number of ways, as illustrated in Figure 12.1, including direct modeling of equal groups, counting, and arrays of equal row size. Often the choice of a model is influenced by a story problem. To determine how many oranges 6 classes need if there are 34 students in each class, students may model 6 sets of 34 counters. If the problem is about a garden's area (6 cm $\times$ 34 cm), then some form of an array is likely. But each representation is appropriate for thinking about 6×34 regardless of the context, and students should get to a point at which they select meaningful thinking tools to think about multiplication.

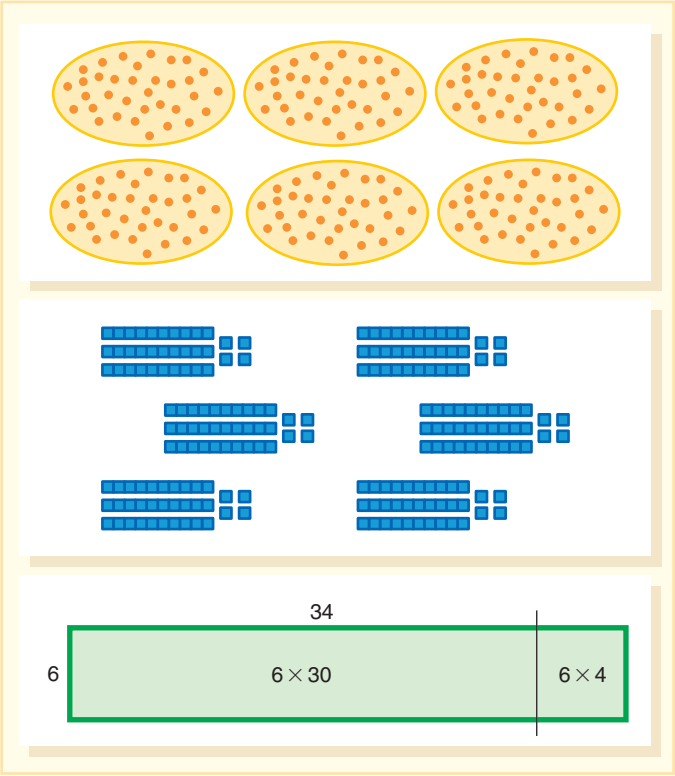


FIGURE 12.1 Different ways to model 6×34 that focus on equal groups

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

How students represent a product influences the way in which they will solve it. At first, the six equal groups of 34 students in a class might suggest doubling 34, which equals 68, and then because there are three groups of those, $68 + 68 + 68 = 204$.

Students initially may approach larger multiplication problems as repeated addition. This may result in a long column of the same number to be added. These early strategies based on repeated addition are neither efficient (for example 78×234) nor useful (think about multiplication of fractions) (Chin et al., 2019; Devlin, 2011; Larsson et al., 2017; Säfström et al., 2019). To fade this approach, encourage students to recognize that if they add two numbers that are the same, the next two will have the same sum and so on. This doubling

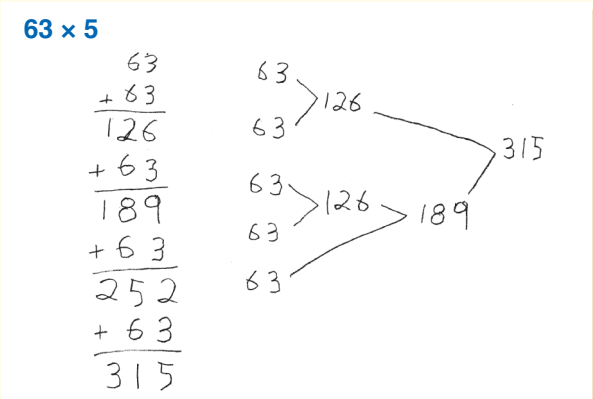


FIGURE 12.2 Two students use adding on and a doubling strategy

strategy can become the principal approach for many students (Flowers & Rubenstein, 2010/2011). Doubling capitalizes on the distributive property, whereby doubling 47 is double 40 + double 7, and the associative property, in which doubling 7 tens—or $2 \times (7 \times 10)$—is the same as doubling 7 and then multiplying by 10, or $(2 \times 7) \times 10$. Figure 12.2 illustrates two strategies students may use.

Decomposition Strategies

Students decompose (partition, split, or break apart) numbers in a variety of ways that reflect an understanding of place value, at least four of which are illustrated in Figure 12.3.

Another example of by tens and ones is to think about how six groups of base-ten materials with a groups size of 34 might be broken into tens and ones: 6 times 3 tens or 6×30 and 6 times 4 ones or 6×4 and add those two products together. Some students use the tens individually: 6 tens make 60. So that's 60 and 60 and 60 (180); then add on the 24 to equal 204. The by-decades decomposition strategy (which can be extended by hundreds, by thousands, etc.) is the same as the standard algorithm except that students always begin with the largest values and create partial products. This mental math strategy is very powerful.

The decomposing so far has involved breaking one or both factors' sums, which is the distributive property in action. You can also decompose into factors. For example, for 27×4, it could also be $27 \times 2 \times 2$, then 54×2 which equals 108. A common use of this type of decomposing is the halve-and-double strategy, which is particularly useful when 5 is a factor. Here is how it works:

$$\begin{aligned} 48 \times 5 &= 24 \times 2 \times 5 \\ &= 24 \times 10 \\ &= 240 \end{aligned}$$

Pearson eText
Video Example 12.2
In this video consider the strategy Ally uses as she solves a multiplication problem mentally.

Decomposition Strategies for Multiplication

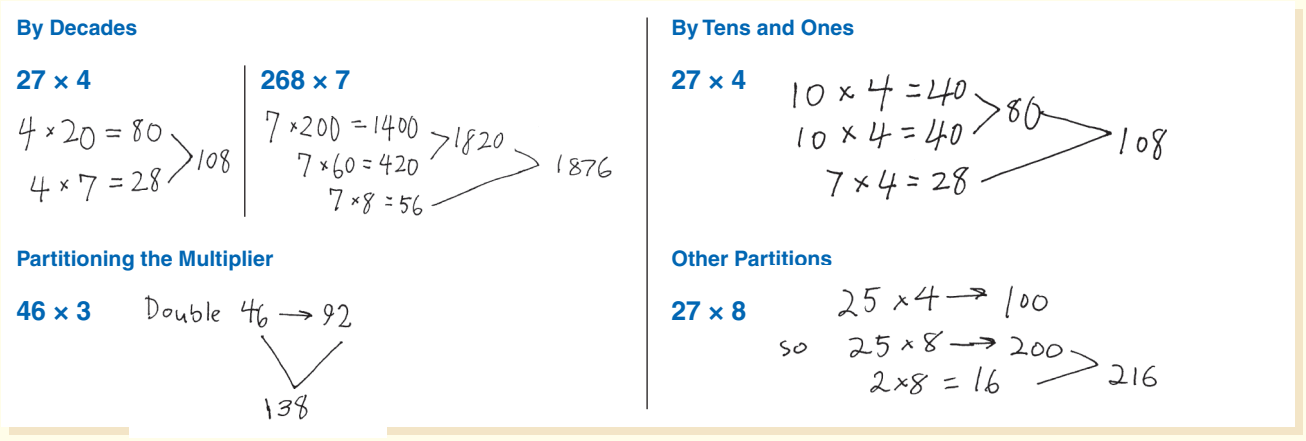


FIGURE 12.3 Four different ways to apply the distributive property to decompose factors

In the end, you have halved the 48 and doubled the 5. A student might explain it more simply, as illustrated in Figure 12.4.

As you may notice, all decomposing strategies rely on knowledge of the distributive property.

Activity 12.1 Halve and Double

CCSS-M: 4.NBT.B.5

Have pairs work together on the following problems using their knowledge of decomposing and the halve-and-double strategy.

25 × 8 is the same as _____ × 4

140 × 5 is the same as _____ × 10

20 × 54 is the same as _____ × 108

50 × 14 is the same as 100 × _____

Challenge students to create their own.

Pearson eText Activity Page: Halve and Double

Compensation Strategies

Students and adults look for ways to manipulate numbers so that the calculations are easier. In Figure 12.5, the problem 27 × 4 is changed to an easier one, and then an adjustment or compensation is made. The second example shows the half-then-double strategy, in which one factor is cut in half and the other factor doubled. This approach is often used when a 5 or a 50 is involved. Because these strategies depend on the numbers in the problem, they can't be used for all computations. However, they are valuable approaches, especially for mental math and estimation.

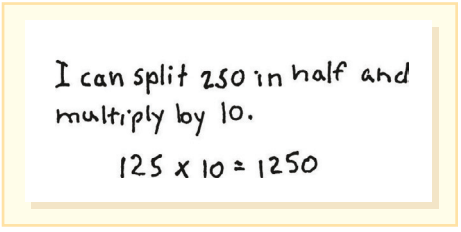


FIGURE 12.4 Using the halve-and-double strategy for the problem 250 × 5

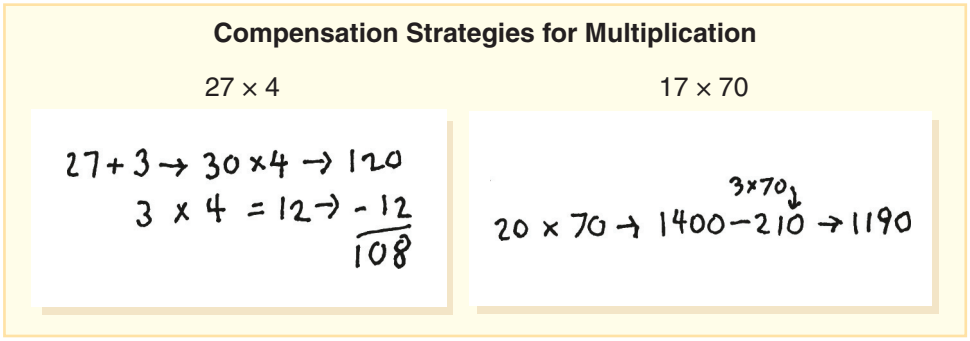


FIGURE 12.5 One factor is adjusted to compensate for a change in the other factor

Teaching Reasoning Strategies

Students need many experiences to help them visualize the strategies and to practice using the strategies. Recall that Chapter 4 describes various routines (including number talks, number strings, and worked examples). Each of these can be used to help students become fluent in using these strategies. Here we elaborate on one—number strings.

Number Strings. Number strings (also called *cluster problems)* are a series of expressions to guide students explicitly from easier to more complex computations. Creating number strings helps students think about ways that they can decompose numbers into easier parts. The strategy of breaking the numbers apart and multiplying the parts—using place-value knowledge coupled with the properties of the operations—is an extremely valuable technique for flexible

Pearson eText
Video Example 12.3
In this video consider the strategy Rachel uses to solve 45 × 36.

computation and prepares students for understanding the standard algorithm. For example, to find 34×50, students might think about and record answers for the following number string:

$$3 \times 50$$
$$4 \times 50$$
$$10 \times 50$$
$$30 \times 10$$
$$30 \times 50$$

You may wonder if I need to use them all, but students can analyze to see which ones they think will help them find the product (there are multiple options). They can also add other problems to the string that might be useful by thinking about which component as broken apart through place value might come in handy. First, have students make an estimate of the final product for the original problem before carrying out the problems in the number string. For example, 3×50 and 10×50 may be helpful in thinking about 30×50. The results of 30×50 and 4×50 combine to give you 34×50. After experiences with teacher-created number strings, you can have students brainstorm as a class a string of related problems to help them solve the original problem. Then, students need opportunities to make up their own set for a given product.

Pause & Reflect

Try making up a number string for 86×42. Include all possible problems that you think might be helpful. Then use your set to find the product. Is there more than one way? ●

Were these problems in your number string? Did you use others?

$$2 \times 80 \quad 4 \times 80 \quad 2 \times 86 \quad 40 \times 80$$
$$6 \times 40 \quad 10 \times 86 \quad 40 \times 86$$

All that is required to begin the number string is that your collection of problems eventually leads to a solution.

Multiplication of Larger Numbers

As students move from strategies with a single-digit factor to two-digit factors (or more), a good place to begin is with products involving multiples of 10 and 100. This focus supports the importance of place value and an emphasis on the actual number (number-based) rather than on the separate digits (digit-based) (Hickendorff et al., 2019). Consider the following problem:

A Scout troop wants to package 400 battery packs as a fundraising project. If each pack will have 12 batteries inside, how many batteries are the Scouts going to need?

Students can use $4 \times 12 = 48$ to figure out that 400×12 equals 4800. Make sure you discuss how to say and write "forty-eight hundred."

Be alert to students who simply tack on zeros without understanding why. They may say "to multiply by 100, just add two zeros on the end of the number." But very soon this rule will expire (Karp, Bush, & Dougherty, 2014) as students try to solve 2.5×10, a problem for which this rule will not work. Try problems in which tens are multiplied by tens, such as 30×60 or 210×40 to support thinking about larger products.

Students should then shift to problems that involve any two-digit numbers, not just those that are multiples of 10. A problem such as the next one can be solved in many ways.

The parade had 23 clowns. Each clown carried 18 balloons. How many balloons were there altogether?

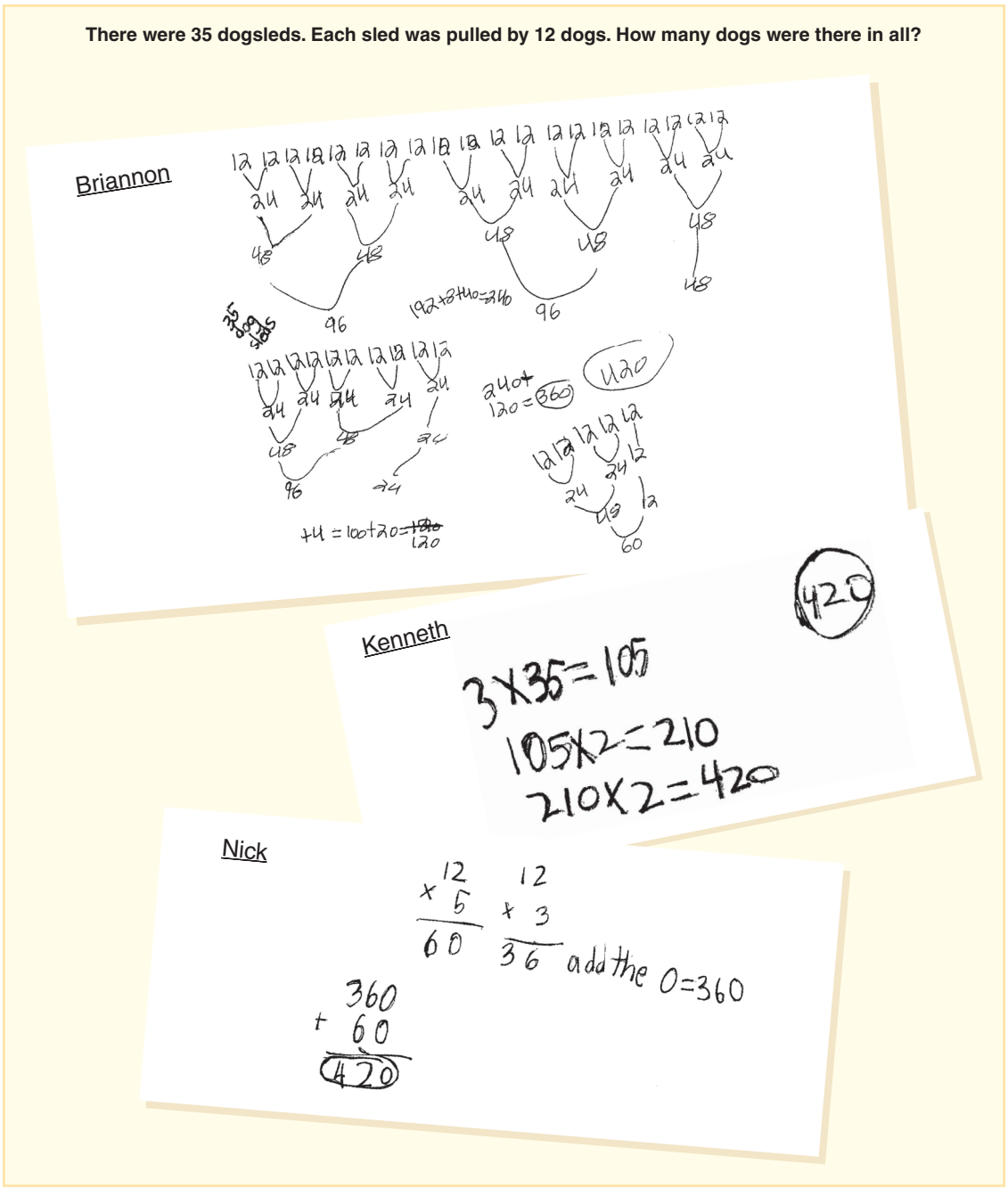


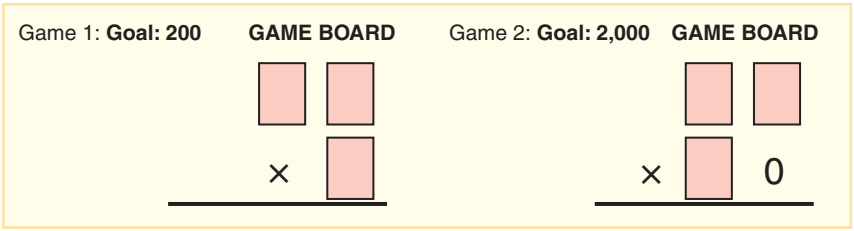
FIGURE 12.6 Three students solve a multiplication problem, using reasoning strategies

Some students might look for smaller products such as 23×6 and then add that result three times. Another strategy is to find the answer to 23×20 and then subtract 23×2. Others will calculate four separate partial products: $10 \times 20 = 200$, $8 \times 20 = 160$, $10 \times 3 = 30$, and $8 \times 3 = 24$. Two-digit multiplication is both complex and challenging. But students can solve these problems in a variety of interesting ways, many of which will contribute to the development of the standard algorithm. Figure 12.6 shows three fourth-grade students' work prior to instruction on the standard algorithm. Kenneth's work shows how he *decomposes* the factor 12 into $3 \times 2 \times 2$. Briannon is using a *doubling strategy*. She may need to see and hear about strategies other classmates developed to move toward a more efficient approach. Nick's method is conceptually very similar to the standard algorithm. As students like Nick begin partitioning numbers by place value, their strategies are often like the standard algorithm but without the traditional recording schemes.

Activity 12.2 Make the Most of It

CCSS-M: 4.NBT.B.5

Give a pair of students two game board templates and a deck of playing cards with the kings and jacks removed. Queens = 0 and Aces = 1. Each student draws three cards of their own and places them on the template to try to reach the largest product. The score is the product. First student to reach 200 in Game 1 wins and the goal for Game 2 is 2000. What strategies do students find that help them make the greatest product? Where should you put the largest card you draw and why? Does your strategy change for Game 2?



Pearson eText Activity Page: Make the Most of It Game

Pearson eText
Video Example 12.4
In this video consider how two students solved 251×12. What did you notice about how the second student solved the problem? How could you follow up with this student?

Algorithms for Multiplication

Learning Outcome 12.2 **Identify a variety of models and recording approaches for developing the multiplication algorithm.**

So far, this chapter has explored reasoning strategies. This section focuses on developing an understanding for algorithms. This includes *partial products* and the *standard algorithm*. Both of these involve an expansion of the distributive property. Partial products is number based—focusing on larger place values first; the standard algorithm is digit-based—focusing on smaller place values first (and using shorthand notations).

The standard multiplication algorithm is probably the most challenging of the four algorithms when students have not had numerous opportunities to explore their own strategies first. As with other algorithms, as much time as necessary should be devoted to conceptual development using concrete contexts and manipulatives, semi-concrete representations, and recording written symbols blended into that process (the CSA model).

Concrete: Contexts and Manipulatives

The multiplication algorithm can be developed with a variety of models, but when you move to two-digit multipliers, the area model has distinct advantages. It supports both the communitive and distributive properties. Let's begin with an example: 6×47. Start with the context of a garden and give students a drawing of a rectangular shape 6 centimeters by 47 centimeters. What is the area of the garden? Let students solve the problem in groups, using base-ten materials to cover the shape before discussing it as a class.

As shown in Figure 12.7, the rectangle can be sliced or separated into two parts: the rectangle formed by the tens rods and the rectangle formed by the unit cubes. Students will naturally focus on the larger quantity first, noticing that they have 6 rows of 4 tens, or 24 tens, and 6 rows of 7 units or 42 units. Each section is referred to as a *partial product*. By adding the two partial products ($240 + 42$), you get the total product or area of the rectangle.

The concept of partial products helps build a conceptual understanding to multi-digit multiplication. The partial products strategy also works well with students with special needs in math (Flores & Milton, 2020). Notice the use of the base-ten language "6 ones times 4 tens

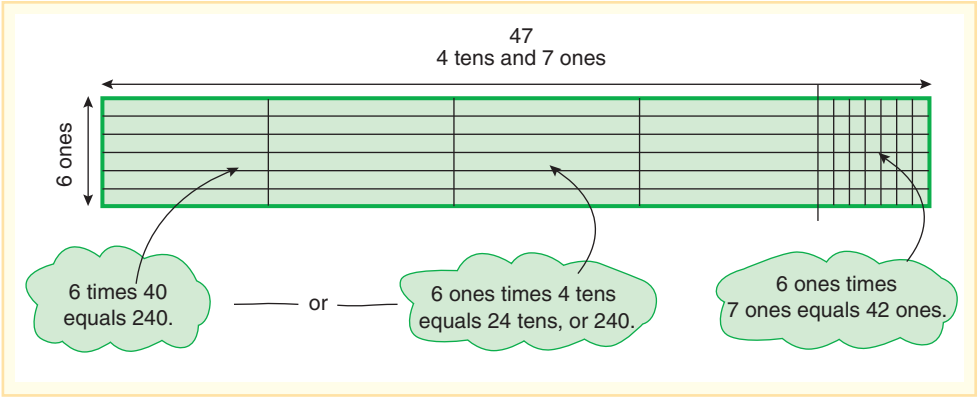


FIGURE 12.7 A rectangle covered with base-ten materials, modeling two-digit-by-one-digit multiplication

equals 24 tens" tells how many tens pieces are in the big section. To say "6 times 40 equals 240" is also correct and tells how many units or square centimeters are in the section. What you wish to avoid is saying "6 times 4" when the 4 actually represents 40.

Building different areas with base-ten pieces strengthens student understanding of the distributive property: $6 \times 47 = 6(40 + 7)$. A written record of the garden problem might look like this:

$$\begin{aligned} 6 \times 40 &= 240 \\ 6 \times \ 7 &= \ 42 \\ 6 \times 47 &= 282 \end{aligned}$$

This *area model*, which can be thought about as a connected array, is an important visual representation that can support students' multiplicative understanding and reasoning (Barmby, et al., 2009). The area model uses a row and column structure to organize equal groups and offer a visual demonstration of the commutative and distributive properties (unlike the number line). It is considered the "bar model for multiplication" (Säfström et al., 2019, p. 8529).

Activity 12.3 Carpet Tiles

CCSS-M: 4.NBT.B.5

Carpet Tiles provides a context for exploring larger multiplication problems with the *area model.* Give each group of students a large piece of paper that is either cut out or drawn to precise dimensions between 25 centimeters and 60 centimeters. This is the floor that is going to be covered in carpet squares. A hundreds piece (flat) is a carpet square, and represents 100 square units. Students determine the total square units of carpet that will cover the floor.

Pearson eText Blackline Master: Base-Ten Materials
Pearson eText Blackline Master: 1-Centimeter Grid Paper

For a rectangle that is 36 centimeters by 47 centimeters, for example, most students will first cover it with as many hundreds pieces as possible by putting the 12 hundreds in the upper corner. This placement will leave narrow regions on two sides that can be covered with tens pieces and a final small rectangle that will hold ones. Especially if students have had earlier experiences with finding products in arrays, figuring out the size of each sub-rectangle and combining them to find the size of the whole rectangle is relatively straightforward. Figure 12.8 shows the four regions.

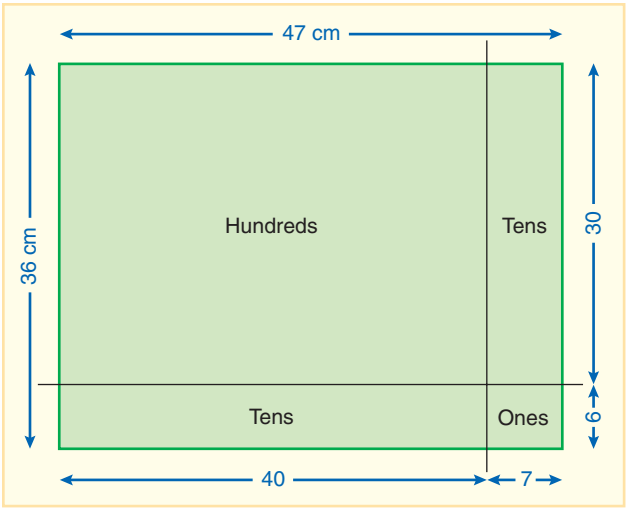


FIGURE 12.8 Ones, tens, and hundreds pieces fit into the four sections of this 36×47 rectangle

Activity 12.4 Build It and Break It

CCSS-M: 4.NBT.B.5

Select a problem such as 23×18. Use base-ten materials or 1-centimeter grid paper to build or sketch the corresponding area model. Then, have students show and record as many ways as possible to slice the array into pieces. For example, they could cut the array into $23 \times 10 + 23 \times 8$. Have students label the partial products. (If using base-ten materials, give students small pieces of paper to record the partial products.) Ask students questions such as, "What other vertical or horizontal slices can be made? For which slice options is the multiplication of partial products the easiest?" Before launching this activity, provide students, particularly multilingual learners and students with special needs in math, a labeled visual of an array (or area model) that includes the terms *array, area model, slice, vertical,* and *horizontal.* In a wrap-up discussion, be sure to focus on the vocabulary of the key concepts (*distributive property, decompose, strategy,* etc.).

Pearson eText Blackline Master: Base-Ten Materials
Pearson eText Blackline Master: 1-Centimeter Grid Paper

Semi-Concrete

While base-ten materials serve an important role in understanding partial products, it takes a long time to build large problems. Drawings become an efficient way to continue to visualize partial products.

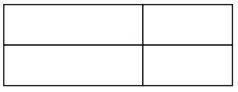
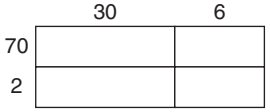
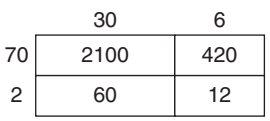
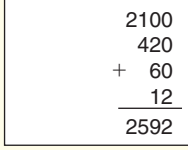
	1. Create an array according to the number of digits in the problem (in this case, a 2 × 2).
	2. Label the sides of the array with the factors decomposed by place value.
	3. Multiply the row by the column and put the product in the corresponding space.
	4. Add the partial products.

FIGURE 12.9 An open array model with partial products

Open Array. The open array (Fosnot & Dolk, 2001; NCTM, 2014a) is a semi-concrete representation of the area model and can be successfully used after students experience several constructions of area models with base-ten materials. Starting with a blank rectangle, students can mark off areas (the number of subdivisions depends on the number of digits in the factors) that align with the use of the distributive property. Students record partial products inside each subdivision. Note that the dimensions of the open array are not usually drawn to scale and therefore are often not precisely proportional. But the model can be productively used to think about the multiplication as an area of a rectangle, which aligns to the standard algorithm. The open array will also connect to future work with multiplication with fractions (Lannin, Chval, & Jones, 2013), algebraic terms (Benson, Wall, & Malm, 2013; Lischka & Stephens, 2020), and the history of mathematics through explorations of Egyptian and Russian peasant multiplication. Figure 12.9 shows four steps in the process of developing an open array for the problem 72×36:

Pearson eText Blackline Master: 2-Digit Open Array Template

Activity 12.5 Open Array Foldables

CCSS-M: 4.NBT.B.5

Rather than draw an open array, students can make a foldable. Give students a piece of colored paper (it can be a smaller piece for the activity) and a multiplication problem. Students fold their paper to match the size of the problem—if it is a two-digit by two-digit, they will have four partial products, so they fold the paper into four sections. If it is a three-digit by a two-digit, then they fold their paper into six sections. On the borders, they can record the parts of the factors. Then, inside each box, they record the partial products. On a separate piece of paper, they add the partial products (in any order they like). This activity can be replicated in a Learning Center with a stack of colored paper for folding and a set of multiplication problem cards.

Pearson eText Activity Page: Set of Multiplication Problem Cards

***TECHNOLOGY* Note.** Computer versions of the area model can ease some of the difficulties of physically covering rectangular grids with base-ten materials. Go to PhET at https://phet.colorado.edu/sims/html/area-model-multiplication/latest/area-model-multiplication_en.html for an area model that can be adjusted using a grid model or hundreds, tens, and ones pieces. Also, the Math Learning Center (https://apps.mathlearningcenter.org/partial-product-finder/) has the Partial Product Finder to compose arrays, use partial products, and apply the distributive property. ■

Moving to the Abstract

As students have used manipulatives and open arrays, they have also recorded equations. Eventually, students move to abstract notation, perhaps using mental images. This is true for both partial products and the standard algorithm. You can see partial products recorded in Step 4 of Figure 12.9. This is considered the partial products strategy. Ask students to record this in the opposite way, starting with the ones place times the ones place, and they are getting closer to using the standard algorithm.

Figure 12.10 illustrates a progression from partial products (a), to the standard algorithm without regrouping (b) and the standard algorithm with shorthand notation (c). This final form is the most abstract and therefore error-prone (see Table 12.4 for details). To avoid errors, encourage students to record partial products, which can be recorded in any order. To help students develop a recording scheme, provide multiplication recording sheets with base-ten columns on which students can record problems.

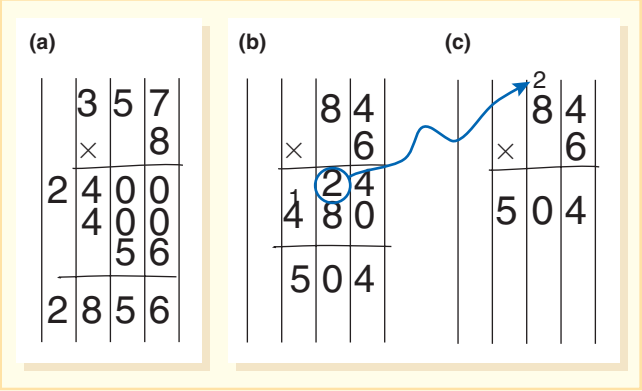


FIGURE 12.10 Symbolic notation for partial products and the standard algorithm

Pearson eText Blackline Master: Multiplication and Division Recording Sheets

As the standard algorithm is introduced, revisiting area models with base-ten materials is a good way for students to see the parts and connect them to the regrouping (See Figure 12.11). Several variations in language might be used as you talk about moving to the abstract from the concrete models. Consider 36×47. Note that base-ten language connects the four partial products to the written format of the standard algorithm. In the partial product 30×40 if base-ten language is used—*3 tens times 4 tens equals 12 hundreds*—the result tells how many hundreds pieces are in that section. Avoid saying "three times four," which promotes thinking about digits rather than actual numbers. It is an important generalization for students to uncover that a product of *tens times tens equals hundreds.*

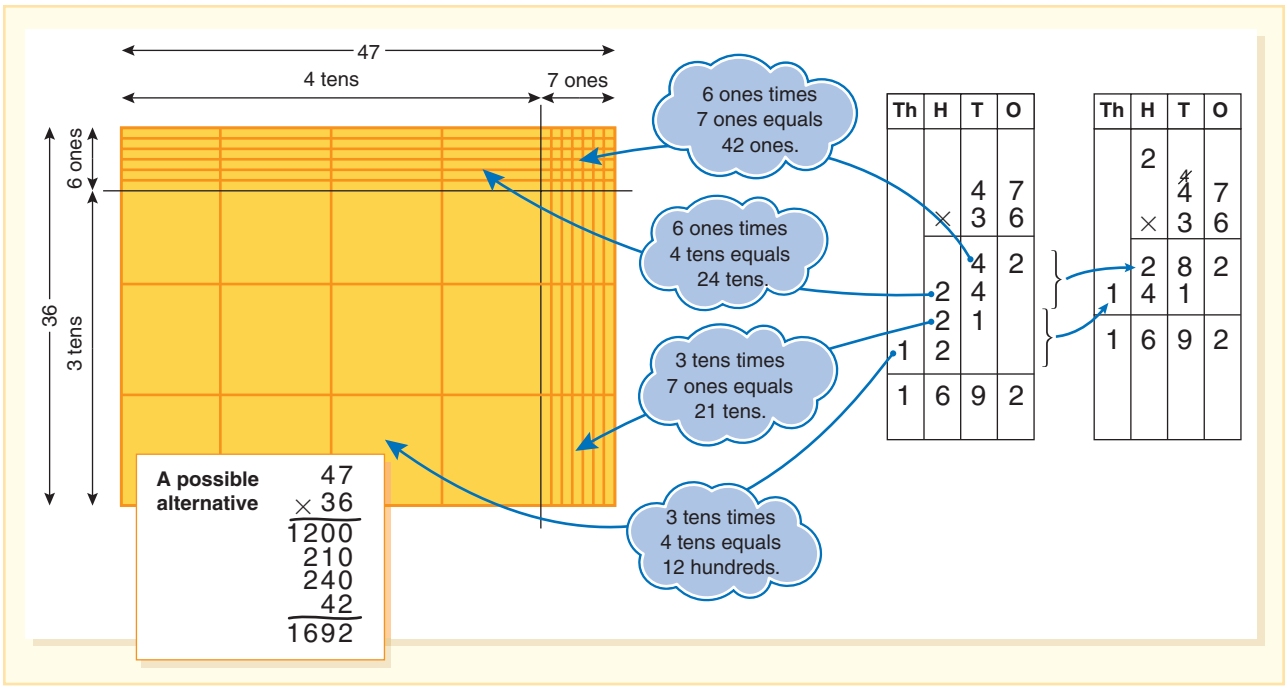


FIGURE 12.11 Connecting the area model with base-ten materials to the written standard algorithm

Figure 12.11 also shows the recording of four partial products in the order of the standard algorithm and how these products can be collapsed to two lines if small digits representing the regrouping are used to record the trade. Here the second exchange or trade technically belongs in the hundreds column but is often written elsewhere, which is again a source of errors. The lower left of the figure shows another alternative with all four products written. Using this approach, multiplying numbers such as 538×29 results in six partial products, but far fewer errors!

While students need to learn the standard algorithm, they must not abandon their reasoning strategies or their use of open arrays to support their thinking. Think about the problem 13×7. What is an efficient way to solve it? Partial products of 70 and 21 can be generated through multiplication and added mentally—more efficient than the standard algorithm. One study by Keiser (2010) found that, given the preceding problem, only 11 percent of the sixth graders used the standard algorithm. When given two-digit by two-digit multiplication problems, fewer than 20% of students chose to use the standard algorithm and less than half of that group got the correct answer (Keiser, 2010). The array or area model was most often selected and the most accurate. Remember that fluency includes efficiency, flexibility, and accuracy. As you are working on fluency with multiplication, continue to encourage students to use reasoning strategies and visuals. The key is have them make decisions based on the numbers in the problem and not just assume the standard algorithm is going to be the best option, as it often is not.

Pearson eText
Video Example 12.5
In this video author Jennifer Bay-Williams explains the use of partial products to multiply multidigit whole numbers.

Reasoning Strategies for Division

Learning Outcome 12.3 **Recognize how understanding place value and the properties of operations support the learning of reasoning strategies in division.**

Division computation strategies with whole numbers are developed starting in third grade, with the most common expectation in state standards that students will know the division algorithm in the fifth grade. Although there is considerable research on whole-number operations, division often receives the least attention (Camos & Baumer, 2015; Hickendorff, 2018), which is surprising, because division is challenging for students (Auer et al., 2016; Schulz, 2018), and knowledge of division supports students' understanding of fractions (Namkung, Fuchs, & Koziol, 2018).

TABLE 12.2 PROGRESSION OF DIVISION STRATEGIES

Strategy	Characterized by:
Direct Modeling: Partition or Fair Share (partitive model of division)	Fairly sharing the dividend into a number of equal groups
Direct Modeling: Repeated Subtraction (measurement [also called quotative] model of division)	Subtracting the divisor or multiples of the divisor (equal-sized groups) from the dividend
Think-Multiplication (also called Missing Factor)	Emphasizes the inverse relationship of multiplication and division by asking, for example, what number times 5 is equal to 135?
Decomposition Strategies	Decompose the dividend to create sub-problems (partial quotients) that are easier to carry out
Standard Algorithm	A sequenced, step-by-step process used to calculate a division situation

You can support student understanding of division by attending to the learning progression in Table 12.2, which is based on the work of several researchers (i.e., Hulbert et al., 2017; Schulz & Leuders, 2018). Fair sharing will also be discussed in detail in Chapter 14, because it is also the beginning of fraction understanding.

Pearson eText
Video Example 12.6
In this video consider how the student solves 15×12. What's a strength you notice in this student's reasoning?

Direct Modeling

It is important to revisit the models for division learned earlier as the fair sharing (partition) or the repeated subtraction (measurement) methods as we again model these ideas.

Partition or Fair-Share Strategy. Recall that there are two conceptualizations of division: partitive and measurement. Let's look at partition, or fair-sharing strategy, illustrated by this story problem and associated direct modeling.

Eileen's piggy bank has 783 pennies. She wants to share them equally with her 4 friends and herself. How many pennies will Eileen and each of her friends get?

Fair-share problems are often easier to solve with base-ten materials, and they mirror the idea of partitioning that is eventually used in the standard algorithm.

Figure 12.12 shows partitioning strategies two fourth-grade students used to solve division problems. The first example (a) illustrates $72 \div 3$, using base-ten materials and a fair-sharing process. When no more tens can be distributed, a ten is traded for ten ones. Then the 12 ones are distributed, resulting in 24 in each set.

Figure 12.12(b) shows a record of student thinking for the problem $342 \div 4$. The student sets out the base-ten materials and draws a four-column recording chart (to match the divisor) and uses a circle to mimic a paper plate image used previously to share pieces physically. After noting that there are not enough hundreds for each group to get one, she splits 2 of the 3 hundreds in half, putting 50 in each of the four columns. That leaves 1 hundred, 4 tens, and 2 ones. Trading the 1 hundred for tens (now having 14 tens), she puts 3 tens in

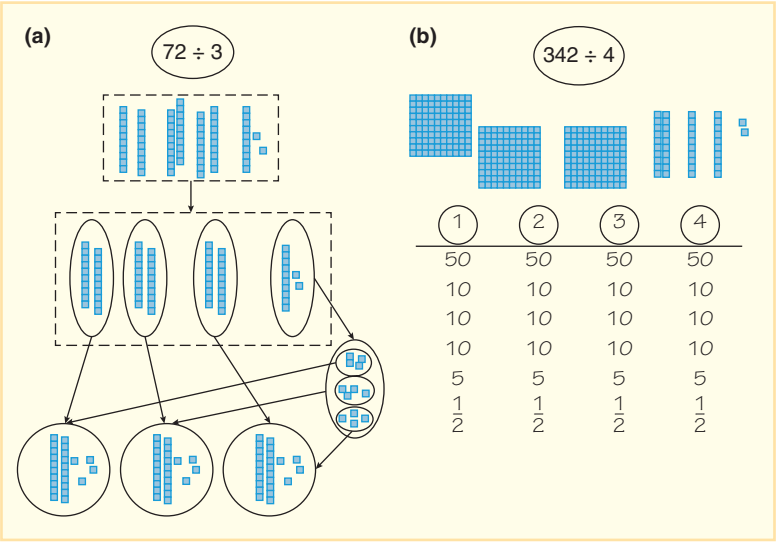


FIGURE 12.12 Students use both models and symbols to solve division tasks

Source: Based on *Developing Mathematical Ideas: Numbers and Operations, Part I, Casebook*, by Deborah Schifter, Virginia Bastable, and Susan Jo Russell. Copyright © 2000 by the Education Development Center, Inc. Published by Pearson Education, Inc.

each group, recording 30 in each column. Now she has 2 tens and 2 ones, or 22. She knows that 4×5 equals 20, so she adds 5 to each group, leaving 2 left over. She mentally shares the 2 remaining pieces, recording $\frac{1}{2}$ in each column.

Repeated Subtraction. Here is a problem that illustrates the measurement model of division solved here with repeated subtraction:

Jumbo the elephant loves peanuts. His trainer has 625 peanuts. If he gives Jumbo 20 peanuts each day, how many days will the peanuts last?

Here the students would subtract groups of 20 until there are no more full groups to subtract. This can be in clusters of 20, such as subtracting 10 groups of 20 three times and then calculating one more group of 20 with 5 remaining. See more about repeated subtraction when we discuss algorithms.

Teaching Reasoning Strategies

After students have experienced sharing (partitive) and repeated-subtraction (measurement) problem types concretely, they will have the conceptual understanding to employ reasoning strategies. For division, there are not very many reasoning strategies. Two are described here.

164 stickers; 6 stickers to a bag. How many bags?

I want to put 6 stickers in each bag.
So 6 x ? = 164
I estimate 6 x 10 = 60, and double that is 120.
Another 60 is 180, which is too high.
My answer is between 20 and 30 bags.
I have 44 more stickers to put in bags.
6 x 7 = 42 with 2 left over.
My answer is 27 bags with 2 stickers left over.

FIGURE 12.13 Student uses a think-multiplication strategy

Source: Based on *Developing Mathematical Ideas: Numbers and Operations, Part I, Casebook*, by Deborah Schifter, Virginia Bastable, and Susan Jo Russell. Copyright © 2000 by the Education Development Center, Inc. Published by Pearson Education, Inc.

Think-Multiplication (Missing-Factor Strategy). The student in Figure 12.13 solves a division problem that involves a measurement situation: How many bags with 6 stickers in each can be made if there are 164 stickers? She wants to find out how many groups of six are in 164. As a first step, she estimates and tries 6×20 by multiplying 6×10 and doubling the answer. Then, she tries adding another group of 10 and recognizes that is too high. She knows the answer is more than 20 and less than 30. She thinks about how many sixes are in 44, which she knows is 7 with 2 left over. So, her answer is 27 bags with 2 stickers left over.

This approach emphasizes the inverse relationship of multiplication and division. Sometimes this strategy is paired with a compensation approach when you slightly change the dividend to a higher or lower number to estimate a multiple of the divisor (in this case, thinking about 164 as 180 or 120).

Pause & Reflect

Before reading further, think about the quotient of $318 \div 7$ by trying to figure out *what number times 7* (or *7 times what number*) is close to 318 without going over. Do not use the standard algorithm. ●

There are several ways to think about beginning to solve this problem. For instance, because 10×7 equals 70 and 100×7 equals 700, the answer is between 10 and 100. You might start with multiples of 10. Forty sevens are 280. Fifty sevens are 350. So, 40 is not enough and 50 is too many. The answer has to be forty-something. At this point, you could test numbers between 40 and 50 or add on groups of seven. Or, you could notice that 40 sevens (280) leaves you with $20 + 18$ or 38. Five sevens will be 35 of the 38 with 3 remaining. In all, that's $40 + 5$ or 45 with a remainder of 3. Remember that remainders to be part of the quotient should be reported as fractions so the full answer is $45\frac{3}{7}$.

This missing-factor approach is likely to be developed by some students if they are solving measurement problems such as the following:

Owen can put 6 pictures on one page of his photo album. If he has 82 pictures, how many pages will he need?

Alternatively, you can simply pose a task such as 82 ÷ 6 and ask students, "What number times 6 would be close to 82?" Notice that the missing-factor strategy is equally good for one-digit divisors and for two-digit divisors.

Let's look how an open array can support students' thinking about a division problem by means of the think-multiplication strategy. Using the problem 342 ÷ 6, look at two useful steps in Figure 12.14. First, we use the open array to think about the missing factor by putting the known factor at the left of the rectangle and considering an estimate of how many hundreds, tens, and ones are needed for the other factor (see the top half of Figure 12.14). Then, using place-value understanding, partial factors are selected to come as close as possible to the largest multiplier, and the multiplication is carried out underneath (see bottom half of Figure 12.14). In every case, the resulting product gets subtracted from the dividend so the remaining amount can be distributed. If students underestimate, they can just create another section to the array to accommodate. Then they add the partial factors to get the quotient (and the remainder when appropriate).

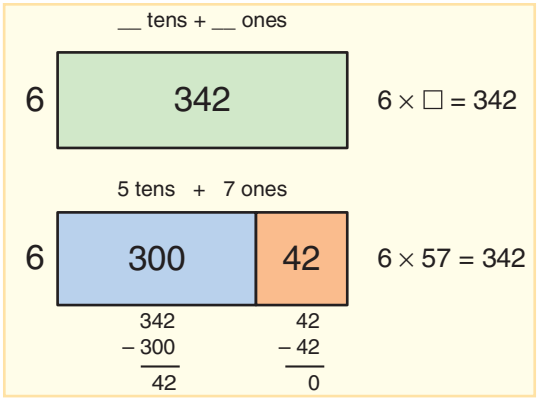
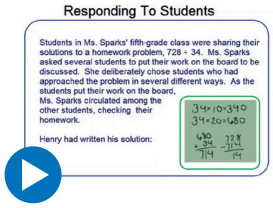


FIGURE 12.14 Using an open array to support student thinking with the think multiplication strategy

Decomposition Strategy. When considering division problems with a 3-digit dividend and a 2-digit divisor, students can manipulate the numbers to work toward their advantage. These strategies involve trying to identify a relationship between the dividend and divisor, sometimes by breaking numbers apart to a more mentally convenient form. For example, if asked to solve 157 ÷ 13, students could alter that problem by decomposing the dividend to (130 ÷ 13) + (27 ÷ 13) to get an answer of (10 + 2) with a remainder of 1. Students must be reminded again that here it is the dividend that can be broken apart by just grouping. The divisor is the number of required subsets and that can't change (in this case, 13). If students need more explicit proof, use direct modeling with base-ten materials and paper plates (as described in Chapter 8) to show how the materials (dividend) can be grouped differently for sharing, but the number of paper plates must remain stable.

You can decompose the divisor, but it is through multiplication, not addition, such as when you have 270 ÷ 15 conducting a chain of divisions by decomposing the 15 into 5 × 3, which can be thought of as 270 ÷ 5, and then that answer (54) is divided by three, as in 54 ÷ 3.


Responding To Students
Students in Ms. Sparks' fifth-grade class were sharing their solutions to a homework problem, 728 ÷ 34. Ms. Sparks asked several students to put their work on the board to be discussed. She deliberately chose students who had approached the problem in several different ways. As the students put their work on the board, Ms. Sparks circulated among the other students, checking their homework.
Henry had written his solution:

Pearson eText
Video Example 12.7
In this video, consider Henry's solution to 728 divided by 34 and how this method compares to the standard division algorithm. How would you use Henry's method to solve the problem 348 divided by 23?

Instructional Routines

Students need many experiences with routines learned previously in Chapter 4 as they cement their newly learned reasoning with division.

Number Strings Another approach to developing think-multiplication strategies for division is to use the number strings routine (sometimes called cluster problems). Following are examples of two number strings for two division problems (written in bold at the bottom of the columns):

Number String 1	Number String 2
100 × 4	10 × 72
500 ÷ 4	5 × 70
25 × 4	2 × 72
6 × 4	4 × 72
527 ÷ 4	5 × 72
	381 ÷ 72

Notice that it is useful to include division and multiplication problems in the string. In the first example, 400 ÷ 4 could easily replace 100 × 4, and 125 × 4 could replace 500 ÷ 4. The idea is to capitalize on the inverse relationship between multiplication and division. Number strings provide students with a sense that problems can be solved in different ways and with different starting points.

Pause & Reflect

Solve 514 ÷ 8 in two ways, beginning with different first steps. Do your approaches converge before you reach your solution? ●

For example, here are four possible starting points for 514 ÷ 8:

8 × 10 **400 ÷ 8** **8 × 60** **80 ÷ 8**

When you first ask students to solve problems using two strategies, they often use an inefficient method for their second approach (or revert to a standard algorithm). For example, to solve 514 ÷ 8, a student might perform a sequence of repeated subtractions (514 × 8 = 506, 506 × 8 = 498, 498 × 8 = 490, and so on) and count how many times they subtracted 8. Others will actually draw 514 tally marks and loop groups of 8. These students need more experiences to develop the flexibility to think of other, more efficient methods. Posing a variety of starting points can nudge students into other, more profitable alternatives, as will class discussions where flexible approaches are shared.

Worked Examples Can Showcase Student Thinking. Fraction notation can help make sense of the decomposition strategy, but the notation may be difficult for students to understand. That is a good time to use a worked example.

Consider Shenada's method for dividing.

$$345 \div 15 = \frac{345}{15} = \frac{150 + 150 + 45}{15} = 10 + 10 + 3 = 23$$

Notice that this problem can also be solved with the think-multiplication strategy. A student might say, "I know ten 15s equals 150, so twenty 15s equals 300 and three more 15s gives me a total of 23 groups of 15 in 345." That is the nature of strategies: Students can bring together ideas from each strategy as they make sense of the situation. You can also pose both approaches and ask students to compare the two approaches.

Algorithms for Division

Learning Outcome 12.4 **Explain the development of the standard algorithm for division and ways to record students' thinking.**

The standard algorithm for multi-digit division is the method whereby students make estimates, multiply, subtract, and repeat the process. Most state standards suggest that the division algorithm with one-digit divisors is developed in fourth grade with extension to two-digit divisors in fifth grade. The standard algorithm for multi-digit division is frequently expected to be learned in sixth grade. Students who are still struggling in grades 5 and beyond with single-digit divisors can also benefit from the following conceptual development. Out of the standard algorithms for the four main operations, the division algorithm appeared to be the one least well understood by teachers (Raveh et al., 2016). So this section is designed to help you understand this algorithm for yourself, as well as to help you teach students so that they will understand it. The key is to begin with a variety of number-based reasoning strategies and then progress to digit-based procedures through realistic problem solving (Hickendorff et al., 2018).

Concrete: Contexts and Manipulatives

Division with two-digit divisors is the one standard algorithm that starts with the left-hand or biggest pieces rather than the smallest pieces (the ones). The conceptual basis for the algorithm most often taught is the partition or fair-share method, the method we will

explore in detail in this section. Contexts are important in thinking about fair-sharing. Use a context like the one in Activity 12.6, or select a context meaningful to your own students.

Activity 12.6 Sharing Sea Shells

CCSS-M: 4.NBT.B.6

Explain to students that they are sharing a large collection of shells (e.g., 500ish) with their classmates (e.g., 26 friends). Ask: How would you efficiently give each friend the same number of sea shells? Listen for student ideas. If it doesn't come up, introduce the idea that rather than passing them out one at a time, you conservatively estimate a quantity for each friend. For example, you know you have enough to give each friend 10 shells. Ask: How many shells have you distributed? Could you fairly share more shells? How many might you give each friend on your second round?

The sea shell activity gives us good ideas for sharing equal groups in division with multiple digits. First, always underestimate how much can be shared. You can always pass out some more. To avoid overestimating, always pretend there are more sets (or people) to share with than there really are. For example, if you are dividing 312 by 43 (sharing among 43 sets or friends), pretend you have 50 instead. Round *up* to the next multiple of 10. You can easily determine that 6 pieces can be shared among 50 sets, because 6×50 is a familiar product. Therefore, because there are really only 43 sets, you can reason that giving *at least* 6 to each is a good start.

Traditionally, if a division problem such as $583 \div 4$ was posed, we might hear someone say, "4 goes into 5 one time." Not surprisingly, initially this approach may be quite mysterious to students. How can you just ignore the "83" and keep changing the problem? Preferably, you want students to think of the 583 as 5 hundreds, 8 tens, and 3 ones, not as the independent digits 5, 8, and 3. In any case, without a context, this type of discussion makes little sense. Use realistic contexts that lend to sharing. Consider the following problem:

We have 5 cartons, 8 boxes, and 3 energy bars to share evenly among 4 schools. **The energy bars are bundled in boxes of ten with 10 boxes (100 bars) to a carton.**

In this context, it is reasonable to share the cartons first until no more can be shared. Those remaining cartons are unpacked, and the boxes shared and so on.

Pause & Reflect

Try the distributing or sharing process yourself, using base-ten materials, four paper plates, and the problem $583 \div 4$. Talk through the process without using goes-into language. Think about sharing equally by place value.

Language plays an enormous role in supporting conceptual thinking about the standard division algorithm. Most adults are so accustomed to the goes-into language that it is hard for them to let it go. For the problem $583 \div 4$, here is some suggested language:

- I want to share 5 hundreds, 8 tens, and 3 ones among these four schools. There are enough hundreds for each school to get 1 hundred. That leaves 1 hundred that I can't share in its current form.
- I'll trade this remaining hundred for 10 tens. That gives me a total of 18 tens. I can give each school 4 tens and have 2 tens left over. Two tens are not enough to go around to the four schools in their current form.
- I'll trade the 2 tens for 20 ones and combine them with the 3 ones I already had for a total of 23 ones. I can give 5 ones to each of the four schools. That leaves me with 3 ones as a remainder. In all, I gave out to each school 1 hundred, 4 tens, and 5 ones with 3 left over. ●

Activity 12.7 Left Overs

CCSS-M: 4.NBT.B.6

Use the Left Over game board; have students work in pairs and place their game pieces at the start of the path. The first player spins the Left Over spinner. The amount on the spinner is used as the divisor for the number on the start square. If there are leftovers after the division, the player moves the number of spaces that equals the remainder. If there is no remainder, the player stays in the same position. The winner is the first to reach the end of the path. You can differentiate the game by using easier (or harder) divisors (see blank spinners on the Left Over Spinner activity page).

Pearson eText Activity Page: Left Over Game Board
Pearson eText Activity Page: Left Over Spinner

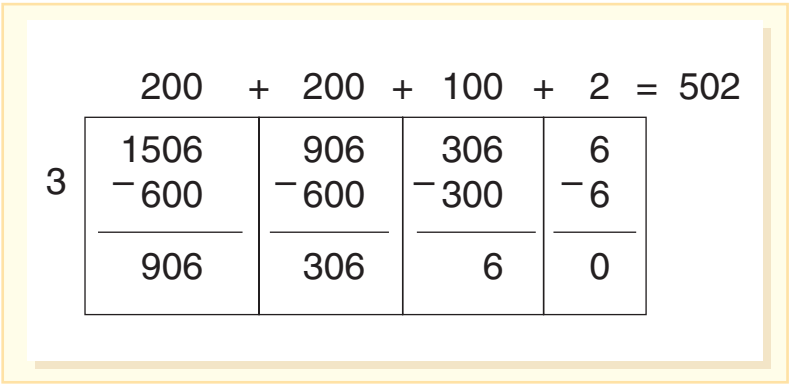


FIGURE 12.15 Using a bar diagram to show partial quotients

Semi-concrete

You can look at division with an eye to partial quotients by using a version of a bar diagram model blended with the repeated subtraction approach. Look at the problem 1506 ÷ 3. If we use multiplication facts we know as a guide, then we can repeatedly subtract partial products and record them through an approach that uses measurement division (see Figure 12.15).

Moving to the Abstract

The recording scheme for the standard division algorithm is not completely intuitive. You will need to be explicit in helping students learn to record the action of fair sharing with models symbolically. There are essentially four steps:

1. *Share* and record the number of pieces put in each group.
2. *Record* the number of pieces shared in all. Multiply to find this number.
3. *Record* the number of pieces remaining. Subtract to find this number.
4. *Trade* (if necessary) for smaller pieces and combine with any of the same-sized pieces that are there already. Record the new total number in the next column. The final subtraction step finds the remainder.

When students initially model problems with a one-digit divisor, steps 2 and 3 seem unnecessary. Explain that these steps really help when you don't have the pieces there to count.

Explicit-Trade Method. Figure 12.16 details each step of the recording process just described. On the left is the standard algorithm. To the right is an explicit-trade method that matches the actual action with the models by explicitly recording the trades. Instead of the traditional bring-down step of the standard algorithm, traded pieces are crossed out, as is the number of existing pieces in the next column. The combined number of pieces is written in this column, using, in this case, a two-digit number. In the example, 2 hundreds are traded for 20 tens, combined with the 6 that were there for a total of 26 tens. The 26 is, therefore, written in the tens column. Be sensitive to the fact that the notation for the traditional division algorithm differs by different cultures (Ortiz-Laso & Diego-Mantecón, 2020).

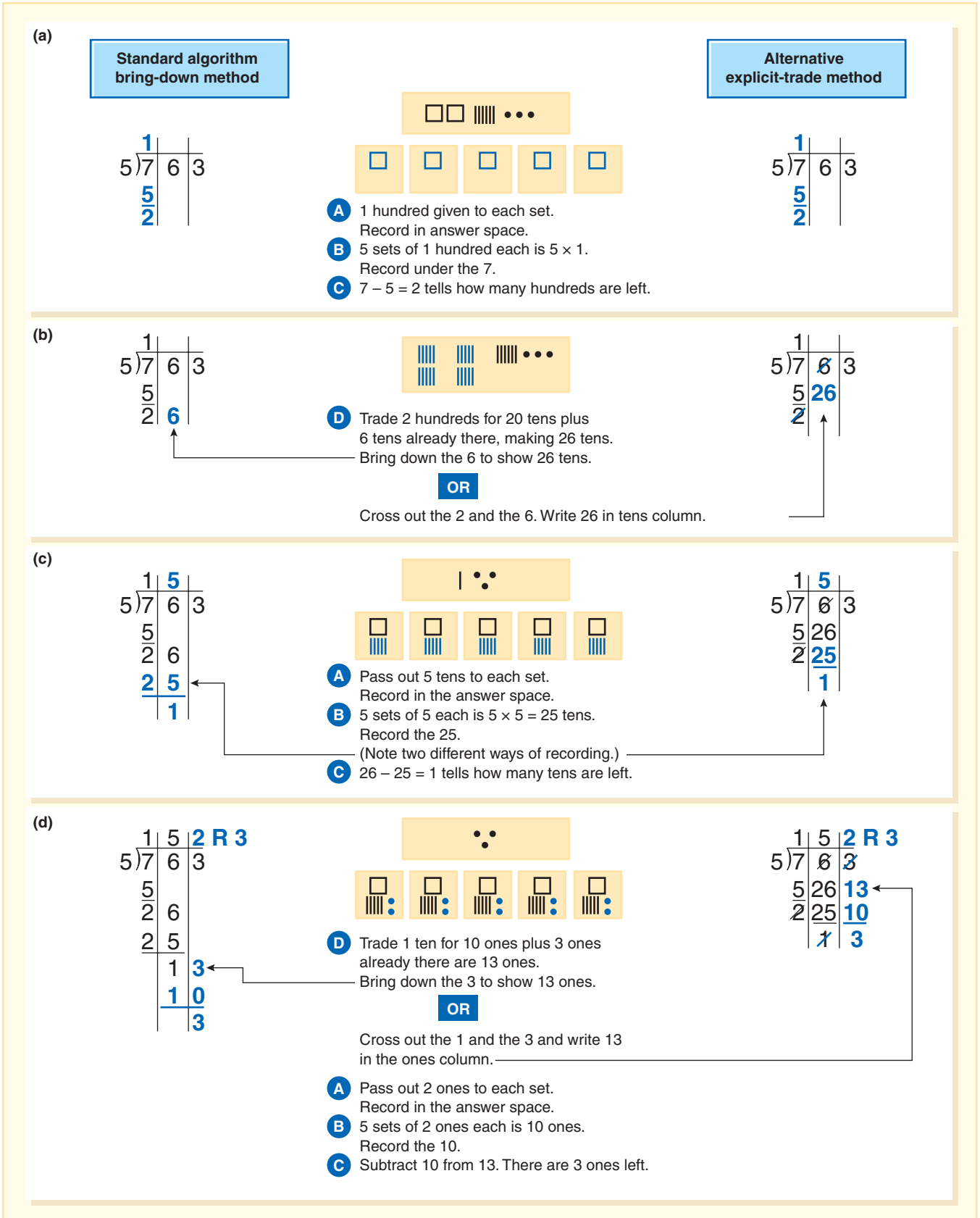

(a)
Standard algorithm bring-down method
Alternative explicit-trade method
1
5)7 6 3
5
2
A 1 hundred given to each set.
Record in answer space.
B 5 sets of 1 hundred each is 5 × 1.
Record under the 7.
C 7 – 5 = 2 tells how many hundreds are left.
(b)
6
26
D Trade 2 hundreds for 20 tens plus
6 tens already there, making 26 tens.
Bring down the 6 to show 26 tens.
OR
Cross out the 2 and the 6. Write 26 in tens column.
(c)
1 5
2 6
2 5
1
A Pass out 5 tens to each set.
Record in the answer space.
B 5 sets of 5 each is 5 × 5 = 25 tens.
Record the 25.
(Note two different ways of recording.)
C 26 – 25 = 1 tells how many tens are left.
25
(d)
1 5 2 R 3
1 3
1 0
3
13
10
D Trade 1 ten for 10 ones plus 3 ones
already there are 13 ones.
Bring down the 3 to show 13 ones.
OR
Cross out the 1 and the 3 and write 13
in the ones column.
A Pass out 2 ones to each set.
Record in the answer space.
B 5 sets of 2 ones each is 10 ones.
Record the 10.
C Subtract 10 from 13. There are 3 ones left.

FIGURE 12.16 The standard algorithm and explicit-trade method are connected to each step of the division process

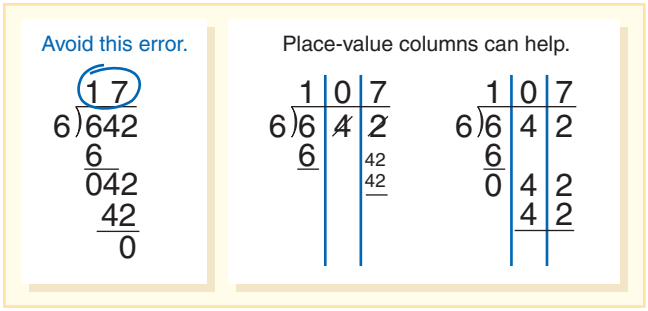


FIGURE 12.17 Using lines to mark place-value columns helps students remember to record internal zeros

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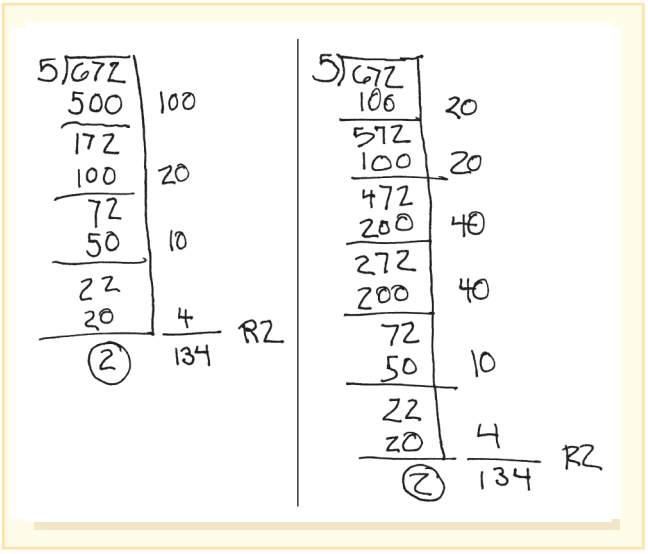


FIGURE 12.18 The numbers on the right side of the column indicate the number of groups of the divisor being subtracted from the dividend

Students often find this *explicit-trade method* easier to follow. (The explicit-trade method is a successful approach developed by John Van de Walle and tested with students in grades 3 to 8. You will not find it in other textbooks.) Blank Multiplication and Division Recording charts with wide place-value columns are highly recommended for this method. By spreading out the digits in the dividend when writing down the problem, you help students avoid the common problem of leaving out an internal zero in a problem (see Figure 12.17).

Pearson eText Blackline Master: Multiplication and Division Recording Charts

Repeated-Subtraction Algorithm. A well-known way to record division is an algorithm based on repeated subtraction where the divisor is subtracted from the dividend in groups of any amount. This sequential approach records the missing factor, with partial products written in the column to the right of the division computation (see Figure 12.18). Students may prefer this approach, especially students who bring this approach from other countries and students with special needs in math who can select facts they already know and work from that point. This method is also useful with multi-digit divisors.

Two-Digit Divisors

Commonly during fifth grade, students should explore whole-number division with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and the inverse relationship between multiplication and division. Many states suggest students should "[i]llustrate and explain the calculation by using equations, rectangular arrays, and/or area models" (NGA Center & CCSSO, 2010, p. 35). This topic is particularly important, because knowledge and understanding of division with a two-digit divisor is a predictor of mathematical success in high school (Siegler et al., 2012).

Recording the Ideas Symbolically. Using these ideas about reasoning with division situations with two-digit divisors, both the standard algorithm and the explicit-trade method of recording are illustrated in Figure 12.19. The rounded-up divisor, 70, is written in a little think bubble. Then use 70 to estimate, but multiply what you share (in this case 5 tens) by 63. Next, in the ones column, share 8 more with each set. Oops! 88 left over, so just give 1 more to each set. This is a good moment to remind students that the remainder cannot be larger than the divisor. You will notice the rounding up has an advantage to get students started, because it is easier to use the multiples of 70 than those of 63 and compare them to 374. For students who need additional concrete support, return to sharing base-ten materials (thousands, hundreds, tens, and ones). Work through the problem one step at a time, using a think-aloud as you say exactly what each recorded step represents.

This approach has proved successful with students who are learning division for the first time and students in middle school in need of additional experiences. It reduces the mental load of making choices and essentially eliminates the need to erase. If an estimate is too low, that's okay. And if you always round up, the estimate will never be too high. The same is true of the explicit-trade notation.

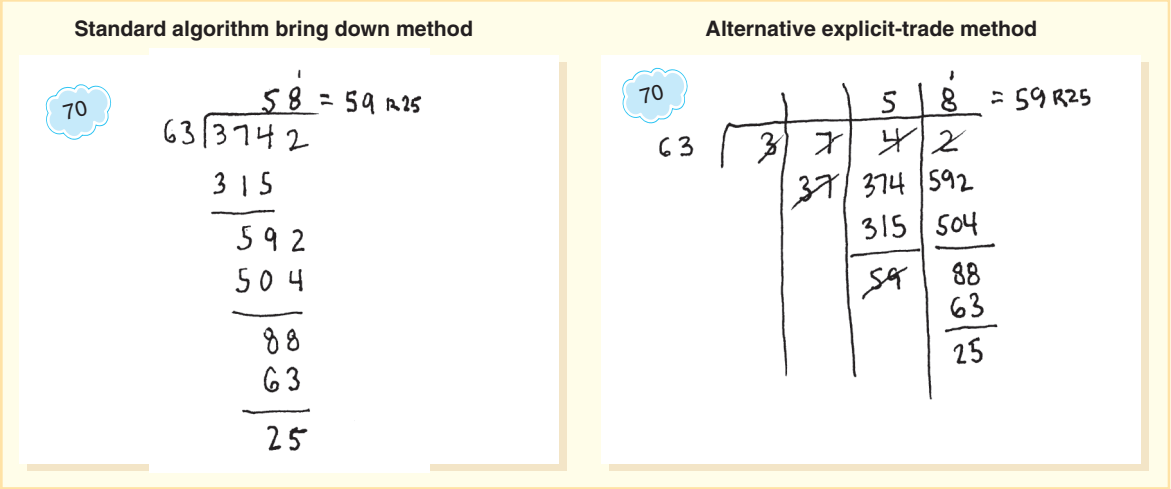


FIGURE 12.19 Recording for the standard algorithm and explicit trade method

Connect to Place Value and Multiplication. With a two-digit divisor, it can be a challenge to estimate the right amount to share at each step. First, start by having students think about place value by asking, "Will the answer be in the thousands? Hundreds? Tens?" To decide, multiply the divisor by 10s, 100s, or 1000s. So, for the problem $3842 \div 14$, decide first whether the answer is in the thousands, hundreds, or tens. This may be done mentally or in writing: $14 \times 10 = 140$ $14 \times 100 = 1400$ $14 \times 1000 = 14000$. By electing hundreds in this case, the focus becomes how many hundreds? One idea is to create a doubling sidebar chart that starts with a benchmark multiple of the divisor and then doubles each subsequent product (Martin, 2009). For this problem, students will develop a chart of 100, 200, 400, and 800 times 14 (see Figure 12.20). Can you see how knowing 100×14 can help you figure out 200 and other products? Using a doubling sidebar chart can help students find the products of the divisor multiplied by 100 through 900 by adding products in combinations. For example, to know 300×14, add the products of 100 times and 200 times the divisor or subtract 100×14 from 400×14. Knowing these products will logically help the student also know what 10 through 90 times the divisor is! This scaffolding process builds number sense and helps students estimate more successfully while providing an accessible division process.

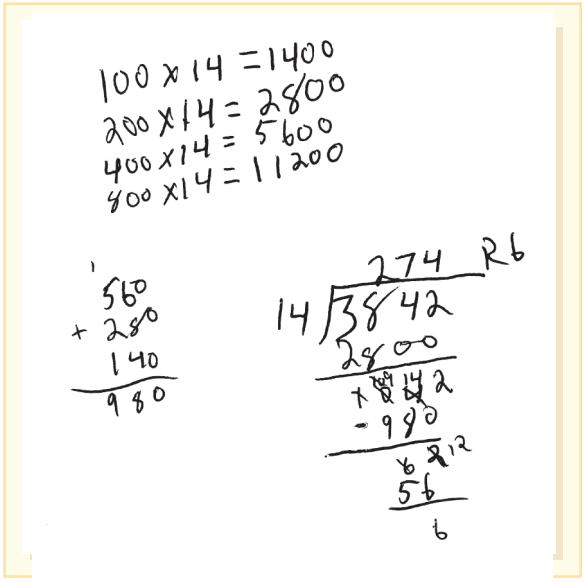


FIGURE 12.20 A student uses doubling to generate useful estimates in a sidebar chart

Activity 12.8 Double, Double—No Toil or Trouble!

CCSS-M: 5.NBT.B.6

Select a division problem with a two-digit divisor, such as $936 \div 18$. Display a corresponding sidebar chart (Martin, 2009) (or give students a Sidebar Chart template) of the products of the divisor (in this case, 10, 20, 40, and 80×18). Then students think about the division, using a think-multiplication approach to identify the missing factor and repeated subtraction, using the chart to support the estimation process. When working with students with special needs in math, you may want to start in a more structured way by first supplying them with the appropriate sidebar chart with the products filled in. Then, on the next day, supply them with the chart and have them fill in the products, moving toward their independent creation and completion of the chart. This transition moves students in an organized and systematic way to working independently.

Pearson eText Blackline Master: Sidebar Chart

Research by Keiser (2012) analyzed middle school students' strategies for solving two division problems with two-digit divisors. Only four of 91 students used the standard algorithm, and of those four students, only two reached the correct answer. Largely, students relied on repeated subtraction or the think-multiplication strategy. Errors also occurred, For example, students solving $95 \div 16$ broke it into $(95 \div 10) + (95 \div 6)$, demonstrating a common confusion. Students incorrectly thought that when using the distributive property, you can break apart the divisor, when instead you can split the dividend into components. This points to the fact that just teaching the algorithm is not the answer to helping all students do division well.

FORMATIVE ASSESSMENT Notes. Conduct a diagnostic interview to explore students' level of understanding of the division algorithm in more detail. Use an observation look-for tool to record students' responses. Begin by having the student complete $115 \div 9$ and ask them to discuss verbally the connections between what was done with the models and what was written symbolically. If there is difficulty explaining, have the student use base-ten materials to model the problem directly and attempt to link the actions to the procedure. ■

Pearson eText Teacher Resource: Observation Look-for Tools

Computational Estimation

Learning Outcome 12.5 **Identify ways to teach computational estimation for multi-digit multiplication and division as a way to develop students' flexibility and ability to recognize reasonable answers.**

Computational estimation is "determining an approximate but satisfactory solution to arithmetic computations with the use of mental mathematics" (Desli & Lioliou, 2020, n.p.). Particularly in multiplicative situations, computational estimation is useful in daily events such as calculating a tip or figuring out miles per gallon of gasoline. As students move to a more technological world where calculations are frequently carried out by devices, never before has judging reasonableness been more important. Those who can estimate well will rarely be misled by a mistake in using a calculator and they can check answers to problems for reasonableness!

Computational estimation skills in multiplication and division round out a full development of flexible and fluent thinking with whole numbers that build students' number sense and reasoning skills. Many states' standards expect that fourth graders should use estimation strategies to judge reasonableness, including the use of rounding. But at all grades, estimation is central to the development of numbers sense. In particular, when students are using standard algorithms that are less intuitive and more error-prone, they need to have methods to determine whether their answer is reasonable.

Teaching Computational Estimation

What estimate would you give for 27×325? If you use 20×300, you might say 6000. Or, you might use 25 for the 27, noting that four 25s make 100. Because $325 \div 4$ is about 81, that would make an estimate of 8100. If you use 30×300, your estimate is 9000, and 30×320 gives an estimate of 9600. Is one of these right or best? No. They each give a value that gives an idea of the magnitude of the answer. Yet sometimes, one method is a better fit for the numbers in the problem.

Students often feel less comfortable estimating than they do computing. You must encourage students to think flexibly about their approach and not worry about getting the closest estimate. With time and experience, students will get better at what estimation strategy best suits the situation at hand. In contrast, if you just tell students to use a given strategy (e.g.,

round each number to one significant digit and multiply), they won't develop the skills to pick different strategies purposefully for different circumstances.

Scaffolding Estimation. Because students are uncomfortable with estimation, they may try to calculate an exact answer quickly and then round the answer (Hanson & Hogan, 2000). A good way to begin is to ask over/under type questions. Compare how a student might respond to these two prompts:

1. Estimate: $7 \times \$87$.
2. Is $7 \times \$87$. Over or under 1000? Over or under 700? Over or under 400?

To support student reasoning further when asking over/under, have students pair-share and explain how they know. You can add a context to support the reasoning, such as the cost of tickets. Will \$400 be enough for 7 tickets? 700? 1000?

A second option is to ask over/under questions. In this case, you begin by asking, "What number are you sure is less than the answer? What number are you sure is greater than the answer?" For example, for $4{,}531 \div 25$, a student might say, "I know the answer is more than 100 because $25 \times 100 = 2500$. And, I know the answer is less than 200 because 25×200 equals 5000."

A third option is to pose different ranges and ask students to pick a range. Students choose whether they think the answer is between \$100 and \$400, \$500 and \$800, or \$900 and \$1200. Narrow your ranges as students become more adept.

The next two activities provide a format for computational estimation in which a specific numeric response is not required.

Activity 12.9 High or Low?

CCSS-M: 4.NBT.A.3; 4.NBT.B.5

Display a computation and three or more possible computations that might be used as an estimate. The students' task is to decide whether the estimation will be higher or lower than the actual computation.

Multiplication Example: 736×18		Division Example: $363 \div 8$	
750×10	730×15	$320 \div 8$	$400 \div 8$
700×20	750×20	$360 \div 9$	$400 \div 10$

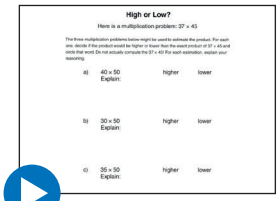
Also explore the High or Low activity page for additional problems and the Expanded Lesson: High or Low for an instructional plan for this activity.

Pearson eText Activity Page: High or Low?
Pearson eText Expanded Lesson: High or Low?

Computational Estimation Strategies

Just like with computation strategies, there are many ways to estimate. Earlier in this chapter you learned ways to compute mentally. In this section, we describe ways to estimate mentally. Mental calculations using estimations are more complex than just the application of a procedure in that they require a deep knowledge of how numbers work (Hartnett, 2007).

Estimation strategies are specific approaches that produce approximate results. As you work through the strategies in this section, you may recognize many of the same approaches students developed from their reasoning methods. Table 12.3 describes the three most useful, generalizable computational estimation strategies. Estimation strategies, like reasoning strategies, are meant to be flexible, and their usefulness depends on the numbers in the problem. When estimating, avoid presenting division problems by using the computational form $7\overline{)3482}$, because this format tempts students to just solve the problem and not estimate!



Pearson eText
Video Example 12.8
In this video consider how the student problems on the High or Low? multiplication activity page.

TABLE 12.3 COMPUTATIONAL ESTIMATION STRATEGIES

Estimation Strategy	Description	Examples
Front End	This method focuses on the leading, or left-most, digit(s) in a number (ignoring the rest of the number as if the remaining digits are all zeros). After this estimate is made, an option to be more accurate is to look at part of the number that was ignored and adjust the estimate. This method is one of the easiest for students to learn (Star & Rittle-Johnson, 2009).	• **446 × 7:** 400 × 7, or 2800. Adjusting could lead to raising the estimate to 3000. • **452 × 23:** 400 × 20, or 8000. Adjusting might raise estimate to 9000. • **3482 ÷ 7**: 3400 ÷ 7 (note that the front end must be the first two digits). The closest 7s multiple is 3500, so the front-end estimate is about 500.
Rounding	Rounding is explained in Chapter 10. As an estimation strategy, however, you do not need to follow rounding rules. The goal is to think flexibly about rounding to get a reasonable estimate. You can round one or both numbers, and you do not always round to the nearest. For example, for 34 × 64, rather than round both down, you can round one up and one down to get a more accurate estimate. With division, rounding goes to a compatible number, so division is addressed in the next strategy. This method is the most commonly used strategy, with 1 out of 3 students and 1 out of 2 adults choosing it (Desli & Lioliou, 2020, np).	• **13 × 37** (Figure 12.21(a)): 10 × 37. If one factor can be rounded to 10, 100, or 1000, the resulting product is easy to determine without adjusting the other factor. • **7 × 485** (Figure 12.21(b)): 7 × 500. If one factor is a single digit, you can just round the other factor. This is an over-estimate. To be more accurate, you can adjust. • **46 × 83** (Figure 12.21(c)): 50 × 80. This example lends to rounding both numbers. Rounding rules result in one number rounding up and one rounding down. • **85 × 28:** 80 × 30. Rather than round both up, 28 is rounded up because it is very close to 30; 85 is rounded down to get a more accurate estimate.
Compatible Numbers	This method also refers to changing either or both numbers to ones that would be easier to compute mentally. With multiplication, this can look like rounding, but you may round to a number like 25. This approach is particularly effective with division because you select a dividend and divisor that fit a basic fact combination. For division, focus first on the divisor, then pick a convenient dividend.	• **413 × 24:** 400 × 25. A compatible option because 4 × 25 equals 100, so a good estimate is 10,000. • **265 ÷ 6:** 240 ÷ 6. With 6 as the divisor, look for a multiple of 6—240 or 300 are both reasonable options. • **697 ÷ 78:** 640 ÷ 80. Look at the divisor first, rounding to 80, then think what a convenient number close to 697 is. How about 640? So 80 is a reasonable estimate.

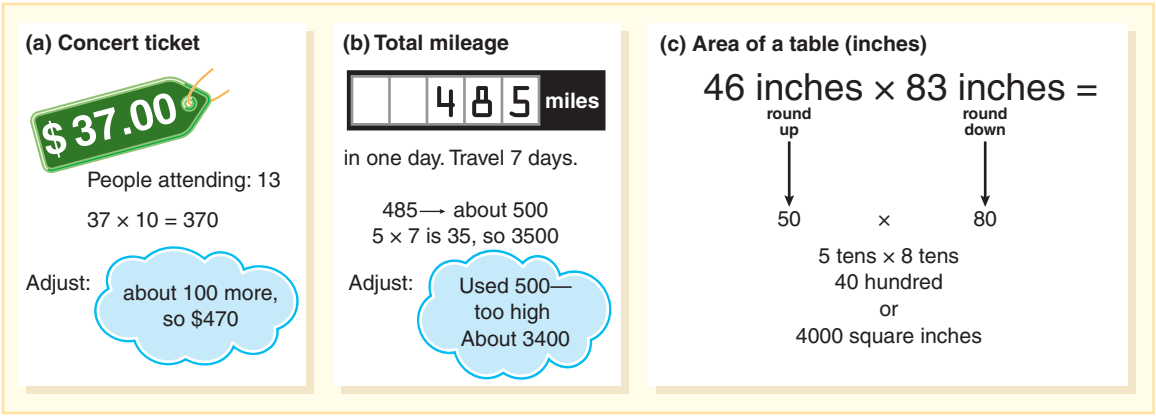


FIGURE 12.21 Rounding in multiplication

FORMATIVE ASSESSMENT Notes. One method to assess computational estimation is to prepare about three estimation exercises for display on a projection device. Show one exercise at a time, perhaps for 20 seconds, depending on the task. Students write their estimate immediately and indicate whether they think their estimate is low or high—that is, lower or higher than the exact computation. They are not to do any written computation. Continue until you are finished. Then show all the exercises and have students write down how they did each estimate. They should indicate why they think the estimate was reasonable or not. By only doing a few estimates but having the students reflect on them in this way, you receive more information than you would with just the answers to a longer list. ■

Activity 12.10 What Was Their Method?

CCSS-M: 4.NBT.A.3; 4.NBT.B.5

This is a quick routine that uses worked examples. Post one worked example. Examples: Juan's work: $139 \times 43 =$ _____ Estimate: 6000. Anetria's work: $516 \div 23 =$ _____. Estimate: 20.
Ask: What was their method?

Give some think time and then have students share. After students determine the student's strategy, ask, "Was that a good approach? Is the estimate larger or smaller than the actual answer, and how do you know? Could it be adjusted? What other options might you suggest?"

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Activity 12.11 Jump to It

CCSS-M: 4.NBT.B.5; 4.NBT.A.3

Students begin with a start number and use mental estimation to find how many times they will need to multiply that start number (estimate of jumps) to reach the goal (going over or under the amount). The numbers can be differentiated to meet the needs and experiences of your students; use the Jump to It activity page or use these to get you started:

Jump Number	Goal	Estimate of Jumps	Was Estimate Reasonable?
5	72		
11	97		
7	150		
14	135		
47	1200		

To check estimates on the calculator, students can enter 0 [+] [jump number] and press [=] once for every estimated jump, or multiply [jump number] [×] [estimate of jumps]. Students with special needs in math may need to have a number line close by to mark their goal number with a sticky dot and use another color dot to mark their first estimate. This strategy supports students in the process of deciding whether they need to increase or decrease their estimation of the number of jumps.

Pearson eText Activity Page: Jump to It

Activity 12.12 Hit the Target

CCSS-M: 4.NBT.B.5

This calculator game focuses on estimation, using multiplication. First, pick a start number. Pairs of students (or the whole class with a projection) take turns entering the start number, [×], a number of choice, and [=] to try to make the result land in the stated target range. The following multiplication example illustrates the activity:

Start Number: 17
Target: 800 – 830

If the first number tried is 25, pressing 17 [×] 25 gives 425, which is not in the target range. Then the calculator is passed to the partner, who clears the screen and picks a different number—for example, a number close to 50 because the first product was about half of the numbers in the target range. A second estimate might be 17 [×] 45, or 765, which is closer but still not in the target range. The calculator goes back to the first person. Continue to clear each estimate and try again until someone gets a product that hits the target range. Figure 12.22 gives examples of start numbers and target ranges.

Pearson eText Activity Page: Hit the Target

Multiplication:

START		TARGET
67	→	1100–1200
143	→	3500–3600
39	→	1600–1700

FIGURE 12.22 Possible starting numbers and targets for Hit the Target

Common Challenges related to Multiplication and Division Computation

Explore common student challenges in Table 12.4 to help you interpret what students may be thinking and thereby support your lesson planning.

TABLE 12.4 COMMON CHALLENGES RELATED TO MULTIPLICATION AND DIVISION COMPUTATION

Common Challenge	What It Looks Like	How to Help
1. Ignoring internal zeros in multiplication.	When given the problem: 4005 × 9 3645 The student just writes the products of 9×5 and then 9×4, ignoring the multiplications of 0 tens and 0 hundreds.	• Focus on the meaning of a zero in any number by starting with a number like 4005 and asking how that would be shown with materials. Explicitly discuss the role of a 0 in the number. • Never refer to 0 as a placeholder. This terminology gives the impression that it is not a numerical value and is there just to fill a space. • Discuss whether this answer is reasonable by thinking about 4000×10.
2. Use of an approach that mimics addition by considering the tens and ones separately.	When multiplying 28×36, the student calculates 20×30 and then adds to that the product of 8×6.	• Return to direct modeling, using an array. Work backward to links between the distributive property of multiplication over addition and the listing of the partial products. Then ask the student to show where the partial products they calculated are found in the array. What's missing?
3. Misinterpretation of the regrouped number in a multiplication problem.	When given the problem: ^{3}37 × 5 305 The student puts the regrouped 3 above the tens column and adds it to the "3" there before they multiply. So instead of multiplying 5×7 and then 5×30, the student is multiplying 5×7, and then 5×60, for an incorrect answer of 305.	• Go back to the base-ten materials and use an area model to show how the partial products are derived. • Consider using an open array model so the actual amounts the student should be multiplying are apparent. • Have the student move away from just using the algorithm until they can explain it.
4. Student breaks apart the divisor instead of the dividend (mis-applies the distributive property).	Students solving $95 \div 16$ broke the problem into $95 \div 10 + 95 \div 6$. Students thought that when using the distributive property, you could break apart the divisor, when instead, you can decompose the dividend.	• Go back to using smaller numbers and show—with paper plates to represent the groups—that the number of groups remains stable; it is the quantity of items that need to be distributed equally that can be broken up into different collections to reach the correct answer. • This challenge can be persistent so be ready for it by presenting the possible confusion early for class discussion. Don't wait for students to demonstrate this mix up.
5. The internal zero in a division problem is ignored.	When given the problem $8002 \div 2$, students will say the answer equals 41.	• Have students estimate the answer before starting: Will the answer be in the tens? Hundreds? Thousands? • Start with a smaller number such as $202 \div 2$ and model the problem with base-ten materials and two paper plates. Then link the action to the algorithm.
6. When dividing, no contextualized solution is given.	Students calculate a solution to the division accurately but do not use the remainder to adjust the final answer to match the situation.	• Have students calculate and then reveal several solutions for them to choose from that include solutions that ignore the context. Discuss.

RESOURCES FOR CHAPTER 12

WRITING TO LEARN

Assess your understanding and application of chapter content.

1. Draw sketches showing how 4×23 could be modeled in two ways, using base-ten blocks, an open array, or an area on a grid.
2. What would you do if your students seemed to persist in using repeated addition for multiplication problems without really doing any multiplication?
3. Which division concept, measurement, or partition is easier for direct modeling? Which is the one used to develop the standard multi-digit division algorithm?
4. Use the standard algorithm for $735 \div 6$ and then repeat the process, using the text's suggestion of recording trades explicitly. With the two algorithms side by side, explain every recorded number in terms of what it stands for when sharing base-ten blocks.

5. Describe each of the three estimation strategies suggested and include an example of when you might use each.
6. You notice a student is estimating by first doing the computation and then rounding the answer. Why might the student be using this strategy? What experiences might you plan to improve the student's ability to estimate?

LITERATURE CONNECTIONS

Literature often provides excellent contexts for computation and estimates, as in the following two engaging examples.

Is a Blue Whale the Biggest Thing There Is?

Wells (2005)

This intriguing book is about large objects and large distances. The data in the book allow students to make other comparisons, such as the number of fourth-grade students who would have the same weight or volume as a blue whale or would fill the gymnasium. These comparisons are perfect opportunities for estimations and discussions about how much precision is necessary to make a meaningful comparison.

How Many Guinea Pigs Can Fit on a Plane?

Overdeck (2017)

This book answers fascinating questions, with many of the situations requiring multiplication and division skills. Enjoy the possibilities when using such inquiries as: How many bees does it take to make a jar of honey? or How long would it take to run around the world?

RECOMMENDED READINGS

Articles

Benson, C., Wall, J., & Malm, C. (2013). The distributive property in grade 3? *Teaching Children Mathematics*, *19*(8), 498–506.

Don't let the title influence you; this article is not just about third graders! The focus is on linking visual representations, using area of rectangular regions for the understanding of the power of the distributive property.

Books

Lannin, J., Chval, K., Jones, D., & Dougherty, B. (2013) *Putting Essential Understanding of Multiplication and Division into Practice in Grades 3-5*. Reston, VA: NCTM.

This book explores ways to get students to reason about multiplication and division through the use of a variety of strategies.

CHAPTER

13 Algebraic Thinking

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **13.1** Identify strategies to build strong relational understanding of equivalence.
- **13.2** Describe ways to connect number and algebraic thinking, including using properties of the operations to build number sense and procedural fluency.
- **13.3** Explain the meanings of variable and how to develop these meanings conceptually.
- **13.4** Illustrate and describe how to use patterns to develop functional thinking across K–8.
- **13.5** Define mathematical modeling and describe grade-appropriate examples of it.
- **13.6** Describe ways to connect algebra to other big ideas in mathematics.

Contrary to what some people might say, algebraic thinking is essential for making mathematics useful in daily life and permeates most mathematics. Algebraic thinking is about noticing patterns and making generalizations. It begins in kindergarten and is an important part of doing mathematics at every grade. Algebraic thinking *extends* arithmetic so that students see underlying structure and properties and use these ideas to reason quantitatively. There are five big ideas within algebraic thinking (Knuth, Stephens, Blanton, & Gardiner, 2016). They are:

1. Equivalence, expressions, equations, and inequalities
2. Generalized arithmetic
3. Variable
4. Functional thinking
5. Proportional reasoning

This chapter focuses on 1–4 as they develop across grades K–8 (5 is the focus of Ch. 17). These early algebraic thinking experiences set students up for success in their future formal study of algebra and beyond (Blanton et al., 2015; Blanton, Stroud, et al., 2019).

BIG IDEAS

- The equal sign means that the quantities on both sides are the same. The = and other symbols must be understood.
- Algebraic thinking involves generalizing arithmetic.
- Properties of addition and multiplication are generalized constructs; students need to understand and use these properties to solve problems (not just identify them).

- Functional thinking starts in early elementary school and grows into one of the biggest ideas in algebra. It includes looking for patterns and relationships. Functional thinking is strengthened by using multiple representations (e.g., equations, tables, and graphs).
- Mathematical modeling is the process of using mathematics to analyze and provide insight into real-world issues, dilemmas, and other phenomena.

Equivalence, Expressions, Equations, and Inequalities

Learning Outcome 13.1 **Identify strategies to build strong relational understanding of equivalence.**

As early as kindergarten, students explore the concept of equivalence as they consider whether one group of objects has more, less, or the same quantity as another group of objects. Notice that when exploring equivalence, the two quantities are either equal, or one quantity is greater than the other quantity. This is true with numbers (e.g., 6 and 15) as well as with expressions (e.g., $4 + 11$ and $5 + 10$). While it may seem obvious that we use an equal sign when the quantities or equations are the same and an inequality sign when they are not, the meaning of these symbols are often not understood (Blanton et al., 2018). This section focuses on developing such understandings.

The Meaning of the Equal Sign

The equal sign is one of the most important symbols in elementary arithmetic, in algebra, and in all mathematics. At the same time, research dating from 1975 to the present suggests that $=$ is a very poorly understood symbol (Blanton et al., 2018; Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012; Stephens, Ellis, Blanton, & Brizuela, 2017) and rarely represented in U.S. textbooks in a way to encourage students to understand the equivalence relationship—an understanding that is critical to understanding algebra (McNeil et al., 2006).

Students may think about the equal sign in three ways, each developmental in nature (Stephens, Knuth, Blanton, Isler, Gardiner, & Marum, 2013).

1. *Operational view:* Students in this phase view the equal sign as meaning "do something."
2. *Relational —computational view:* Students understand that the equal sign symbolizes a relation between answers to two calculations, but they only see computation as the way to determine whether the two sides are equal.
3. *Relational—structural view:* Students view the equal sign as a balance and can use relationships between the two sides of the equal sign to reason about equivalence.

Does this remind you of the Continuum of Understanding from instrumental to relational (see Figure 2.2)? If so, good for you! Skemp's (1978) continuum is reflected in this developmental progression for understanding the equal sign!

Pause & Reflect

For the following equations, figure out the value of x, using reasoning. (Try to solve it without applying traditional steps of adding/subtracting the same quantity from both sides.) As you solve, think about how students with the three views of the equal sign described earlier might solve each problem.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

$$(1)\ 21 + 5 = n + 19 \qquad (2)\ 5x + 12 = 6x$$

Compare these three students' explanations for figuring out the value of n in the first equations:

> I noticed that this side of the equation has a 21 [points at the 21] and the other side has a 19 [points at the 19]. Because 19 is two less than 21, the missing number has to be two more than 5. So, the n equals 7.

I added 21 and 5 and got 26. So, n equals 26.

I added 21 and 5 and got 26. The other side also has to equal 26, so I subtracted 19 from 26 and got 7.

Each of these students' responses reflects one of the three views. Can you tell which? The first explanation represents view 3; the second, view 1; and the third student, view 2. How might a student operating with each view reason about the second equation? A student operating at view 1 may be stuck because the equation is not set up in the form to do an operation on the left side. Students operating with view 2 and 3 may respond like this:

I see I can subtract $5x$ from both sides, so then I have $12 = x$. So, x equals 1 (view 2).

The right side has one more x than the left side. And the right side has an extra 12, so that extra x must be 12. So, x equals 12 (view 3).

Standards for Mathematical Practice
MP7. Look for and make use of structure

In both examples, students in view 3 notice the structure of the equations and use that structure to reason about the answer. Unfortunately, far too many students have an operational view of the equal sign.

Pause & Reflect

In the following expression, what number do you think belongs in the box?

$$8 + 4 = \square + 5$$

How do you think students in the early grades or in middle school typically answer this question? ●

In a classic study, no more than 10 percent of students from grades 1 to 6 put the correct number (7) in the box. The more common responses were 12 and 17 (add all of the numbers) (Falkner, Levi, & Carpenter, 1999). These students are stuck in view 1, an operational understanding of the equal sign.

As early as kindergarten, students demonstrate an operational understanding of the equal sign (Blanton et al., 2018). The operational view is reinforced when equations students encounter always come in this format: $5 + 7 =$ _____ or $8 \times 45 =$ _____. Naturally, students see $=$ as signifying "and the answer is" rather than as a symbol to indicate equivalence (Carpenter et al., 2003; Kieran, 1981; McNeil & Alibali, 2005). An operational view is also the result of (1) an over-focus on answers, with no focus on generalizations, (2) not noticing properties of number and operation at play, and (3) a lack of understanding of variable notation (Carraher & Schliemann, 2007).

Instruction on the meaning of the equal sign does make a difference! Figure 13.1 displays the results of students participating in an intervention that focused on a relational view (view 3) of the equal sign.

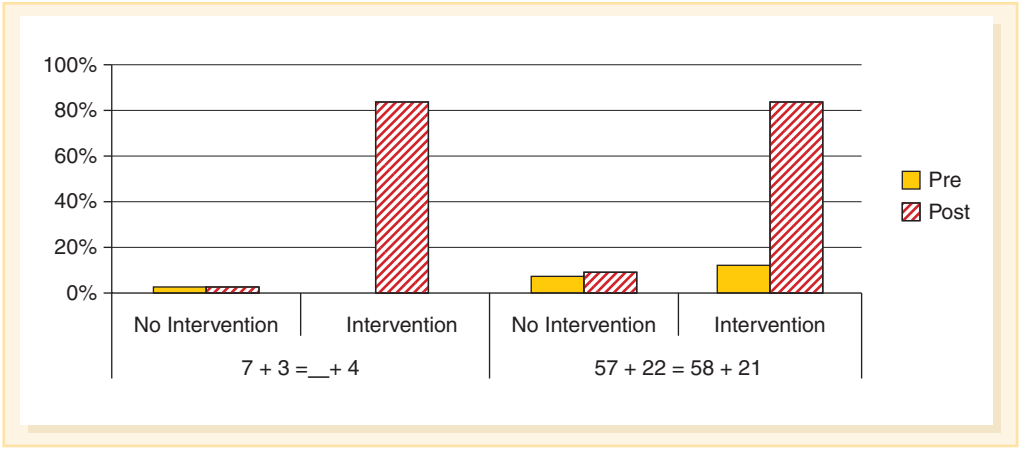


FIGURE 13.1 A third-grade intervention focused on the meaning of the equal sign

M. Blanton, S. Stephens, E. Knuth, A. M. Gardiner, I. Isler, & J. Kim (2015b). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal of Research in Mathematics Education, 46*(1), 39–87.

The Meaning of Inequalities

Inequalities are also poorly understood and have received less attention than the equal sign, likely because inequalities are not as prevalent in the curriculum or in real life. But as noted earlier, comparing quantities leads to two options: Are these two quantities equal, or is one greater than the other? Students encounter many real-life situations that are greater than or less than situations (Tent, 2000). Examples include:

- At the grocery store, the express lane is "10 items or fewer."
- At an amusement park, the sign says, "You must be at least 42 inches tall" to ride.
- At the airport, a bag must weigh 50 pounds or less (and carry-on bags have length and width limits).
- Party event places often have price structures or capacity limits such as, "For up to 10 guests."
- The teacher says, "No more than four people at the game center at a time."

To develop the meaning of inequalities, connect familiar contexts to $<$, $\leq$, $>$, and $\geq$ symbols. For example:

Ernesto has more than 6 cousins.
How many cousins might Ernesto have?

Students can share answers that are possible (e.g., 7, 9, 12) and answers that are not possible (e.g., 3, 5, 6). Students can represent the situation symbolically. In this case, $n > 6$ or $n \geq 7$ reflect the situation. In so doing, students are able to connect the meaning of equals (e.g., Ernesto has 8 cousins) to inequalities (typically, the $\leq$ and $\geq$ are not introduced until middle school). The next sections focus on activities that support a conceptual understanding of the equal sign and inequalities.

Developing the Concept of Equivalence

Helping students understand the idea of equivalence can and must be developed concretely. This includes emphasizing the notion of an equation as a balance on both sides, which can be reinforced through balancing activities, true/false questions, and writing equivalent expressions or equations.

Balance. Physical and virtual balances support student conceptual understanding of the equal sign (Bajwa & Perry, 2021; Homburg et al., 2021). For example, you can use pattern blocks and a balance. Place a yellow hexagon on one side and ask students what shapes might go on the other side to return the balance to being level: What will weigh the same as one hexagon? This experience focuses on balancing objects, which is also the focus of Activity 13.1.

After exploring the balancing of concrete objects, explore numbers. Use an image of a balance as a reminder of the meaning of the equal sign. Bajwa & Perry (2021) found that an image of a balance was actually more effective than a (virtual) moving balance at increasing student understanding of the equal sign. Activity 13.2 can be implemented at any grade, using the types of numbers and operations being explored at that grade.

Activity 13.1 Seesaw Students

CCSS-M: 1.OA.D.7; 2.NBT.A.4

Ask students to raise their arms to look like a seesaw (palms up). Explain that you are going to give some situations, and they will show with their arms how the objects will balance: You have big juicy oranges, all weighing the same, and tiny apples, all weighing the same. Examples:

- One orange in your left hand (students tilt to lower left side)
- One orange in both hands (students level off to show balance)
- Keep those oranges, plus an apple in your right hand (students tilt to lower right side)
- Put fruit down
- Two apples in each hand (balance)
- Remove one apple from left hand (students tilt to lower right side)
- An orange and an apple in left hand, two apples in right hand (students tilt to lower left side)

This is a particularly important activity for students with special needs in math, who may be challenged with the abstract idea of balancing values of expressions.

After acting out several seesaw examples, distribute the Seesaw Situations activity page. Students can cut and paste the fruits, redraw them, or write words (e.g., 2 oranges) to show ways to balance the seesaws, and one way that does not balance. Conclude the activity by writing equations (and inequalities) to match the students' responses. For example, a balance might be written as $a + a = a + a$. An inequality might be $a < o + o$. Seesaw situations can be increased in challenge by adding information such as two apples weigh the same as one orange. Then, many more options are available for considering what will balance.

Pearson eText Activity Page: Seesaw Situations

Activity 13.2 Tilt or Balance?

CCSS-M: 2.OA.A.4; 3.OA.A.4; 5.NBT.A.3a, b; 6.EE.A.4

This activity can be a daily routine (posting one or two and discussing), a full lesson (see Expanded Lesson), and/or a set of cards can placed in a learning center for ongoing opportunities to explore equivalence. Distribute or display a balance (see Balance Scales Placemat). Then share two expressions. Ask students whether, if those quantities were placed on the balance, they would tilt (one is greater than the other) or balance (they equal the same amount). When writing the two expressions, include ones that can be solved using relational understanding (see Sets A and B for examples):

Set A Examples (Whole Numbers):	**Set B Examples (Beyond Whole Numbers):**
$5 + 3$ and $3 + 5$	$\frac{1}{2}$ and $\frac{1}{8} + \frac{1}{4}$
$32 - 17$ and $31 - 18$	$\frac{3}{4} + \frac{1}{2}$ and $\frac{7}{8} + \frac{1}{2}$
7×17 and $5 \times 17 + 17$	$4(^{-}35)$ and $70(^{-}2)$
$92 \div 4$ and $80 \div 4 + 12 \div 4$	

Ask students to write either an equation ($=$) or inequality ($>$ or $<$), using the symbols and saying verbally what they mean. The key is to include examples for which students can determine the answer by analyzing the relationships on the two sides rather than by doing the computation. Visuals and manipulatives can be a support for students with special needs in math. To challenge students, use missing-value expressions. Ask students to find three values, one that balances, one that tilts down on the left side, and one that tilts down on the right side.

Pearson eText Activity Page: Tilt or Balance Equation Cards Set A
Pearson eText Activity Page: Tilt or Balance Equation Cards Set B
Pearson eText Activity Page: Balance Scales Placemat
Pearson eText Teacher Resources: Tilt or Balance

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

***TECHNOLOGY* Note.** There are some excellent virtual balance and virtual hanger activities:

- NCTM Illuminations Pan Balance—Numbers or Pan Balance—Expressions provide a virtual balance, where students can enter what they believe to be equivalent expressions (with numbers or symbols).
- SolveMe Mobiles (https://solveme.edc.org/mobiles) offers dozens of problems to solve, based on what is hanging from a mobile.
- PhET Equality Explorer (phet.colorado.edu) provides virtual pans, and students place shapes on the two sides to create equations and/or inequalities. ■

A balance is also effective in helping students understand how to solve equations involving variables. For example, Figure 13.2 shows two ways students might solve for x for the equation $4 - 6x = 3 + 3x$, using a simple sketch of a balance (as a reminder that they must preserve equivalence). A note about the final step: Students may recognize that 1 is the identity and therefore know that x must be $\frac{1}{9}$. Wonderful! Then they do not need to divide to solve for x. Recall that worked examples are effective in helping students make sense of mathematics. You can use these two examples (removing the text), ask students to fill in the text, and then compare the two options. This helps students develop flexibility. The notion of preserving balance also applies to inequalities—but what is preserved is imbalance. In other words, if one side is more than the other side and you subtract 5 from both, the one side is still more.

As students move to middle school, they continue to explore situations involving inequalities with increased complexity, like the example here:

You have $100 for purchasing gift cards for your 5 friends. You want to spend the same on each, and you will also need to spend $10 to buy a package of card-holders for the gift cards. Describe this situation with symbols.

Pause & Reflect

How might you write this inequality? How might students write it? What difficulties do you anticipate? What questions might you pose to help students build meaning for the symbols? ●

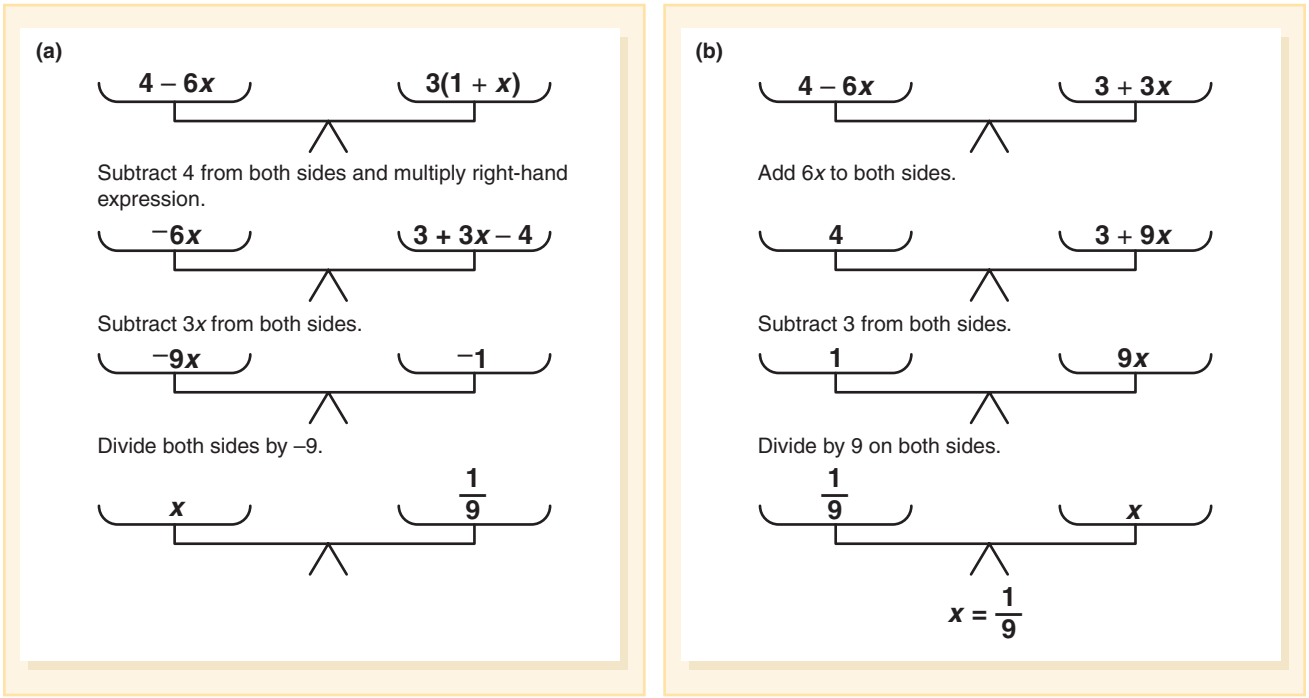


FIGURE 13.2 Using a balance scale to build understanding of solving equations

Students might record the situation in any of the following ways (using a for the amount of money for the gift):

$5a + 10 \leq 100$	$10 + 5a \leq 100$	$100 \geq 10 + 5a$	$100 \geq 5a + 10$

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

They may also make these inequalities without the equal signs: $<$ and $>$. Ask students to *compare* what it means to say "less than" or "less than or equal to." Ask students to *connect* the inequalities with the situation. For example, the first statement directly translates to "5 gift cards and $10 for a package of holders must be less than or equal to $100." The final example directly translates to "I have $100, which must be more than or the same as the cost of 5 cards and the holders."

True/False Sentences. True or false statements can help build relational understanding and address student misconceptions (Carpenter et al., 2003; Fonger et al., 2018). True-false statements can make an engaging number talk or be part of a full lesson. The visual of a balance can support student reasoning. When posting true/false sentences, just like with Tilt or Balance, include situations wherein students can determine whether they are equal by using relational–structural thinking (view 3). Reinforce that the equal sign means "is the same as" by using that language when you read the symbol. Inequalities should be explored in a similar manner.

Activity 13.3 True or False?

CCSS-M: 1.OA.B.3; 1.OA.D.7; 1.NBT.B.4; 2.NBT.B.5; 3.OA.B.5; 4.NBT.B.5; 5.NF.A.1

Introduce true/false sentences or equations with simple examples to explain what is meant by a true equation and a false equation. Then put several equations on the board, some true and some false:

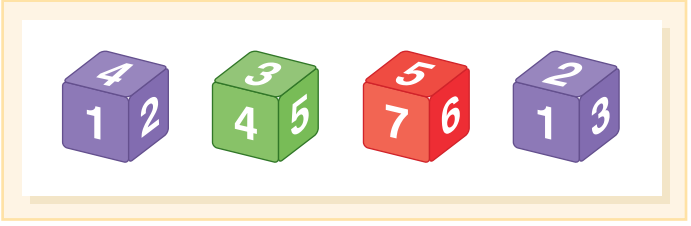
$19 + 7 = 20 + 6$	$30 - 9 = 31 - 10$	$0.4 \times 15 = 0.2 \times 30$
$345 + 95 = 100 + 340$	$345 + 95 = 100 + 340$	$1210 - 35 = 1310 - 45$
$35 = 17 + 15$	$5 \times 14 = 10 \times 7$	$^{-}35 + 45 = 35 + {}^{-}40$
$318 = 318$	$\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$	$\frac{3}{8} = 3 \times \frac{1}{8}$

Listen to the types of reasons that students use to justify their answers and plan additional equations accordingly. Invite students to write equations on the front of 3 by 5 cards, with the answer on the back, and they can exchange cards with peers. Multilingual learners and students with special needs in math will benefit from first explaining (or showing) their thinking to a partner before they share with the whole class.

Activity 13.4 Make a Statement!

CCSS-M: 1.OA.D.7; 1.OA.D.8; 2.OA.A.1; 3.OA.A.4; 4.NF.B.3a; 5.OA.A.2; 6.EE.B.8

Ask students to write their own true/false statements to challenge their classmates. This works for both equations and inequalities! To support student thinking, provide dice with numerals on them. They can turn the dice to different faces to try different possibilities.



Ask students to write three equations (or inequalities) with at least one true and at least one false sentence. For students who need additional structure, in particular students with special needs in math, provide the Make a Statement recording page, which provides options for writing equations and inequalities. Students can trade their set of statements with other students to find the false statement. Interesting equations/inequalities can be the focus of a follow-up whole-class discussion.

Pearson eText Activity Page: Make a Statement Recording Page

When students write their own true/false sentences, they often are intrigued with the idea of using large numbers and lots of numbers in their sentences. This encourages them to create sentences involving relational–structural thinking.

Write equivalent expressions or equations. Rather than always asking students to solve a problem (like $45 + 61$ or 4×26), ask students to find an equivalent expression (Blanton, 2008). For $45 + 61$, students might write $45 + 61 = 40 + 66$. For a multiplication problem, students might write $4 \times 26 = 4 \times 25 + 4$ or $4 \times 26 = 2 \times (2 \times 26)$. Activity 13.5 provides an opportunity for students to write equivalent expressions while also learning the Making 10 strategy for basic facts (based on Fosnot & Jacob, 2010).

Activity 13.5 Ten and Some More Equations

CCSS-M: 1.OA.C.6; 1.OA.D.7; 2.OA.B.2

Students create equivalent expressions by matching expressions in the form of $10 + 1$ to $10 + 8$ with other expressions they create with playing cards. Use the Equation Cards activity page as a game board, or cut them into cards, as illustrated in Figure 13.3. Give each pair of students a deck of playing cards (remove face cards, aces, and tens). Each partner draws one playing card and together they write an addition expression (e.g., $8 + 5$). If the cards drawn have a sum less than 10, they draw a third card. They decide which equation card is equivalent to their expression, using a reasoning strategy. In this case, they break apart 5 into $2 + 3$ and then determine that $8 + 5 = 10 + 3$. Focus discussion on this idea of reasoning rather than on computing, as a way to draw attention to generalizations, properties, and meaning of the equal sign.

Pearson eText Activity Page: Equation Cards

Pearson eText Activity Page: Ten and Some More Equations Recording Page

In Chapter 8, you read about the meaning of the operations. An important idea in that chapter was including story situations, where a part is unknown (not just the result). As students interpret these stories, you have an opportunity to build a connection to the equal sign and support a relational understanding. For example, think of how you might write equations for this story:

On Monday, the class counted 42 flowers in the school garden. This included 15 tulips and some daffodils. How many daffodils were in the school garden?

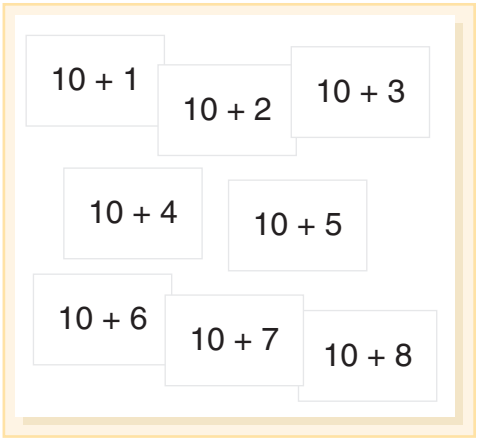


FIGURE 13.3 Ten and some more cards

An equation that follows this story is: $42 = 15 + n$ (or a blank, question mark, or box can replace the variable). Another way to support a relational understanding is regularly to write equations that represent two equivalent expressions. For example, in a number talk, students might be asked to solve $\frac{1}{4} \times 26$. A student responds: I found half of 26, which equals 13, and then found half of 13. To record this student's thinking symbolically, you can write:

$$\frac{1}{4} \times 26 = \frac{1}{2}\left(\frac{1}{2} \times 26\right)$$

Then, as you revoice, you can use conceptual, relational language: [Audrey] used the idea that one-half of one-half is one-fourth, so she found one-half of 26 and then one-half of thirteen. This teacher move supports student understanding of the equal sign *and* the relationships among numbers and forms of numbers (e.g., $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, or $0.5 \times 0.5 = 0.25$).

Looking at equivalent expressions that describe a context is an effective way to bring meaning to numbers and symbols. The classic task in Activity 13.6 involves such reasoning.

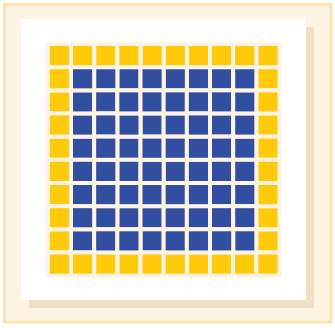


FIGURE 13.4 How many ways can you find to count the border tiles of an 8×8 pool?

Activity 13.6 Border Tiles Expressions—Part 1

CCSS-M: 5.OA.A.1; 5.OA.A.2; 6.EE.A.2b

Distribute the Border Tiles Expressions—Part 1 activity page and color tiles. Ask students to build an 8×8 square array, representing a swimming pool with a border patio (see Figure 13.4). Ask students to find at least two ways to determine the number of border tiles used without counting them one by one. For each idea, students record the numeric expression and connect the expression to the diagram. For example, they may see 10 squares across the top and across the bottom and 8 squares on either side. This thinking might be written as an equation as: $10 + 10 + 8 + 8 = 36$ or $(2 \times 10) + (2 \times 8) = 36$.

Each of the following expressions can likewise be traced to looking at the tiles in various groupings:

$$4 \times 9$$

$$4 \times 8 + 4$$

$$4 \times 10 - 4$$

$$100 - 64$$

(More expressions are possible because students may use addition instead of multiplication.) Ask students to compare the different expressions and discuss whether they are all correct (and therefore equivalent) expressions for describing the general rule.

Pearson eText Activity Page: Border Tiles Expressions—Part I

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Notice that this activity involved numeric expressions—a good place to start. This prepares students for experiences in two important areas: (1) the equal sign $(=)$ and inequality signs $(<, \leq, >, \geq)$ and (2) variables, discussed in the next two sections.

Generalized Arithmetic and Properties of the Operations

Learning Outcome 13.2 **Describe ways to connect number and algebraic thinking, including using properties of the operations to build number sense and procedural fluency.**

Algebra is often referred to as generalized arithmetic. For students to generalize an operation or pattern, they must look at several examples and notice what is happening across the problems, gaining insights into the structure of the number system (Goldenberg et al., 2015). Here we share several ways to connect arithmetic to algebra.

Number Combinations

Looking for generalizations in sets of problems can begin in kindergarten with decomposing numbers and continue through the early grades as students use strategies to add and subtract. For example, students learn basic facts by generalizing the idea of Making Tens. This strategy is later used for making tens (e.g., $29 + 29$) or making hundreds (e.g., $198 + 345$) or making wholes (e.g., $8.9 + 6.5$).

Standards for Mathematical Practice
MP7. Look for and make use of structure

Odd and even numbers provide an excellent context for exploring the structure of the number system (Stephens, Blanton, Knuth, Isler, & Gardiner, 2015). Ask students to explore different numbers, finding how many pairs and how many leftovers. Students can use manipulatives such as connecting cubes (Figure 13.5(a)) and record data in a table (Figure 13.5(b)) to generalize what makes a number even or odd.

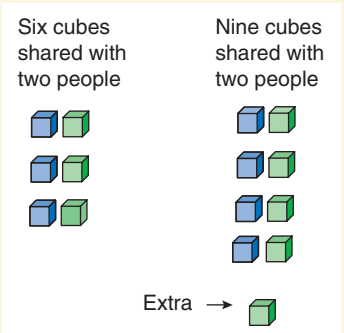
Odds and evens can also be explored using a calculator.

Pause & Reflect

Decide whether these statements are always, sometimes, or never true:

- Adding three consecutive numbers is the same as multiplying the middle number by 3.
- One-half is greater than one-fourth. ●

Slight shifts in how arithmetic problems are presented can open up opportunities for generalizations (Blanton, 2008; Mata-Pereira & da Ponte, 2017). For example, instead of a series of unrelated computation problems, consider a list that can lead to a discussion of a generalization:



(a) Sharing cubes fairly with two people

Number	Number of Pairs I can Make	Number of Leftovers
4		
5		
6		
7		
8		
9		

(b) Exploring pairs of numbers and their left-overs

FIGURE 13.5 Exploring odd and even numbers to generalize mathematical relationships

$3 \times 7 = ?$ $6 \times 7 = ?$

Activity 13.7 Broken Calculator: Can You Fix It?

CCSS-M: 1.OA.B.3; 2.OA.C.3;2.NBT.B.5; 3.OA.D.9

Distribute calculators to every student. In partners, have students select one of these two problems to explore. They must decide whether it is possible to create the even or odd number, share an example of how to do it (if it is possible), and finally prepare a justification or illustration to describe why it does or doesn't work (in general).

1. If you cannot use any of the even keys (0, 2, 4, 6, 8), can you create an even number in the calculator display? If so, how?
2. If you cannot use any of the odd keys (1, 3, 5, 7, 9), can you create an odd number in the calculator display? If so, how?

Invite early finishers to take on the other problem or write their justifications, using variables.

In discussing the relationship, students notice that 6 is twice 3, and therefore the answer is twice as much. This strategy can be applied to any of the $\times 6$ basic facts, helping students with what is often some of the more challenging facts to learn.

Number strings are good ways for students to look for and describe patterns across the problems, patterns that build an understanding for the operation, and related algorithms:

$\frac{1}{2} \times 12 =$ $\frac{1}{4} \times 12 =$ $\frac{1}{8} \times 12 =$ $\frac{3}{4} \times 12 =$ $\frac{3}{8} \times 12 =$

Once students have solved sets of related problems, focus attention on what you want students to generalize:

What do you notice across the problems?

Why is the pattern true?

When will this pattern be true?

In this set of problems, such a discussion can help students understand the relationship between the numerator and the denominator and what that means in multiplication of fractions situations.

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

Algorithms

All algorithms are generalized procedures, based on our base-10 number system. For example, the addition algorithm for whole numbers is based on the generalized idea that 10 ones *is the same as* 1 ten. Therefore, 1 ten can be traded for ten ones and vice versa. Developing algorithms conceptually is discussed in detail in other chapters. Of course, these algorithms, along with reasoning strategies, are all based on the properties of the operations, which are the most important generalizations in mathematics.

Properties of the Operations

Properties are generalized algebraic rules that support the understanding of how numbers or variables can be added or subtracted. Explicit attention to making sense of these properties helps students become more flexible and efficient in how they combine and break apart numbers.

The properties are essential to computation (Blanton, Levi, Crites, & Dougherty, 2011). Table 13.1 provides a list of the properties students must know, including how students might describe the property. In most state standards, properties of the operations are one of the only topics mentioned in each grades 1 through 8. Historically, instruction has focused on identifying which property is reflected in an equation. That level of thinking is insufficient.

TABLE 13.1 PROPERTIES OF THE NUMBER SYSTEM

Properties of the Operations		
Name of Property	**Symbolic Representation**	**How a Student Might Describe the Pattern or Structure**
Addition		
Commutative	$a + b = b + a$	"When you add two numbers in any order, you will get the same answer."
Associative	$(a + b) + c = a + (b + c)$	"When you add three numbers, you can add the first two and then add the third or add the last two numbers and then add the first number. Either way, you will get the same sum."
Additive Identity	$a + 0 = 0 + a = a$	"When you add zero to any number, you get the same number you started with."
	$a - 0 = a$	"When you subtract zero from any number, you get the number you started with."
Additive Inverse	$a - a = a + (-a) = 0$	"When you subtract a number from itself, you get zero."
Inverse Relationship of Addition and Subtraction	If $a + b = c$ then $c - b = a$ and $c - a = b$	"When you have a subtraction problem, you can think-addition by using the inverse operation."
Multiplication		
Commutative	$a \times b = b \times a$	"When you multiply two numbers in any order, you will get the same answer."
Associative	$(a \times b) \times c = a \times (b \times c)$	"When you multiply three numbers, you can multiply the first two and then multiply the answer by the third or multiply the last two numbers and then multiply the answer by the first number. Either way, you will get the same product."
Multiplicative Identity	$a \times 1 = 1 \times a = a$	"When you multiply one by any number, you get the same number you started with."
Multiplicative Inverse	$a \times \frac{1}{a} = \frac{1}{a} \times a = 1$	"When you multiply a number by its reciprocal, you will get one."
Inverse Relationship of Multiplication and Division	If $a \times b = c$ then $c \div b = a$ and $c \div a = b$	"When you have a division problem, you can think-multiplication by using the inverse operation."
Distributive (Multiplication over Addition)	$a \times (b + c) = a \times b + a \times c$	"When you multiply two numbers, you can split one number into two parts (7 can be $2 + 5$), multiply each part by the other number, and then add them together."

Instead, your instruction should build upon identification to focus on using and applying the properties. In the following paragraphs, we will discuss how to identify, use, and apply the properties of the operations.

The Commutative Property for Addition. Although the commutative property may seem obvious to us (simply reverse the two piles of counters on the part-part-whole mat), it is not initially apparent to young students. This property is essential in solving problems such as counting on from the larger number, learning basic facts (if you know $3 + 9$, you also know $9 + 3$), and doing mental mathematics (Baroody, Wilkins, & Tiilikainen, 2003; Goldenberg et al., 2015). The following problems present two contexts, but the addends are the same.

Tania is on page 32 in her book. Tomorrow, she hopes to read 15 more pages. What page will she be on if she reads that many pages?

The milk tray in the cafeteria had 15 cartons. The delivery person filled the tray with 32 more milk cartons. How many milk cartons are now on the tray?

After students solve both tasks, ask them to share what they notice about the two problems. Compare the situations and the equations. Ask them whether the situations could be worded in a different way to change how they would write the equations. Such situations help students understand the commutative property. Similarly, pairing visuals with equations helps students recognize the commutative property (Billings, 2017). The three explorations in Activity 13.8 offer three contexts for exploring number combinations in which students will notice patterns, such as noticing that if $2 + 5$ is a solution, so is $5 + 2$.

Activity 13.8 Birds, Turtles, and Boats

CCSS-M: 1.NBT.C.4; 2.NBT.B.9; 3.OA.D.9

Select a context, such as birds, frogs, boats, or any countable object that is of interest and familiar to your students. For example, consider these contexts:

There are 5 birds in the backyard; some are in a tree, and some are at a feeder. How many birds might be in the tree and at the feeder? Find all the ways (see the Birds in the Backyard activity page and Figure 13.4.)

Seven turtles are on two logs on the side of the creek. How many could be on each log? (See Turtles on Two Logs activity page.)

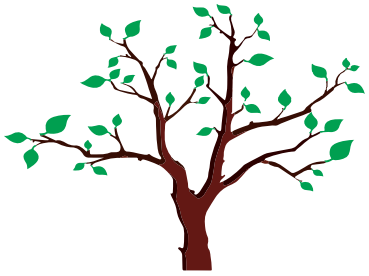
The dock has six fishing boats. Some boats are at the dock, and some boats are at sea. What options are possible? (See Fishing Boats activity page.)

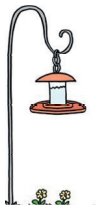
Pearson eText Activity Page: Birds in the Backyard
Pearson eText Activity Page: Turtles on Two Logs
Pearson eText Activity Page: Fishing Boats

Provide counters so that students can model the situations. They may want to use ten frames or the artwork at the top of the activity page. In each of these situations, students are finding combinations of 5 or 6 or 7 to generalize ideas. Ask questions such as, "What do you notice about the ways birds can be in the tree or at the feeder?" Students may notice, for example, that there could be 2 in the tree and 5 at the feeder or the reverse (the commutative property). They may also notice that if there are 5 objects, there are 6 ways; if there are 7 objects, there are 8 ways, and so on.

Name _____ Date _____

Birds in the Backyard





1. There are 5 birds in the backyard. How many birds might be in the tree and at the feeder? Record a different way in each column. Find all the ways. [There may be extra columns]

Tree												
Feeder												

2. Now there are 7 birds. How many birds might be in the tree and at the feeder? Find all the ways. [There may be extra columns]

Tree												
Feeder												

3. Today there are 10 birds in the backyard. How many birds might be in the tree and at the feeder? Find all the ways. [There may be extra columns]

Tree												
Feeder												

4. What patterns do you notice about the possible ways to have birds in the backyard?

FIGURE 13.6 Birds in the Backyard activity page

In this first-grade vignette, the teacher is helping students reason about the commutative property.

Teacher: [*Pointing at* $5 + 3 = 3 + 5$ *on the board.*] Is this equation true or false?
Carmen: True, because $5 + 3$ is 8 and $3 + 5$ is 8.
Andy: There is a 5 on both sides and a 3 on both sides and nothing else.
Teacher: [*Writing* $6 + 9 = 9 + 6$ *on the board.*] True or false?
Class: True! It's the same!
Teacher: [*Writing* $25 + 48 = 48 + 25$ *on the board.*] True or false?
Children: True!
Teacher: Who can describe what is going on with these examples?
Rene: If you have the same numbers on each side, you get the same thing.
Teacher: Does it matter what numbers I use?
Class: No.
Teacher: [*Writing* $a + 7 = 7 + a$ *on the board.*] What is a?
Michael: It can be any number because it's on both sides.
Teacher: [*Writing* $a + b = b + a$ *on the board.*] What are a and b?
Children: Any number!

Notice how the teacher is developing the commutative property of addition in a conceptual manner—focusing on generalizing, not on memorizing. After developing an understanding of this property, use its correct name. Nicknames such as "flip flop" can later confuse students (Karp, Bush, & Dougherty, 2016).

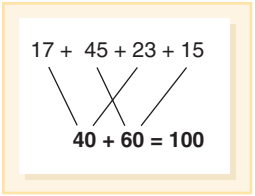
The Associative Property for Addition. It is actually this property in action when using break apart to add numbers, for example, applying the Near-Doubles strategy for basic facts.

$$\begin{aligned} 6 + 7 &= 6 + (6 + 1) \\ &= (6 + 6) + 1 \\ &= 12 + 1 \\ &= 13 \end{aligned}$$

Where the commutative property involves ordering of the addends (or the factors), the associative property involves ordering of operations (Barnett & Ding, 2019). Together, these properties allow for a great deal of flexibility when adding more than two numbers.

$$\begin{aligned} 17 + 45 + 23 + 15 &= 17 + 23 + 45 + 15 \,[\text{Commutative Property}] \\ &= (17 + 23) + (45 + 15) \,[\text{Associative Property}] \\ &= 40 + 60 \\ &= 100 \end{aligned}$$

While this series of equivalent expressions illustrates the properties clearly, students often record their thinking this way:



Commutative Property of Multiplication. The commutative property of multiplication is not as easy for students to notice as it is for addition. That is because 3 groups of 8 objects (3×8) look different from eight groups of 3 objects (8×3). However, when arrays or areas are turned 90°, they can effectively illustrate that the product is the same (see Figure 13.7(a)). Like with addition, understanding commutativity reduces the number of basic facts to be learned in half. Hence, playing games or engaging in activities that use arrays can build students' understanding of commutativity and support basic fact fluency.

Number strings can be used to focus on the commutative property, as illustrated in these two sets of problems:

Set 1:	35 $\times 52$	52 $\times 35$	23 $\times 46$	46 $\times 23$
Set 2:	$\frac{1}{6} \times 12 =$	$12 \times \frac{1}{6} =$	$\frac{2}{3} \times 12 =$	$12 \times \frac{2}{3} =$

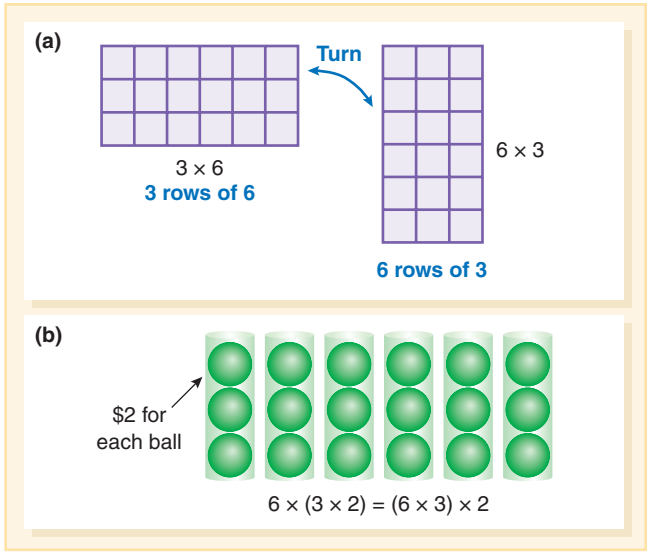


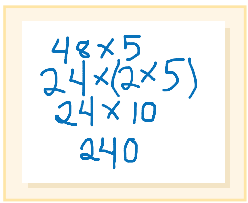
FIGURE 13.7 A model for the commutative property for multiplication (a), and an illustration of a problem, showing the associative property of multiplication (b)

Associative Property of Multiplication. Like addition, this property is put to use to break apart factors, making it very useful in applying strategies to multiply. Contexts and visuals can help students see how associating factors differently preserves the total quantity (see Figure 13.7(b)).

Each tennis ball costs $2. A can has 3 tennis balls. How much will 6 cans cost? The situation can be thought of (and thereby associated) two ways (Ding, 2010):

1. **First find out the cost for each can and then find the cost of 6 cans: $6 \times (3 \times 2)$**
2. **First find out how many balls are in 6 cans and then find the cost of each ball: $(6 \times 3) \times 2$**

The associative property is at the heart of flexible ways to solve multiplication problems (Barnett & Ding, 2019), as can be seen in the doubling and halving strategy:



Student says, "I broke apart 48 into 24 times 2, and then I could multiply 2 times 5 to get a 10. I know 24 times 10 equals 240."

To help make the associative and commutative properties visible and useful, use:

- Concrete situations and visuals to justify how the property works.
- Parentheses to help students focus on the order.
- Precise language, such as for $(2 \times 4) \times 5$, say, "Multiply two by four and then multiply by five."
- Counterexamples, such as $(4 \times 6) \times 2$ and $4 + (6 \times 2)$.

Activity 13.2, Tilt or Balance, and Activity 13.3, True/False, effectively address these important ideas.

Zero and Identity Properties. Factors of 0 and, to less extent, 1 can cause conceptual challenges for students. For example, students may confuse what happens when adding zero and what happens when multiplying by 0. Contexts can build an understanding. Here are some examples of stories involving 0.

There are 7 turtles on the large log and 0 on the small log. How many turtles are on two logs (see Activity 13.8)?

How many grams of fat are there in 7 servings of celery with 0 grams of fat in each serving?

Blueberries cost $2.50, and Sydni bought 0 containers at the grocery store. How much did Sydni spend?

Additive Inverse. This property becomes very important when students begin solving equations with unknowns. Early experiences with additive inverses can build a conceptual foundation, such as in Activity 13.9.

Activity 13.9 Five Ways to Zero

CCSS-M: 1.OA.B.3; 1.OA.C.6; 1.OA.D.7; 2.OA.B.2; 3.OA.D.9

Place students in partners. Give each pair a start number (you can use a deck of cards). If they get a 7, they are to write 5 ways to get to 0, using number sentences with that number in the sentence. For example: $7 - 5 - 2 = 0$ or $7 + 3 - 10 = 0$. Be sure students are using correct notation and grouping so that their statements are true. Use counters or a number line to explore and illustrate the possibilities. Ask students to describe what they notice about their equations, looking for generalizable ideas such as inverse relationships, additive identity, and the commutative property of addition. Restating these ideas and pair-shares provide support for multilingual learners.

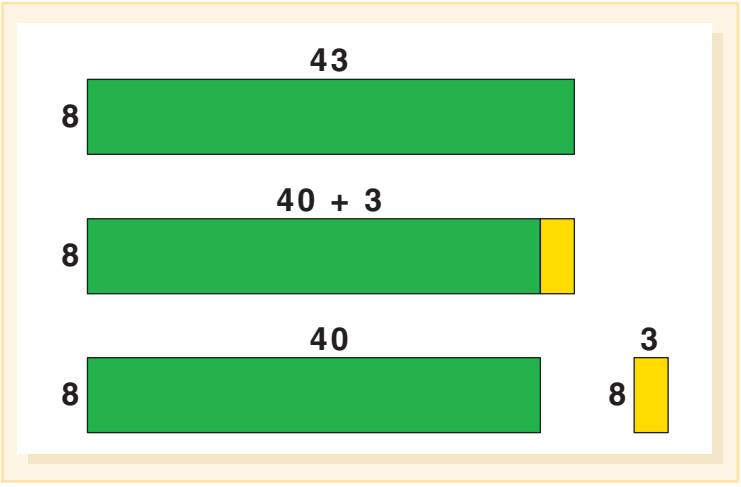
Pearson eText Activity Page: Five Ways to Zero Recording Page

Distributive Property. This property is one of the most powerful ideas in mathematics (Goldenberg, Mark, & Cuoco, 2010). It first appears when learning multiplication facts, as students implement strategies such as Add-a-Group, where they *decompose* one factor into two parts (for which they know the product). For example, a student who knows their 5s facts and 2s facts can apply the distributive property to solve 6×7:

$$6 \times 7 = 6(5 + 2) = 6 \times 5 + 6 \times 2$$

Next, students explore 2-digit by 1-digit multiplication. Again, they can break apart either factor, but they now have the option of breaking apart by place value. For example, 8×43 can be partitioned as illustrated here [symbolically, this is $8(40 + 3)$] and then represented as two rectangles [e.g., $(8 \times 40) + (8 \times 3)$].

As the factors grow in size (e.g., 3-digit by 1-digit and 2-digit by 2-digit), more options for applying the distributive property are possible.



Standards for Mathematical Practice
MP7. Look for and make use of structure

Pause & Reflect

How might you use the distributive property to solve these two problems?

15×12 28×14 ●

The first can be done mentally by breaking apart 15 or by breaking apart 12. The second one can also be broken apart either way, but the partial products are not as easy to do mentally, so breaking apart both factors and finding the four partial products may be necessary.

Base 10 blocks can be used to show the distributive property concretely. Grid paper also supports student understanding of the distributive relationship, because students can see how numbers are decomposed and composed (Maffia & Mariotti, 2020). Eventually, students can draw rectangles freehand to represent problems. Figure 13.8 shows these three illustrations for 15×12 (with different ways to break apart the factors). Challenge students to generalize this idea. Students may describe this break-apart idea in words (at first) and then as symbols: $a \times b = (c \times b) + (d \times b)$, where $c + d = a$. Be sure students can connect the representations to symbols and vice versa.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

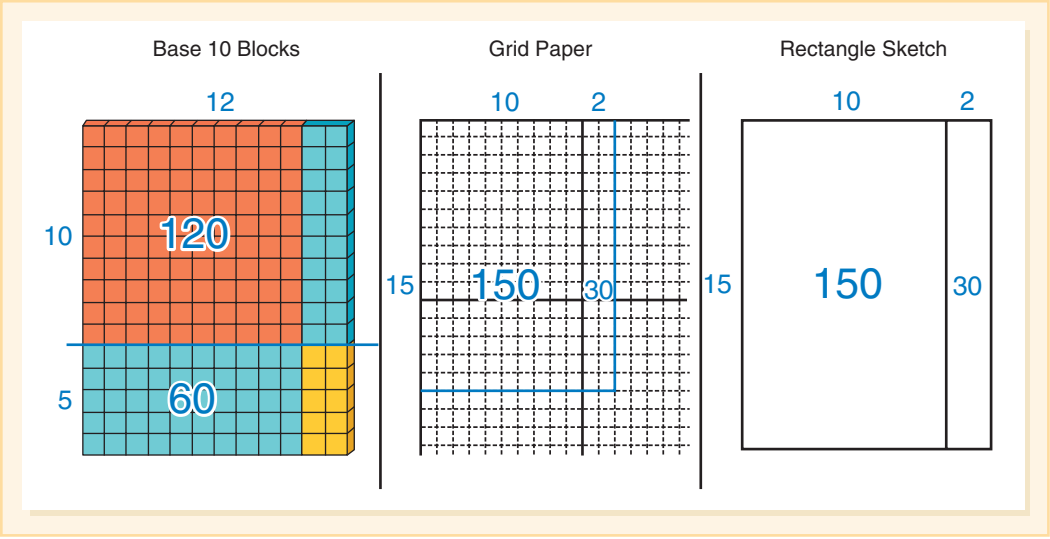


FIGURE 13.8 Representing the distributive property

***TECHNOLOGY* Note.** The DreamBox (dreambox.com) teacher tool, Multiplication: Open Arrays, is an app in which students use a zipper to partition a rectangle to create easier parts to multiply—an application that applies the distributive property and supports student reasoning. The NCTM Illuminations lesson titled "Multiply and Conquer" offers activity pages with illustrations that help students connect the distributive property to partial products (and eventually the standard algorithm). ■

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Conjectures

A conjecture is a proposed generalization to be validated and is essential to mathematical reasoning (Lannin, Ellis, & Elliott, 2011). Asking students whether a statement is always, sometimes, or never true develops their skills at justification and applying properties and takes on misconceptions they might have (Muir, 2015). Sometimes students overgeneralize, thinking something is always true simply because it is true for the cases they have tested (Yopp & Ellsworth, 2017). Conjectures can help students move beyond testing examples to proving, using geometric illustrations, number lines, pictures, or variables. False conjectures are worthy of exploration and perhaps a good place to begin, because students must understand the meaning of a counterexample as it relates to proving conjectures (Vinsonhaler & Lynch, 2020).

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

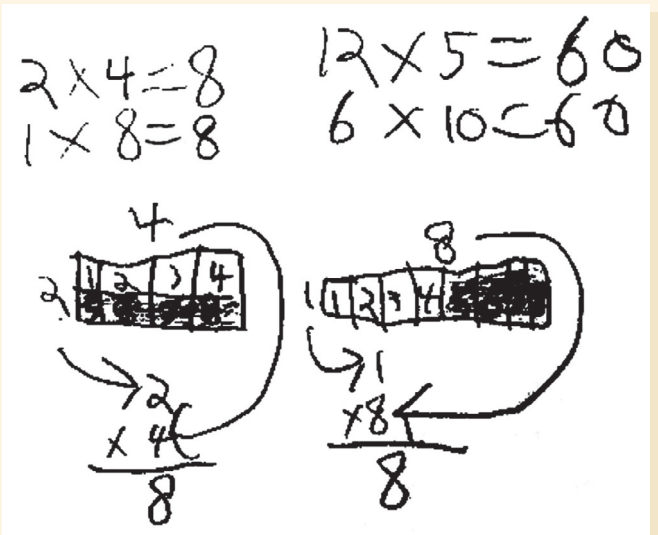


FIGURE 13.9 A student justifies a halve-and-double conjecture

Activity 13.10 Convince Me Conjectures

CCSS-M: 1.OA.B.3; 2.NBT.B.9; 3.OA.B.5; 5.OA.A.1

To begin, offer students a conjecture to explore. For example:

Conjecture: For a multiplication problem, you can take half of one factor and double the other factor and you will get the same product.

Ask students to (1) test examples to see if they think it is always true, and (2) prove or disprove it, preparing a visual or explanation to support their argument (see Figure 13.9). Ask students how they might write the conjecture, using symbols. All students, particularly multilingual learners, benefit from revoicing ideas and using precise language during discussions. Students with special needs in math benefit from the presentation and discussion of counterexamples.

Pearson eText Activity Page: Conjecture Cards, Set A (Grades 1–5)

Pearson eText Activity Page: Conjecture Cards, Set B (Grades 3–8)

Activity 13.11 explores an interesting pattern that leads to having students explain why it works.

Activity 13.11 Diagonal Sums

CCSS-M: 1.NBT.C.4; 2.NBT.B.9; 3.OA.D.9

Provide each student with a hard copy of a hundred chart (Bottom-Up version used here). Students select any four numbers in the chart that form a 2-by-2 square. Tell students that you heard a conjecture that no matter where you are on the hundred chart, the diagonals add to the same sum. For example, on this location, both diagonals equal 37.

23	24
13	14

Have students explore other places on the chart. If they think the conjecture is true, then they must provide a convincing argument for why it works. If students need more guidance, encourage them to explore the numbers in expanded form (e.g., $20 + 3 + 10 + 4$). For younger students or students with special needs, use calculators so that they can explore the pattern without getting bogged down in computations. Conclude by having students present their explanations.

Pearson eText Blackline Master: Bottom-Up Hundred Chart

Pause & Reflect

Standards for Mathematical Practice
MP7. Look for and make use of structure

Before reading further, stop and explore why the diagonal sums described in the previous activities are the same. What questions might you ask students to help them notice relationships between tens and ones? ●

Here are some additional tasks you might explore on a hundred chart that stimulate algebraic thinking. With each one, notice how the first aspect of the task is about *number* and the latter question's aspect is about *generalizations*.

- Pick a number. Move down two and over one. What is the relationship between the original number and the new number? What algebraic expression describes this move?
- Pick a number. Add it to the number to its left and the answer again to its right. Divide by 3. What answer do you get? Can you explain why this works? Can you explain this relationship by using variables?
- Skip count by different values (e.g., 2, 4, 5). Which numbers make diagonal patterns? Which make column patterns?
- Find two skip-count numbers in which one number always lands on top of the other (that is, all of the shaded values for one pattern are part of the shaded values for the other). How are these two skip-count numbers related?

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Asking questions such as "When will this be true?" and "When will it not be true?" and "Why does this work?" requires children to generalize and, consequently, strengthen their understanding of the number concepts they are learning.

Variables

Learning Outcome 13.3 **Explain the meanings of variable and how to develop these meanings conceptually.**

A variable is a letter representing a quantity. Variables can be used to represent a *unique but unknown quantity* (e.g., missing number) or a quantity that varies (e.g., output of a function). Variables can be used and understood in elementary school (Blanton et al., 2011; Blanton, Isler-Baykal, et al., 2019; Carpenter, Franke, & Levi, 2003). Yet, elementary students may not be introduced to variables, and if they are, the focus is solely on variable as an unknown quantity (Blanton et al., 2011; Lloyd, Herbel-Eisenmann, & Star, 2011). Early experiences with variables are important preparation for the more complex mathematical situations students will encounter in middle school.

The letter you (or your students) select as a variable is *arbitrary*. Assigning a letter to match the context can help students make a connection to a context (e.g., let's use b for number of birds), but such matching can also mask the fact that variables are arbitrary. Research suggests that arbitrary variables support student understanding of the abstract role that variables play in representing quantities (Woodward et al., 2012), so vary your choices for variables. Use the common choices of n, x, and y frequently and be explicit about the fact that any letter can be used.

Variable Expressions

Simplifying and recognizing equivalent expressions are essential to working algebraically. Because working with expressions de-emphasizes solving equations and emphasizes the properties of the operations, it is also an important first step in the discussion of variables. The Border Tile problem is helpful for students because it can be used both with and without variables, providing a transition to more abstract thinking.

Standards for Mathematical Practice
MP7. Look for and make use of structure

Activity 13.12 Border Tiles Expressions—Part 2

CCSS-M: 6.EE.A.1; 6.EE.A.2a; b, c, 6.EE.A.3; 6.EE.A.4

Distribute the Border Tiles Expressions—Part 2 activity page and color tiles (recommended). Explain that Marianna's pool patio work has expanded—she is building square pools of different side-lengths. Ask students to prepare variable expressions describing efficient ways for her to determine how many patio tiles are needed for any *square-shaped* pool.

If the square had a side of length p, the total number of tiles could be found in similar ways:

$10 + 10 + 8 + 8$	$(2 \times 10) + (2 \times 8)$	4×9	$100 - 64$
$(p + 2) + (p + 2) + p + p$	$2 \times (p + 2) + (2 \times p)$	$4 \times (p + 1)$	$(p + 2)^2 - p^2$

Finally, challenge students to prepare a variable expression for any *rectangular-shaped* pool. Compare each expression and have students justify why any two expressions are equivalent to each other.

Pearson eText Activity Page: Border Tiles Expressions—Part 2

Students can enter these expressions in the Table function on their graphing calculator and graph them to see whether they are equivalent (Brown & Mehilos, 2010). By looking at these options, the connection can be made for which one is stated the most simply.

Activity 13.13 provides another context that starts with numeric examples and then moves to using variables.

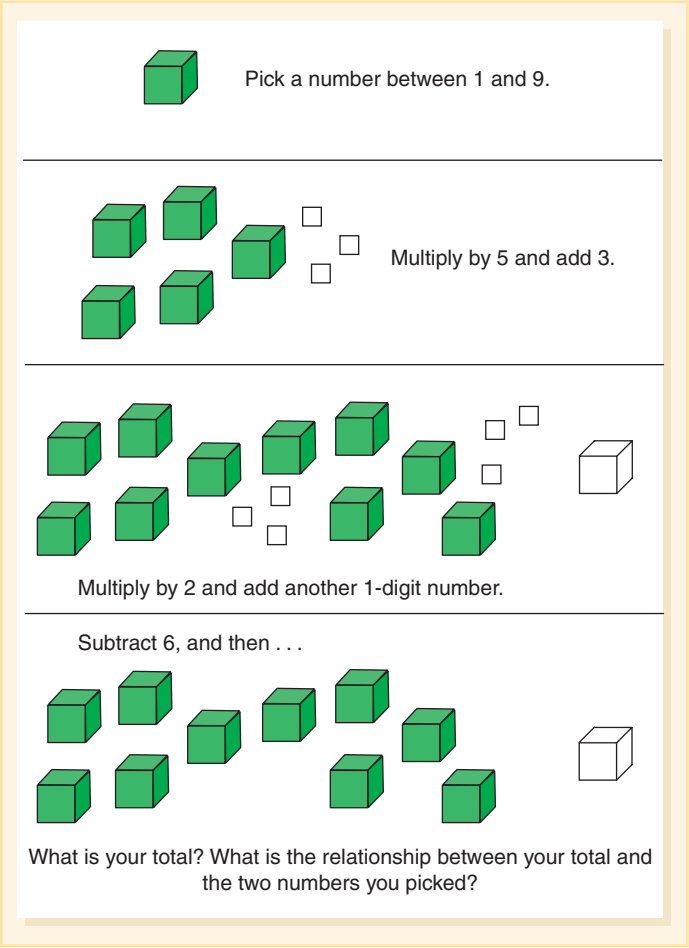

Pick a number between 1 and 9.
Multiply by 5 and add 3.
Multiply by 2 and add another 1-digit number.
Subtract 6, and then . . .
What is your total? What is the relationship between your total and the two numbers you picked?

FIGURE 13.10 Cubes can illustrate the steps in "Solving the Mystery"

Activity 13.13 Solving the Mystery

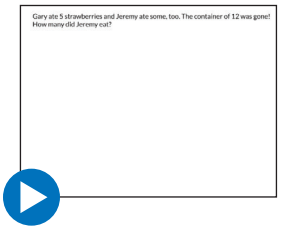
CCSS-M: 5.OA.A.2; 6.EE.A.2a; 7.EE.A.2

The goal (and intrigue) of a number mystery is to figure out why it works! Begin by having students do the following sequence of operations:

Write down any number.
Add to it the number that comes after it.
Add 9.
Divide by 2.
Subtract the number you began with.

Now you can read their minds. Everyone ended up with 5! Ask students, "How does this trick work?" Invite them to explore it, using expressions. [Start with n. Add the next number: $n + (n + 1)$. This is equivalent to $2n + 1$. Adding 9 gives $2n + 10$. Dividing by 2 leaves $n + 5$. Subtract the number you began with n, and the result is 5.] For students with special needs in math or students who struggle with variables, suggest that instead of using an actual number they use an object, such as a cube, and physically build the steps of the problem, as illustrated in Figure 13.10. Next, explore a new mystery, using the Solving the Mystery Activity Page. In this mystery, the result is a two-digit number where the tens place is the first number selected and the second digit is the second number selected (ask students to explain how this happened). See also the More Mysteries activity page. As a challenge, students can generate their own mysteries.

Pearson eText Activity Page: Solving the Mystery
Pearson eText Activity Page: More Mysteries



Pearson eText
Video Example 13.1
In this video consider the type of problem and how the student solves the Strawberry Container problem. What do you notice about her understanding of missing values?

Variables as Unknown Values

What begins as a picture, question mark, or empty box in an equation eventually becomes a variable. In each case, students are asked to figure out what that missing value is. Such explorations should reinforce a relational understanding (view 3) before using computation to figure out the missing value. Encourage students to look at the number sentence holistically and discuss in words what they know about the quantities in the equation.

Many story problems involve a situation in which the variable is a specific unknown, as in the following basic example:

Gary ate 5 strawberries and Jeremy ate some, too. The container of 12 was gone! How many did Jeremy eat?

Activity 13.14 What's Missing?

CCSS-M: 1.OA.D.7; 1.OA.D.8; 2.OA.A.1; 3.OA.A.4; 5.OA.A.2; 6.EE.A.3; 6.EE.B.5

Prepare a set of missing-value equations. Ask students to figure out what is missing and how they know. Notice that the equations are set up so that students do not always have to perform the operations to figure out what is missing. Encourage them to look at the equation and see whether they can figure out what is missing without doing the calculations. Probe to see if there is more than one way to find what is missing.

Here is a sampling of ideas across the grades:

$4 + \square = 6$	$4 + 5 = \square - 1$	$\square + 5 = 5 + 8$	$3 \times 7 = 7 \times \square$
$15 + 27 = n + 28$	$12 \times n = 24 \times 5$	$6 \times n = 3 \times 8$	$15 \times 27 = n \times 27 + 5 \times 27$
$0.5 + a = 5$	$4.5 + 5.5 = a + 1$	$a \times 4 = 4.8$	$2.4 \div a = 48 \div 6$

Using messy numbers such as the following further encourages applying relational understanding to solve:

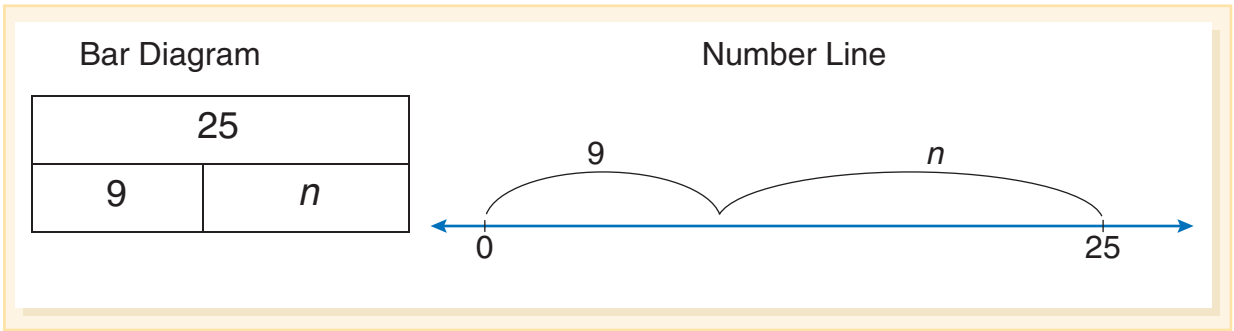
$126 - 37 = n - 40$	$37 \times 18 \div 37 = n$	$20 \times 48 = n \times 24$
$68 + 58 = 57 + 69 + n$	$7.03 + 0.056 = 7.01 + n$	$\frac{3}{10} + n + \frac{1}{10} = \frac{2}{5} + \frac{1}{5}$

Challenge students to create their own missing-value equations! Molina and Ambrose (2006) found that asking students to *write their own* open sentences was particularly effective in helping students solidify their understanding of the equal sign.

Pearson eText Activity Page: Missing-Value Equations

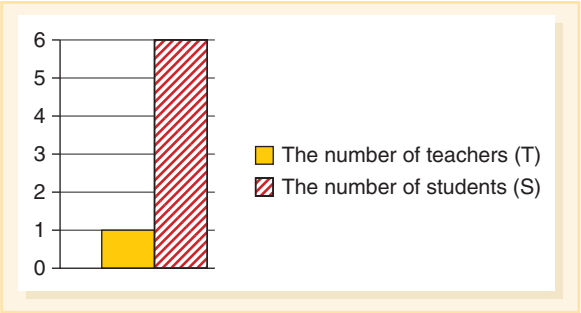
FORMATIVE ASSESSMENT Notes. As students work on these types of tasks, you can interview them one on one (though you may not get to everyone). Listen for whether they are using relational-structural thinking. If they are not, ask, "Can you find the answer without actually doing any computation?" This question helps nudge students toward relational thinking and helps you decide what instructional steps are next. ■

Although students can solve this problem without using algebra, they can begin to learn about variables by expressing the situation in symbols: $9 + n = 25, 25 - 9 = n$ or $25 - n = 25$. These problems can grow in difficulty over time. Illustrations can help students reason about the meaning of the story and serve to help write the related equation (Gavin & Sheffield, 2015). For this task, these two illustrations might be created by students:



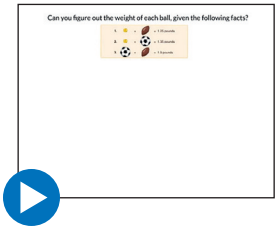
Translating stories to equations involving variables is particularly challenging for students, especially when the word order does not match the symbol order. For example, students often incorrectly write equations such as $6S = T$ for a problem that says, "The number of students

(S) was six times the number of teachers (T)" (Clement, 1982). A bar diagram for this situation looks like this:



Standards for Mathematical Practice
MP4. Model with mathematics

Another strategy that is helpful is to have students compare the variables (without the numbers) before they solve a problem, in this case to reason whether there are more students or more teachers.



Pearson eText
Video Example 13.2
This video shows how one student solves the Ball Weights problem. What is a non-variable way to reason through this problem?

Balances and Scales. Just as balances are an important representation for comparing expressions and exploring equivalence, they are useful for solving for missing values (variables). In Figure 13.11, a series of examples shows problems in which each different shape on the scales represents a different value. When no numbers are involved, as in the top two examples, students can find combinations of numbers for the shapes that make the balances balance. If an arbitrary value is given to one of the shapes, then values for the other shapes can be found accordingly. Problems of this type can be adjusted in difficulty for students across the grades. The last three problems in Figure 13.11 are on scales, and therefore the shapes have an unknown value that can be determined through reasoning.

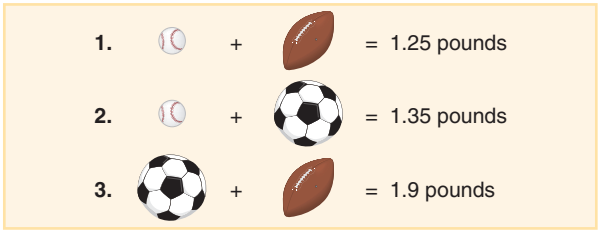
***TECHNOLOGY* Note.** PhET (phet.colorado.edu), mentioned earlier, includes balancing experiences with unknowns in the Equality Explorer simulation. ■

Reasoning with three variables is the focus of Activity 13.15.

Activity 13.15 Ball Weights

CCSS-M: 6.EE.A.4; 7.EE.A.2; 8.EE.C.8b

Students will figure out the weight of three balls, given the following three facts:



Ask students to look at each equation and make observations that help them generate relationships. For example, they might notice that the soccer ball weighs 0.1 pound more than the football. Write this in the same fashion as the other statements. Continue until these discoveries lead to finding the weight of each ball.

One possible approach: Add equations 1 and 2:

Then take away the football and soccer ball, reducing the weight by 1.9 pounds (based on the information in equation 3), and you have two baseballs that weigh 0.7 pound. Divide by 2, so one baseball is 0.35 pounds.

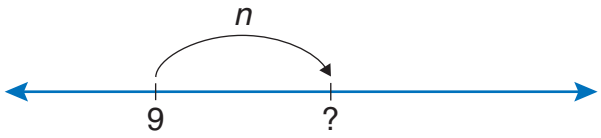
Solving systems of equations has traditionally been presented as series of procedures with little attention to meaning (e.g., solve by graphing, by substitution, and simultaneously). Reasoning strategies, such as the ones used to solve the problems in this section, must be added to this set of options. Additionally, students need to *choose* a method that fits the situation. Finally, they need to understand what that point of intersection means in the context of the problem given.

Variables that Vary

Students need experiences with variables that vary, and pairs of variables that co-vary, early in the elementary curriculum. For example, when describing how many legs for any number of dogs, students might write $L = 4 \times D$, meaning that the number of legs is four times the number of dogs. Importantly, you must emphasize that the variable stands for *the number of*, because students can confuse the variable to be a label (Blanton et al., 2011). Consider this adapted version of the story posed earlier:

Gary ate 9 strawberries and Jeremy ate some, too. How might you describe the total number of strawberries eaten?

Because the total has been removed, the goal becomes writing an expression, in this case $9 + n$. This can also be illustrated on a number line:



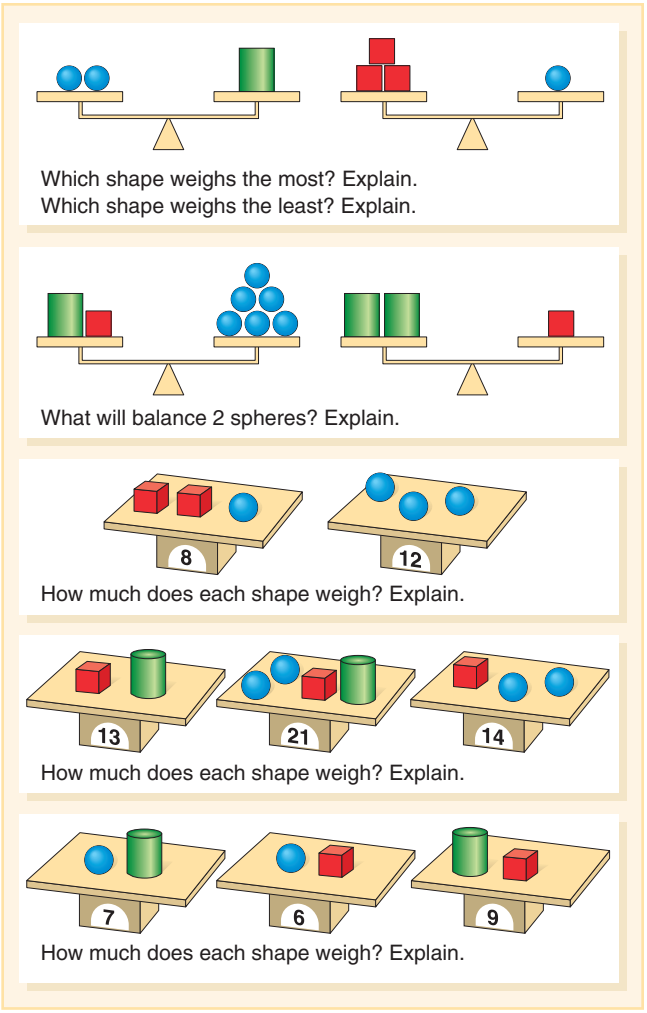


FIGURE 13.11 Examples of problems with multiple variables and multiple scales

Because students have used the number line with whole numbers, using it here is a good way to bridge to algebra. As illustrated in Figure 13.12, finding where variables are in relation to numbers and in relation to other variables helps build meaning (Carraher, Schliemann, & Schwartz, 2008; Darley, 2009).

When students are looking at the number line, ask questions like, "What is the value of x? Can it be any number? If we don't know what x is, how can we place $\frac{1}{3}x$ on the number line?" "Think of a value that x cannot be." Notice that in the two examples, x really can be any positive value. However, if you place $x + 2$ on the number line somewhere close to x, the space between these values is 2, and you can use this distance as a measure to approximate the size of x.

Context is important in writing equations with variables. Compare the two problems here (Blanton, 2008):

1. Annie has \$10. Noah has \$3 more than Annie. How much money does Noah have?
2. Annie has some money. Noah has \$3 more than Annie. How much money does Noah have?

Primary grade students can list possible ways in a table and eventually represent the answer as Annie + 3 = Noah or, more briefly, $A + 3 = N$.

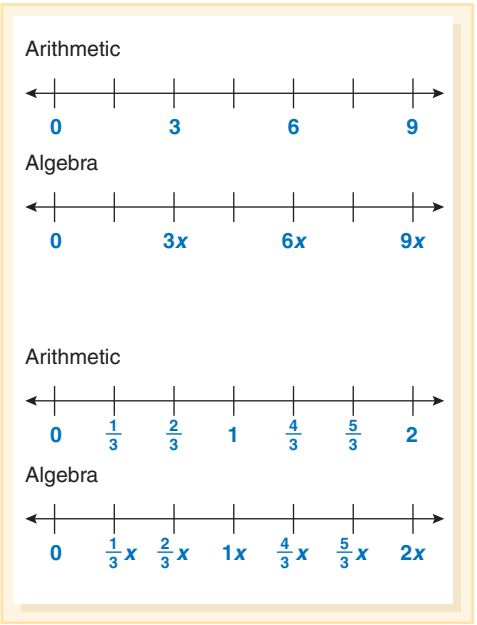


FIGURE 13.12 Using the number line to build meaning of variables

Total $35.00			
$1.75 item		**$1.25 item**	
	$35.00		$0
20		**0**	
	$0		$35
0		**28**	

FIGURE 13.13 A table adapted to include how many and how much for each row

Source: A. Hyde, K. George, S. Mynard, C. Hull, S. Watson, and P. Watson. (2006). "Creating Multiple Representations in Algebra: All Chocolate, No Change," *Mathematics Teaching in the Middle School, 11*(6), 262–268. Reprinted with permission. Copyright © 2006 by the National Council of Teachers of Mathematics. All rights reserved.

Let's revisit Birds in the Backyard (Activity 13.8). After students explore several situations (5 birds, 7 birds, 10 birds) and look at patterns, ask "If I have a birds in the tree, how might you describe how many birds are at the feeder?" Students might answer with this expression: $7 - a$. To go a step further, ask what equation describes the relationship between a birds in the tree and b birds at the feeder. Three equations describe this situation: $a + b = 7$, $7 - a = b$, and $7 - b = a$. Looking at the examples in the tables can help students generalize to these equations involving variables. These equations could be written for any comparison situation—for example, the number of strawberries eaten.

Pearson eText Activity Page: Birds in the Backyard

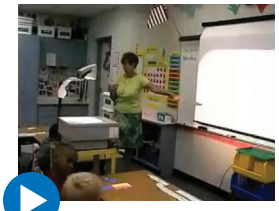
It is also important to include decimal and fraction values in the exploration of variables.

You are buying $1.75 pencils and $1.25 erasers from the school store and want to spend all of $35.00. What equation represents this situation? How many combinations are possible?

Once students have the equation (in this case, $1.75x + 1.25y = 35.00$), ask students to connect each number and variable back to the context. Provide time for them to find solutions, using a table or other strategy. As noted earlier, extra rows or sections on tables can help students organize and make sense of the data. The adapted table in Figure 13.13 helps students keep track of how many (pencils) and how much (cost for the pencils).

Patterns and Functional Thinking

Learning Outcome 13.4 **Illustrate and describe how to use patterns to develop functional thinking across K–8.**



Pearson eText
Video Example 13.3
Notice how the teacher engages students in creating patterns.

Patterns are found in all areas of mathematics. Learning to look for patterns and how to describe, translate, and extend them is part of thinking algebraically. Two of the eight Mathematical Practices begin with the phrase "look for," implying that students who are mathematically proficient pay attention to patterns as much as they do to mathematics (see Appendix A). Functional thinking begins in pre-K–2, when students make observations like, "For each person we add to the group, we add 2 more feet" (Blanton et al., 2011).

Let's look at a modified version of the Birds in the Backyard problem:

Five birds have landed in your backyard, some at the feeder and some in the tree. How many ways might the birds be in the tree and at the feeder?

There are 6 ways for 5 birds to be in the tree and at the feeder, 8 ways for 7 birds, and 11 ways for 10 birds. Upper elementary and middle school students should be able to explain why this is the case: For any number of birds (n), there are $n + 1$ ways, because there can be 0, 1, 2, n birds at the feeder. To help students generalize, ask students questions such as these: "What if there were 340 birds? Would the rule still hold? If there were 20 ways in which the birds could be in the backyard, how many birds are there? Is there a rule for that?"

Repeating Patterns

Repeating patterns are those patterns that have a core that repeats. For example, red-blue could be the core, and a string of beads continues to repeat this pattern: red-blue-red-blue-red.... Non-number patterns can build a foundation for later noticing numeric patterns. Physical materials provide a trial-and-error approach and allow patterns to be extended beyond the few spaces provided on a page. By using a variety of materials such as colored blocks, buttons, and connecting cubes to create and extend their patterns, students begin to generalize ideas of patterns. These can be recorded symbolically; for example, red-blue is an AB pattern because the core has two elements, A and B (see Figure 13.14).

Repeating patterns can be used to strengthen students' understanding of number, for example, multiples, as in Activity 13.16

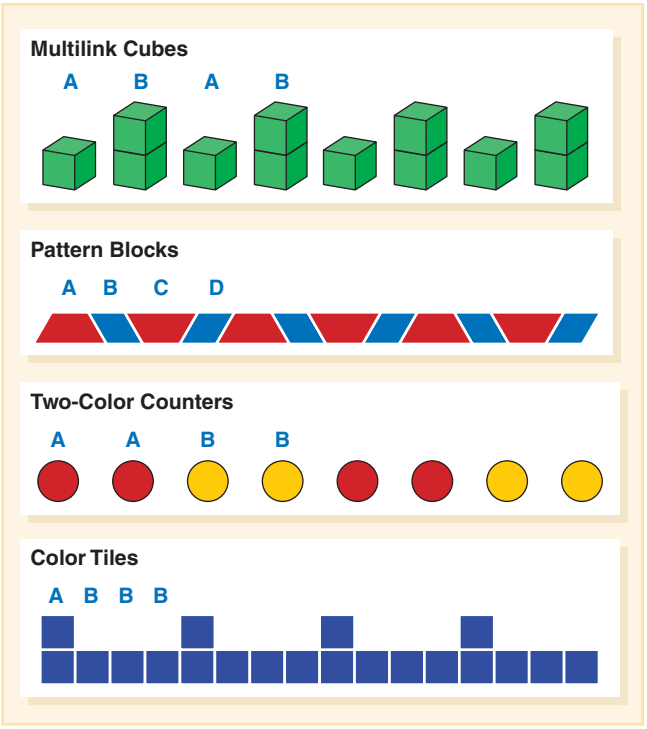


FIGURE 13.14 Examples of repeating patterns with various manipulatives

Activity 13.16 Predict Down the Line

CCSS-M: 2.OA.C.3; 4.OA.A.3, 4.OA.B.4

Provide students with a pattern to extend (e.g., ABC pattern made with colored links). Before students begin to extend the pattern, have them predict exactly what elements (links) will be in, say, the twelfth position. (Notice that in an ABC pattern, the third, sixth, ninth, and twelfth terms are the C element because they are multiples of 3.) After students predict, have them complete the pattern to check. Ask students to justify how they decided what shape goes in that position. Note that this also connects to division with remainders. To find the 100th term for ABC pattern, a student thinks that all multiples of 3 are the C shape, so a remainder of 1 would be the A shape.

Activity 13.17 Hurricane Names

CSSS-M: 4.OA.A.3; 4.OA.B.4; 5.OA.B.3; 6.EE.B.2a

Hurricanes are named such that the first one of the year has a name starting with A, then B, and so on. For each letter, there are six names in a six-year cycle (except those that are retired when a major hurricane has that name). The lists can be found on the National Hurricane Center and Central Pacific Hurricane Center website (https://www.nhc.noaa.gov/aboutnames.shtml). Invite students to select a letter of the alphabet and look up the list of six names. Ask students to answer questions such as these (assume the names do not get retired):

- What years will the hurricanes be the first name on the list? The last name?
- What will the hurricane's name be in the year 2030? 2035? 2050?
- How can you describe in words and symbols the name of a hurricane, given the year?

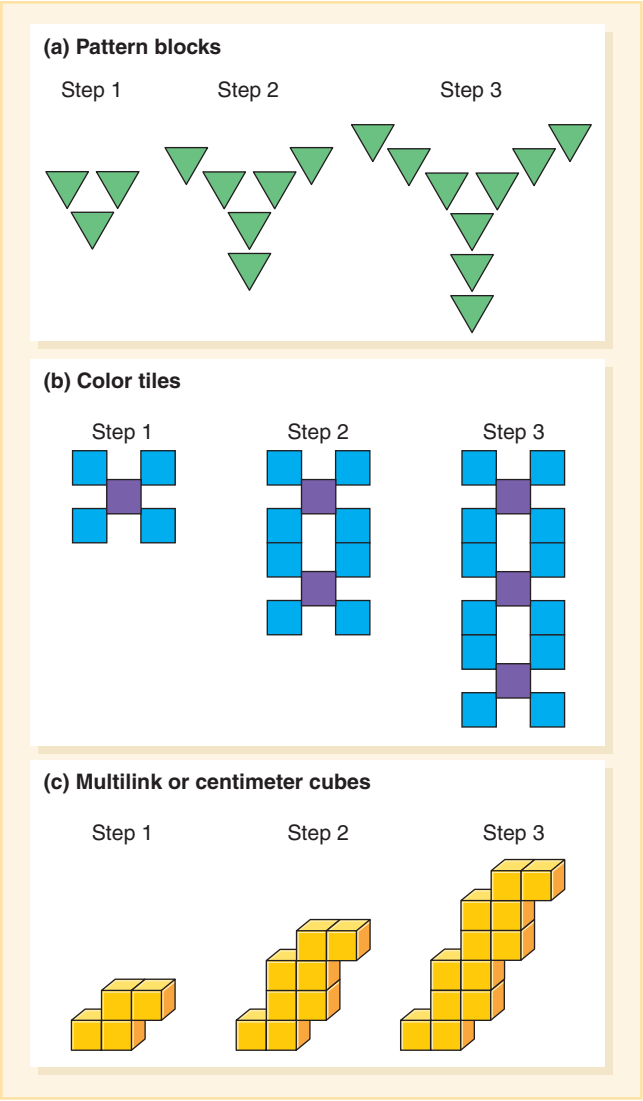


FIGURE 13.15 Geometric growing patterns, using manipulatives

Oral patterns can be recited. For example, "do, mi, mi, do, mi, mi" (ABB) is a simple singing pattern. Body movements such as waving the arm up, down, and sideways can make patterns: up, side, side, down, up, side, side, down (ABBC). Students can clap patterns, either imitating what another student has done or creating their own and having others replicate their pattern (Robinson, 2018). Repeating patterns are everywhere! The seasons, days of the week, and months of the year are just a beginning. Ask students to think of real-life AB patterns—for example, "to school, home from school" or "set table before eating, clear table after eating." Activity 13.17 is a real-life repeating pattern.

Children's books often have patterns in repeating rhymes, words, or phrases, for example, *Pattern Fish* (Harris, 2000). A very long repeating pattern can be found in *If You Give a Mouse a Cookie* (Numeroff, 1985) (or any of this series), in which each event eventually leads back to giving a mouse a cookie, with the implication that the sequence would be repeated.

Growing Patterns

Beginning in the primary grades and extending through the middle school years and beyond, students can explore patterns that involve a progression from step to step. These are called sequences or growing patterns. With growing patterns, the focus is on analyzing how the pattern is changing with each new element in the pattern. Figure 13.15(a) is a growing pattern in which step 1 requires three triangles, step 2 requires six triangles, and so on. Growing patterns are functions—the number of triangles needed is a function of which step it is (in this case, number of triangles $= 3 \times$ design number).

Geometric growing patterns make good exemplars for analyzing functional relationships, because students can manipulate the objects and then see what changes with each new step. Figure 13.15 shows three growing patterns, though the possibilities for visuals and patterns are endless. The questions in Activity 13.18, mapped to the pattern in Figure 13.15(a), can be adapted to any growing pattern and help students begin to reason about functional situations.

Analyzing growing patterns should include the developmental progression of reasoning by looking at the visuals, reasoning about the numerical relationships, and then extending that to a larger (or *n*th) case (Friel & Markworth, 2009). Students' experiences with growing patterns should start with straightforward patterns (such as in Figure 13.15) and continue with patterns that are more complicated (such as the dot pattern in Figure 13.16).

Activity 13.18 Predict How Many

CCSS-M: 5.OA.B.3; 6.EE.C.9; 7.EE.B.4a; 8.F.A.1

Working in pairs or small groups, have students explore a geometric growing pattern. Begin with one-step growing patterns like those illustrated in Figure 13.15. Be sure the specialized meaning of "step" is explained, especially for multilingual learners. The first pattern in this figure is the Triangle Burst. Distribute the activity page and have students work together to complete the page. As a second experience, or more challenging experience, use Predict How Many: Windows (two-step rule). Finally, Predict How Many: Dot Arrays (See Figure 13.16) is an even more challenging rule. If you select two or more of the Predict

How Many activity pages, students can compare these functions. (The first is linear in the form $y = mx$, the second is linear in the form $y = mx + b$, and the third is quadratic.)

Pearson eText Activity Page: Predict How Many: Triangle Burst
Pearson eText Activity Page: Predict How Many: Windows
Pearson eText Activity Page: Predict How Many: Dot Arrays
Pearson eText Expanded Lesson: Exploring Functions through Geometric Growing Patterns

***TECHNOLOGY* Note.** There are several websites that focus on relationships in **functions**. Virtual function machines can be found at Math Playground (mathplayground.com) and Shodor Project Interactive (shodor.org), among others. The Desmos activity, "Number Machines," allows students to undo a series of operations on a final output to discover their initial input. NCTM's Illuminations website has a lesson titled, "The Crow and the Pitcher: Investigating Linear Functions Using a Literature-Based Model." ■

Functional Thinking

Students in early elementary grades can employ functional thinking (Blanton et al., 2015a; Tanish, 2011). Table 13.2 suggests levels of student thinking based on a functional thinking learning progression (Stephens et al., 2017).

Functional thinking is supported through exploring input–output activities with familiar contexts.

The book *Two of Everything* (Hong, 1993) works well because the Haktaks put things "in the pot" (input) and they double as they come "out of the pot" (output). Explain that the pot changes rules on different days (see Magic Pot Mystery Rules activity page).

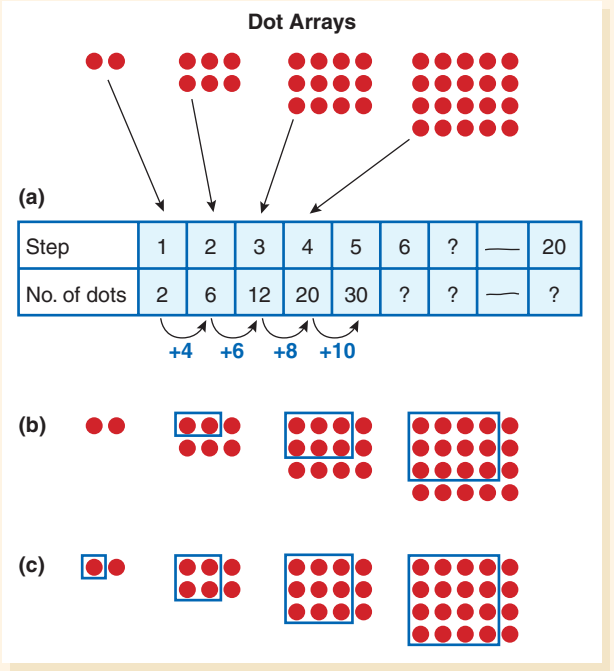


FIGURE 13.16 Analyzing relationships in the dot arrays

Pearson eText Activity Page: Magic Pot Mystery Rules

Shoeboxes or large refrigerator boxes can be turned into input–output machines. Decorate the box to look like a machine and add buttons for "easy," "medium," and "hard" and design functions that are appropriate for the grade of your students (Fisher, Roy, & Reeves, 2013). This activity can become a fun daily routine, trading out the explicit rule.

TABLE 13.2 LEVELS OF STUDENT THINKING WITHIN A FUNCTIONAL THINKING LEARNING PROGRESSION

Level	Description	Example of Student Thinking
No Functional Thinking	Student is not able to notice a relationship or pattern in the situation	[no response]
Variational Thinking	Student notices *recursive patterns* (tells how a pattern changes from step to step)	"The number of tiles goes up 3 with each new growing pattern."
Covariational Thinking: General	Student notices a relationship between input and output	"I notice that the number of tiles is 3 times more than the pattern number."
Covariational Thinking: Correspondence Rule	Student notices an *explicit relationship* and describes a rule in words or symbols	"The rule for figuring out how many tiles is needed is 'times 3'," or "The rule is $y = 3x$."

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

To support student development of functional thinking, ask questions such as these as they explore growing patterns and other covariation situations:

- What is changing? What is staying the same?
- What quantities are you comparing?
- How could you organize your information in a table?
- What relationships do you notice in the table?
- How might the quantities be represented in a graph?
- What relationships do you notice in the graph?
- How can you represent the rule for the machine in words? In symbols?

FORMATIVE ASSESSMENT Notes. Being able to make connections across representations is important for understanding functions, so is an excellent time to use a **diagnostic interview**. Engage students in a task (e.g., Magic Pot Mystery). Then pose questions, asking students to explain where the functional relationship is in the table, graph, equation, and situation. ■

Graphs of Functions

So far, growing patterns have been represented by (1) the physical materials or drawings, (2) a table, (3) words, and (4) symbols. A graph adds a fifth representation, and one that effectively illustrates covariation. Figure 13.17 illustrates what these five representations look like with the context of selling hotdogs.

Figure 13.18 shows the graph for the triangle burst and the dot array growing patterns. Notice that the first is a straight-line (linear) relationship and the other is a curved line that would make half of a parabola if the points were joined.

An engaging way to explore linear situations is with pattern blocks. Perimeter Patterns has a functional (linear) relationship for each shape, and this can be generalized for any polygon.

Pause & Reflect

Which representations do you find most useful in determining the explicit rule or function? Which representation do you think students new to exploring patterns will use? ●

Having graphs of three related growing patterns provides the opportunity to compare and connect the graphs to the patterns and to the tables (see Figure 13.19 and Figure 13.20). For example, ask students to discuss how to get from one coordinate to the next (e.g., up six, over one for the hexagon), where that information can be found by looking at the pattern itself or in a table.

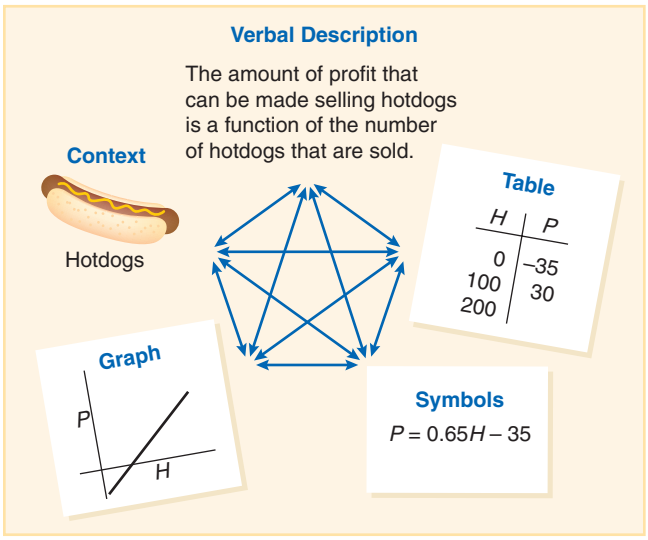


FIGURE 13.17 Five representations of a function for the situation of selling hotdogs

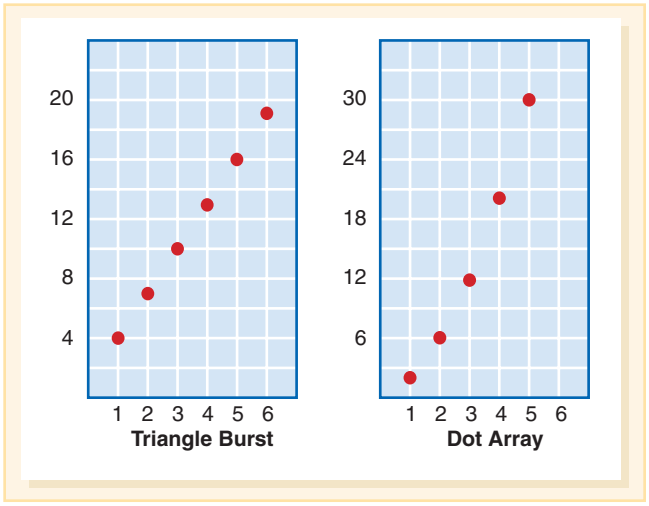


FIGURE 13.18 Graphs of two growing patterns

Activity 13.19 Perimeter Patterns

CCSS-M: 6.EE.C.9; 7.EE.B.4a; 8.F.A.1; 8.F.A.2

Use a document camera or interactive whiteboard to show rows of same-shape pattern blocks (see Figure 13.19). Working in pairs or small groups, have students build each pattern and explore what patterns they notice about how the perimeter grows. Ask: "What is the perimeter of a row with 6 squares? 10 squares? Any number of squares?" Repeat the process with trapezoids and hexagons (or assign different groups of students to work on different shapes). Distribute a coordinate grid. Ask students to create a graph to illustrate the relationship between the number of pattern blocks in the row and in the perimeter (see Figure 13.20).

Pearson eText Blackline Master: Coordinate Grid—Quadrant I

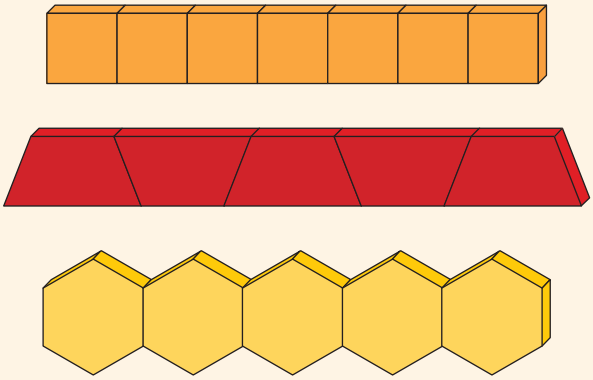


FIGURE 13.19 Same-color rows of pattern blocks.

***TECHNOLOGY* Note.** Function graphing tools permit users to create the graph of almost any function very quickly. Multiple functions can be plotted on the same axis. Digital graphing tools, such as Desmos (https://www.desmos.com), can also be used for these purposes, and add speed, color, visual clarity, and a variety of other interesting features to help students analyze functions. ■

Linear Functions

Linear functions are a subset of functions. In a linear function, the relationship, when graphed, looks like a line. That means the situation or pattern grows at a constant rate.

Pause & Reflect

Think back to example tasks shared in this chapter. Which are linear functions? Which are not linear but are still functions? ●

The examples that involved linear functions include birds/turtles/boats, the geometric growing patterns in Figure 13.15, Predict How Many: Windows (Activity 13.18) and the pattern block perimeters. The dot arrays pattern (also Activity 13.18) is nonlinear (it is quadratic).

Beyond geometric patterns, students need multiple opportunities to explore contexts that are linear, as in Activity 13.20.

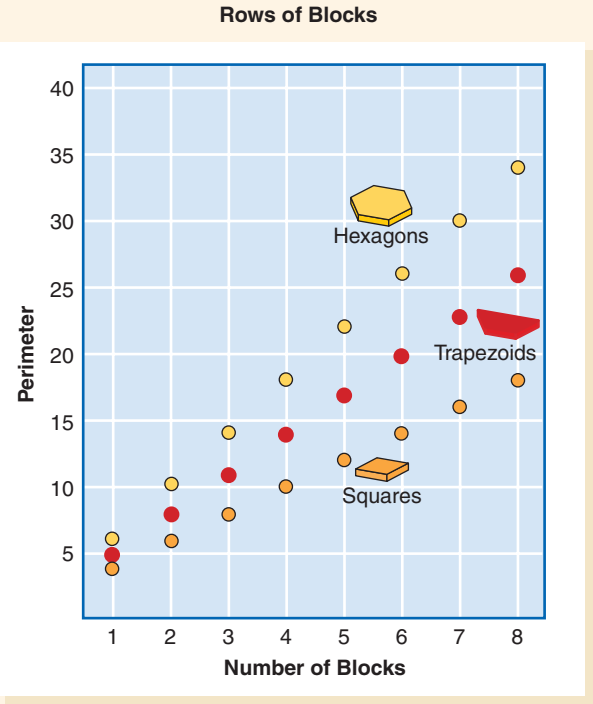


FIGURE 13.20 Graphs of the perimeters of three different pattern-block rows

Activity 13.20 The Grocery Store

CCSS-M: 5.MD.C.5b; 6.EE.A.2c; 6.G.A.2

Ask students about their trips to the grocery store. What do they notice about grocery carts? Ask whether they think it is possible to figure out mathematically the total length of stacked carts for any number of carts. Distribute The Grocery Store activity page. For students with special needs in math or students who benefit from concrete experiences, you can use paper or Styrofoam cups to model the nesting of one inside other. After students have explored the pattern, you can extend it by asking about the short carts (vary the length measurements).

Pearson eText Activity Page: The Grocery Store

Finally, exploring and comparing linear functions in abstract forms (e.g., tables, graphs, and equations) helps students notice what makes a function linear. Two such explorations are described in Activities 13.21 and 13.22.

Activity 13.21 G.E.T.S. Match-It

CCSS-M: 7.EE.B.3, 7.EE.B.4, 8.FA.1, 8.FA.2

Place students in groups of 4. Distribute one G.E.T.S. activity page to each group, or have students fold a piece of paper into fourths and label each section "Graph," "Equation," "Table," and "Situation." Give each group a different linear function, selecting one representation. For example, give each group a different situation (e.g., word problem). Have groups fill in the three missing representations. Once each group has completed all four sections, have them cut them into four little cards. Collect the cards. Shuffle. Have students stand up and come and pick a card. Students walk around the room finding their matches. For a set of ready-made representation cards and more ideas, see Wells (2016/2017).

Pearson eText Activity Page: G.E.T.S. Match-It

Activity 13.22 Is It Linear?

CCSS-M: 8.FA.1, 8.FA.3

This activity zooms in on one representation (e.g., graphs). Prepare a collection of graphs that include examples and non-examples of linear functions (some that are continuous, some discrete, some that go through the origin, some that do not). Have students line up all the ones they think are linear and all the ones that are not. Once lined up, students can do a gallery walk to see if other groups lined up the same graphs. Then return to their own list to reflect on the examples and non-examples. Discuss what is true about graphs that are linear and not linear. This activity can be repeated with equations or tables. (For a set of ready-made graph cards and more ideas, see Baldinger, Campbell, & Graif, 2020).

Standards for Mathematical Practice
MP4. Model with mathematics

From geometric growing patterns to linear situations to abstract representations for linear functions, it is important to focus on the idea that the recursive pattern has a constant rate of change—this is a central concept of linearity (Smith, 2008; Tanish, 2011).

Rate of Change and Slope. Rate involves two different units and how they relate to each other (see Chapter 17). Rates can be seen in a wide range of contexts, such as the geometric model of the pattern block perimeters (design number and blocks) or the rate of growth of a plant (time and height). The rate of growth in a linear situation is the *slope*. For example, the hexagon perimeter growing pattern is represented by $y = 4x + 2$. The rate of change (slope) is 4 because the perimeter increases by 4 with each new piece. All linear functions can be written in this form: $y = mx + b$ (including $y = mx$ when $b = 0$).

Conceptually, then, slope signifies how much y increases when x increases by 1. If it was a physical staircase, slope signifies how tall each step is in relation to the length of one step (Smith, Seshaiyer, Peixoto, Suh, Bagshaw, & Collins, 2013). If a line contains the points (2, 4) and (3, −5), you can see that as x increases by 1, y decreases by 9. So, the rate of change, or slope, is -9. After further exploration and experiences, your students will generalize that you can find the rate of change or slope by finding the difference in the y values and dividing by the difference in the x values. For an interactive online tool that connects linear equations in the form $y = mx + b$ to graphs, try "Graphing Lines" at PhET (https://phet.colorado.edu/).

Zero Slope and No Slope. Understanding these two frequently confused slopes requires contexts, such as walking rates. Consider this story:

You walk for 10 minutes at a rate of 1 mile per hour, stop for 3 minutes to watch a nest of baby birds, and then walk for 5 more minutes at 2 miles per hour.

What will the graph look like for the 3 minutes when you stop? What is your rate when you stop? In fact, your rate is 0, and because you are at the same distance for 3 minutes, the graph will be a horizontal line.

Let's say that you see a graph of a walking story that includes a vertical line—a line with no slope. What would this mean? You traveled a distance with no time passing! Now, even if you were a world-record sprinter, this would be impossible. Remember that rate is based on a change of 1 in the x value.

Proportional and Non-proportional Situations. Linear functions can be proportional or non-proportional. For example, the amount of money earned is proportional to hours worked (assuming an hourly rate), but if the person started with some money already saved, then the situation is no longer proportional because of the additional money (constant). All proportional situations, then, are equations in the form $y = mx$, and non-proportional linear situations are in the form $y = mx + b$. Proportional situations when graphed go through the origin.

The pattern block perimeter patterns are non-proportional. Although you have a constant increase of 4, 2 units (one on each end) must be added. Said another way, you cannot get from the input to the output by multiplying by a factor, as you can in proportional situations. Context is important. The Grocery Store task, for example, asks students to figure out how long grocery carts are when they are pushed together. Students can see that it grows at a constant rate, but that there is extra on the end (so linear, but not proportional).

Finding the explicit relationship is more difficult for non-proportional situations. Students want to use the recursive value (e.g., $+4$) as the factor ($\times 4$). Adding extra rows (or columns) to the table can highlight recursive and explicit relationships (Burton, 2017; Panorkou & Maloney, 2016). Figure 13.21 provides an example for the Predict How Many Windows task from Activity 13.18.

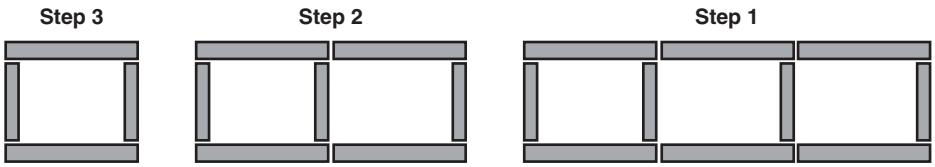


FIGURE 13.21 Expanded table to highlight recursive and explicit relationships

Input: Step Number [x]	Pattern	Output: Number of Tiles [y]	Shorthand
1	4	4	$4 + 0(3)$
2	$4 + 3$	7	$4 + 1(3)$
3	$4 + 3 + 3$	10	$4 + 2(3)$
4	$4 + 3 + 3 + 3$	13	$4 + 3(3)$
x	$4 + 3 + \ldots$ •	?	$4 + (x - 1)(3)$
		Rule: $y = 4 + 3(x - 1)$ or $y = 3x + 1$	

Characteristics of Functions. Table 13.3 summarizes the language used to describe functions, which can also become part of the discussions about functions to build meaning for students.

Parallel, Same, and Perpendicular Lines. In middle school, students explore two or more lines, deciding if they are parallel, same, or perpendicular. Of course, a context is necessary to build understanding.

Larry and Mary each earn $30 a day for the summer months. Mary starts the summer $50 dollars in debt, and Larry already has $20. In week 3, how much more money does Larry have? How much more does he have in week 7? When will Mary and Larry have the same amount of money?

The rates for Larry's and Mary's earnings are the same—and the graphs therefore go up at the same rate—that is, the slopes are the same. The graphs of Larry's earnings ($y = 30x + 20$) and Mary's earnings ($y = 30x - 50$) are parallel. We know this without even making the graphs because the rates (or slopes) are the same. Can you think of what change in Larry's and Mary's situations might result in the same line? Their initial value (and their rate) must both be the same.

Pearson eText
Video Example 13.4
In this video consider how the class engages with Tom's Job Problem. Pay attention to how the teacher uses a table to make the distinction between proportional and nonproportional situations.

TABLE 13.3 CHARACTERISTICS OF FUNCTIONS

Concepts	Description	Examples
Independent and dependent variables	The *independent variable* is the input, or whatever value is being used to find another value. The dependent variable represents the output.	In the Pattern Blocks Perimeter problem, the independent variable is the number of blocks in the string; the dependent variable is the perimeter. You can say that the perimeter of the block structure depends on the number of blocks used.
Discrete and continuous functions	When isolated or selected values are the only ones possible for a context, the function is *discrete*. If all values along a line or curve are solutions to the function, then the function is *continuous*.	Discrete: Pattern Blocks Perimeter problem. Only whole-number values make sense. Continuous: Walking rate, where at any given time, there is a distance walked (even if that distance is 0 meters).
Domain and range	The *domain* of a function comprises the possible values for the independent variable. The *range* is the corresponding possible values for the dependent variable.	In the Pattern Blocks Perimeter problem, the domain and range are all positive whole numbers.

Standards for Mathematical Practice MP5.

Slopes can also tell us when two lines are perpendicular, but it is less obvious. Analyzing perpendicular lines on a coordinate grid illustrates that the slope of one line is the negative reciprocal of the other, a powerful pattern to explore using dynamic geometry software such as GeoGebra.

Mathematical Modeling

Learning Outcome 13.5 Define mathematical modeling and describe grade-appropriate examples of it.

The term *model* can mean different things in teaching mathematics and in mathematics standards documents, including manipulatives, demonstrations, and conceptual models (Felton-Koestler, 2016/2017). Throughout this book, we have seen many examples of these ideas about models and modeling. *Mathematical modeling* has a specialized meaning, which is, "a process that uses mathematics to represent, analyze, make predictions, or otherwise provide insight into real-world phenomena" (COMAP & SIAM, 2016, p. 8). The process answers messy, authentic questions, and it is something that all students, beginning in elementary school, need to learn to solve problems at home, the workplace, and school (COMAP & SIAM, 2016; Usiskin, 2015).

Let's compare possible ways to engage students in multiplying (and other mathematics):

1. $14 \times 25 = 7 \times 50$ True or False? Justify your answer.
2. Randall wants to read 25 books over the summer vacation, which is 9 weeks long. How much does he need to read each week?
3. Corey and Sam would like to make handmade notecards with bows on them and package them in sets to sell at fall craft shows, to make money. How many cards do they need to make to earn the money they want to have?

Pause & Reflect

Which of these tasks are high-quality tasks? Which ones are examples of using models? Which ones are examples of mathematical modeling? ●

While all of these tasks have potential to be high quality and use models, only the last task will engage students in mathematical modeling. If you read this problem and thought, "There isn't enough information here to solve it," then that is your hint that it is a mathematical modeling problem. Questions need to be asked and answered, including:

1. Design and creation of the cards (How big will the cards be? What type of paper? How much ribbon will go on each card? What other supplies might be needed?)
2. Packaging of the cards (Will they tie a set together or put them in a box? How many cards make a set?)
3. Cost (How much money did it cost to make each box? How much is reasonable to make per box (i.e., profit)?)
4. Goal (How many sets might they like to make or how much money does each person hope to earn?)

FIGURE 13.22 THE PROCESS OF MATHEMATICAL MODELING

Mathematical Modeling
IDENTIFY THE PROBLEM
We identify something in the real world we want to know, do, or understand. The result is a question in the real world.
MAKE ASSUMPTIONS AND IDENTIFY VARIABLES
We select objects that seem important in the real-world question and identify relations between them. We decide what we will keep and what we will ignore about the objects and their interrelations. The result is an idealized version of the original question.
DO THE MATH
We translate the idealized version into mathematical terms and obtain a mathematical formulation of the idealized question. This formulation is the model. We do the math to see what insights and results we get.
ANALYZE AND ASSESS THE SOLUTION
We consider: Does the solution address the problem? Does it make sense when translated back into the real world? Are the results practical, the answers reasonable, the consequences acceptable?
ITERATE
We iterate the process as needed to refine and extend our model.

Source: COMAP and SIAM. 2016. *Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME)*. http://www.comap.com/Free/GAIMME/index.html. Accessed July 1, 2017.

Standards for Mathematical Practice MP4.

Imagine how messy this task will get and how much multiplication and other mathematics will be used to lead to a successful card-making endeavor. How will such a solution be reached? By applying the process of mathematical modeling as illustrated in Figure 13.22. And this process is messy as well, because in practice it is not necessarily linear; you are likely to bounce back and forth among the various stages. In mathematical modeling situations, students have to make genuine choices (e.g., what size the cards will be, how long the ribbon will be), and they benefit from working together as they consider options and make decisions. As a teacher, you are helping them learn the modeling process, helping them find information they are seeking, and providing feedback (via assessment) on their skills at implementing the process (not just on their product) (Cannon & Sanders, 2017).

Mathematical modeling is appropriate for investigating real challenges in the world. Invite students to think of a real-world problem they would like to explore. More and more examples are available of such tasks. Examples include:

- Which deal will give the biggest discount (from a newspaper, or providing such options as Buy-One-Get-One half off, 30% off, Buy-Two-Get-One Free)? (www.illustrativemathematics.org/practice-standards/4)
- How many students will be in our school in ten years? (Usiskin, 2015)
- Which conserves more water, taking a bath or taking a shower? (Anhalt, 2014)

Algebraic Thinking across the Curriculum

Learning Outcome 13.6 **Describe ways to connect algebra to other big ideas in mathematics.**

Algebraic thinking permeates all of mathematics. You have already experienced some of this integration—the strong connection between number and algebra (e.g., properties and generalizations). Here we share connections to other strands of mathematics.

Measurement and Algebra

Measurement formulas are functions. Each of these formulas involves at least one functional relationship. For example, the circumference of a circle is $c = 2\pi r$. We can say that the circumference is dependent on the radius. Even nonlinear formulas like volume of a cone $\left(V = \frac{1}{3}\pi r^2 b\right)$ are functions. The following activity explores how the volume of a box varies as a result of changing the dimensions.

Activity 13.23 Designing the Largest Box

CCSS-M: 5.MD.C.5b; 6.EE.A.2c; 6.G.A.2

Give each student or pair of students a piece of card stock. Explain that they are to cut out a square from each corner, using an exact measurement that you will give them. All four squares must be the same size. Assign different lengths for the squares that are cut out (e.g., 2 cm, 2.5 cm, 3 cm, and so on). Explain to students that after they cut out their four squares, they will fold up the four resulting flaps, and tape them together to form an open box. Have students calculate the volume of their box. Then, have them trade boxes and determine the volume of other boxes. The volume of the box will vary depending on the size of the squares (see Figure 13.23). Ask students to record their data in a table.

After students have recorded the data for several boxes, ask them to write a formula that gives the volume of the box as a function of the size of the cutout squares. Use the function to determine what size the squares should be to create the box with the largest volume. Alternatively, make origami boxes, using squares with various side lengths and see what the relationship is between the side length and the volume of the open box. (See DeYoung, 2009 for more on this idea.)

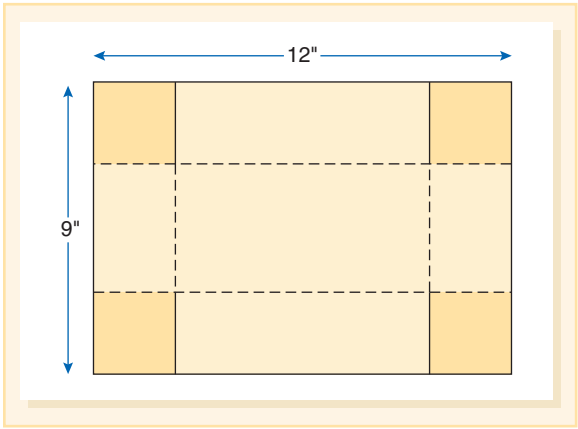


FIGURE 13.23 Cutting squares from cardstock. What size squares will result in an open box with the largest volume?

Data and Algebra

Data can be obtained from sports records, census reports, the business section of the newspaper, and many other sources. Students can gather data such as measurement examples or survey data. The Internet also has many sites where data can be found.

Experiments. There are many experiments that students can explore to see the functional relationships, if any, that exist between two variables. Gathering real data is an excellent way to engage a range of learners and to see how mathematics can be used to describe phenomena.

Data should be collected and then represented in a table or on a graph. The goal is to determine whether there is a relationship between the independent and dependent variables and, if so, whether it is linear or nonlinear, as in the following engaging experiments:

- What is the relationship between the number of [uncooked] spaghetti strands forming a bridge and the amount of weight it can hold (in nickels)? (Kroon, 2016)
- How long would it take for 100 students standing in a row to complete the wave (similar to those seen at baseball games)? Experiment with different numbers of students from 5 to 25. Can the relationship between times and students be used to predict how many students it would take for a given wave time?
- How far will a Matchbox car roll off of a ramp, based on the height the ramp is raised to?
- How is the flight time of a paper airplane affected by the number of paper clips attached to the nose of the plane?
- What is the relationship between the number of dominoes in a row and the time required for them to fall over? (Use multiples of 100 dominoes.)
- Make wadded newspaper balls, using different numbers of sheets of newspaper and a constant number of rubber bands to help hold the paper in a ball. What is the relationship between the number of sheets and the distance the ball can be thrown?
- What is the relationship between the number of drops of colored water dropped on a paper towel and the diameter of the spot? Is the relationship different for different brands of towels? (McCoy, 1997)

Experiments like these are fun and accessible to a wide range of learners. They also provide an opportunity for students to engage in experimental design—a perfect blend of mathematics and science.

Scatter Plots. Often in the real world, phenomena are observed that seem to suggest a functional relationship, but they are not necessarily as clean or as well-defined as some of the situations we have described so far. In such cases, the data are generally plotted on a graph to produce a scatter plot of points. Two very good scatter plot generators can be found online at NCES Kids' Zone.

A visual inspection of the scatter plot may suggest what kind of relationship, if any, exists. If a linear relationship seems to exist, for example, students can approximate a line of best fit or use graphing technology to do a linear regression to find the line of best fit (along with the equation).

Common Challenges Related to Algebraic Thinking

A lack of emphasis on algebraic thinking and connecting algebra to meaningful contexts results in the development of various challenges (e.g., Carpenter et al., 2003; Collins & Dacey, 2011; Kieran, 1981; Sakow & Karaman, 2015). Table 13.4 highlights the most prominent challenges in K–8.

TABLE 13.4 COMMON CHALLENGES RELATED TO ALGEBRAIC THINKING

Common Challenges	What It Looks Like	How to Help
1. Thinking the equal sign means "and the answer is . . ."	For the equation, $8 + 4 = \square + 5$, student puts 12 in the box (or 17). Equations such as $8 = 5 + 3$ and $6 = 6$ do not make sense, because they don't fit with their interpretation of the equal sign.	• Revisit/develop the concept of balance (two-pan balances, true/false equations, open sentences, etc.). • Read the equal sign as "is the same as," or "is equal to." For example, read $4 + 5 = 9$ as "four plus five *is the same as* 9." • Revoice students' use of the word *makes* (e.g., $5 + 12$ makes 17, restate as $5 + 12$ is equal to 17). • Record children's equivalent expressions when they solve problems in different ways (e.g., $8 + 7 = 8 + 2 + 5$).
2. Thinking an equation is written improperly" if it is not in the form of $a + b = c$ (for example).	Student is confused by equations such as: $17 = 17$ (identity statements) $6 + 6 = \square \times 6$ (operations on both sides of the equation) $62 = \square - 24$ (operation on the right side of the equation).	• Use a variety of equation formats all year long. • When recording equations (e.g., that represent story problems), write in more than one way (e.g., as multiplication *and* division).
3. Matching words with equations in solving story problems.	For a statement like "Maria is twice as old as Alonso," the equation $2M = A$ is recorded.	• Begin by asking who is older. • Ask students to substitute numbers to see if the equation fits the situation (and/or make a table). • Provide frequent experiences, such as warm-ups, starting with basic situations and building to more complex statements.
4. Understanding the meaning of zero in addition and multiplication	Student is not sure if the answer is 8 or 0 (or something else) in these problems: $8 + 0 =$ $0 \times 8 =$	• Use story problems that introduce operations involving zero in a meaningful context that students can act out or illustrate.
5. The commutative property of addition is overgeneralized to subtraction	In solving $24 - 7$, subtracts $4 - 7$ as $7 - 4$ for an answer of 23.	• Have students model problems with single-digit numbers and show how you cannot subtract 7 cubes from 4 cubes.
6. Checking only a few examples to decide if something is always true (i.e., generalizable)	Student decides that two odd numbers always add to even number because $5 + 7 = 12$ and $15 + 17 = 32$. Student says "The answer to a division problem is always equal to or less than the original value, using $15 \div 5$ and $15 \div 15$ and $15 \div 1$ as cases.	• Make distinction between testing and proving (see Activity 13.10). • Encourage use of visuals and manipulatives to show it is true for *any* numbers.
7. Thinking a variable is a label rather than a quantity.	Given a problem about cans of juice in a package, a student might interpret $6c$ mistakenly as 6 cans when it is instead 6 times *the number of* cans in the packages. (Note: In measurement, 6 m may be a label—a unit of measure.)	• Define the variable. • Emphasize that it means *the number* of. • Use generic variables (ones that don't always match the context). • Have students write out labels rather than use abbreviations. • Use boxes, blanks, and letters interchangeably in equations, emphasizing that each of these represents missing numbers.
8. Confusing the variable x for a multiplication symbol $\times$ or vice versa.	Student does not know whether to multiply or add $4x + 5$	• Use other variables, such as n. • Point out that once variables are used, the multiplication symbol becomes a dot, or parentheses are used to prevent the confusion.
9. Simplifying expressions involving variables, ignoring the variable or the need for parentheses	Making these errors: $5x + 9 = 14$ (or $5x + 9 = 14x$) $a + b = ab$ $l = 7$; $w = 4 + x$; Area $= 28 + x$	• Use concrete representations, such as a bag with some tiles to represent x ($5x$ is then 5 bags). • Ask students to say the meaning of the expression. • Present these as incorrectly worked examples to discuss why they are not correct.
10. Interpreting inequalities and their solutions.	Student doesn't know which symbols go with statements such as *less than, no more than,* or *at least*. Students don't understand how to interpret $a \leq 5$.	• Use contexts and number lines (e.g., Alison has $5; show how much she might spend at the store). • Have students create contexts to go with inequalities. • Devote more time to problems with inequalities.

RESOURCES FOR CHAPTER 13

WRITING TO LEARN

1. Share three ideas for how to help students move from an operational view of the equal sign to a relational view.
2. How can balances be incorporated into teaching to support student understanding of equations, inequalities, and solving missing-value problems?
3. Generalization is essential to algebraic thinking. Describe what it means and give an example.
4. Select two of the properties and explain how they are useful when doing arithmetic (pick your operation).
5. What conceptions do students need related to variables? Describe an activity or task to support each meaning.
6. Functions are often considered a middle or high school topic. What does functional thinking look like in elementary school?
7. What is mathematical modeling and why is it important?

LITERATURE CONNECTIONS

The following books are excellent books for exploring big ideas of algebra.

Bats on Parade

Appelt and Sweet (1999)

Bats on Parade walk 1 by 1, then 2 by 2, and so on—a growing pattern! For activity pages for this idea and others, see Roy and Beckmann (2007).

Equal Shmequel

Kroll (2005)

A mouse and her friends want to play tug-of-war and must determine how to make both sides equal to make the game fair. In the end, they use a seesaw to balance the weight of the friends.

Patterns at the Museum

Steffora (2011)

Photographs in this informational text illustrate different types of patterns in shapes and colors.

Two of Everything: A Chinese Folktale

Hong (1993)

As described earlier, a pot doubles whatever goes in it (see the Magic Pot Mystery Rules activity page). For more details and handouts while using this book, see Suh (2007b) and Wickett and colleagues (2002).

Pearson eText Activity Page: Magic Pot Mystery Rules

RECOMMENDED READINGS

Article

Chan, H. H. (2015). How do they grow? *Mathematics Teaching in the Middle School*, *20*(9), 548–555.

Dozens of interesting pattern block growing patterns are shared in this article, and students are set up to compare and contrast these patterns, leading to important functional thinking!

Leavy, A., Hourigan, M., & McMahon, A. (2013). Early understanding of equality. *Teaching Children Mathematics*, *20*(4), 247–252.

Nine strategies with corresponding activities are shared for helping students strengthen their understanding of the equal sign.

Molina, M., & Ambrose, R. C. (2006). Fostering relational thinking while negotiating the meaning of the equals sign. *Teaching Children Mathematics*, *13*(2), 111–117.

This article explains conceptual considerations related to the equal sign while simultaneously illustrating the value of errors and misconceptions in creating opportunities for learning.

Stephens, A., Blanton, M., Knuth, E., Isler, I., & Gardiner, A. (2015). Just say YES to early algebra! *Teaching Children Mathematics*, *22*(2), 92–101.

A must-read article that shares tasks, ideas, and assessment items for teaching across each of the five big ideas of algebra.

Books

Essential Understandings Series (*Expressions, Equations, and Functions: Grades 6–8* (2011) and *Algebraic Thinking: Grades 3–5* (2011)). Reston, VA: NCTM.

Each of these books provides a teacher-friendly discussion of the big ideas of algebra. Interwoven are excellent tasks to use with students.

Blanton, M. L. (2008). *Algebra and the elementary classroom.* Portsmouth, NH: Heinemann.

This is an excellent book for teachers at all levels—full of rich problems to use and helpful for expanding the reader's understanding of algebra. Great for a book study.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heinemann.

This book takes a detailed look at helping students in the primary grades develop the thinking and create the generalizations of algebra. The included CD shows classroom-based examples of the ideas discussed.

Fosnot, C. T., & Jacob, B. (2010). *Young mathematicians at work: Constructing algebra.* Portsmouth, NH: Heinemann.

Like the other books in the series, this is a gem. Full of classroom vignettes and examples that will enrich your understanding of how algebra can support arithmetic (and vice versa).

Developing Fraction Concepts

CHAPTER 14

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **14.1** Describe and give examples for fraction constructs and fraction representations.
- **14.2** Describe three types of fraction representations, and what situations and manipulatives fit within each.
- **14.3** Explain foundational concepts of fractional parts, including equal sharing, iterating, and partitioning.
- **14.4** Illustrate the concept of equivalence across fraction models.
- **14.5** Describe strategies for comparing fractions and ways to teach this topic conceptually.

Fractions are important to understand, for life and for future learning in mathematics. Students who do not have foundational understandings, such as knowing the relative size of a fraction, are more likely to struggle with fraction operations, decimal and percent concepts, and algebra (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; National Mathematics Advisory Panel, 2008; Siegler, Fazio, Bailey, & Zhou, 2013). Perhaps because they are not well understood, many people dislike or have anxiety when they see math problems involving fractions. This chapter focuses on meaningful and engaging ways to ensure that students develop a strong understanding of fraction concepts.

BIG IDEAS

- Fractions can and should be represented across different interpretations (e.g., part-whole and division) and different representations: area (e.g., $\frac{1}{3}$ of a garden), length (e.g., $\frac{3}{4}$ of an inch), and set (e.g., $\frac{1}{2}$ of the marbles).
- The number line is a critical representation in developing fraction concepts.
- Fractions are equal shares of a whole or a unit. Therefore, equal sharing activities (e.g., 2 sandwiches shared with 4 friends) build on whole-number knowledge to introduce fractional quantities.
- Partitioning and iterating are strategies students can use to understand the meaning of fractions. Partitioning can be thought of as splitting the whole equally (e.g., splitting a whole into fourths), and iterating can be thought of as making a copy of each piece and counting them all (e.g., one-fourth, two-fourths, etc.).

- Equivalent fractions are ways of describing the same amount by using subunits (different-sized fractional parts).
- Fractions can be compared by reasoning about the relative size of the fractions. Estimation and reasoning are important in teaching fractions with understanding.

Meanings of Fractions

Learner Outcome 14.1 **Describe and give examples for fraction constructs and fraction representations.**

Fraction understanding is developmental in nature and can begin as early as kindergarten with equal sharing experiences (Empson & Levi, 2011). The learning progression outlined in many state standards, based on *Common Core State Standards*, is summarized in Table 14.1. Fraction operations begin in grade 4 and continue through middle school and will be the focus of Chapter 15.

TABLE 14.1 CCSS PROGRESSION OF FRACTION CONCEPTS IN GRADES 1–4

Grade 1	Grade 2	Grade 3 [major topic]	Grade 4 [major topic]
Partition areas into halves, fourths.	Partition areas into halves, thirds, and fourths.	Develop concept of a unit fraction, including counting (iterating) by unit fractions	Decompose and compose fractions, connecting to addition and subtraction.
Describe fair shares.	Justify that an equal part may not be the same shape.	Understand that the size of a fraction is relative to size of the whole.	Connect iteration of fractions to multiplication.
		Use fraction notation to represent quantities between 0 and 1, equal to 1, and greater than 1.	Explain why two fractions are equivalent when the numerator and denominator are both multiplied by the same number.
		Understand fraction equivalence with visuals, including recognizing equal fractions on a number line.	
		Compare fractions that have the same denominator or same numerator.	Use a range of strategies to compare and order fractions (i.e., develop fluency).

Students need significant time and experiences to develop a deep conceptual understanding of this important topic, which is why developing a strong understanding of fractions is a major topic in both grades 3 and 4 in most state standards. Each of the ideas in the table is explored in this chapter. We begin with exploring the various meanings of fractions (constructs) and ways to represent fractions.

Pause & Reflect

Beyond shading a region of a shape to show part-whole, what other meanings do fractions have? And, how else might they be represented? ●

Fraction Constructs

One of the commonly used meanings of fraction is the description of the relationship of a part to a whole (part-whole) (Fuchs et al., 2016). But many researchers suggest an emphasis on other meanings of fractions is critical to students' long-term success with fractions (Clarke, Roche, & Mitchell, 2008; Lamon, 2020; Siebert & Gaskin, 2006).

Part-Whole. This construct is about understanding that a fraction is one or more equal parts of a single object (e.g., two of eight equal parts of a candy bar). Using the part-whole construct is an effective starting point for building the meaning of fractions (Cramer & Whitney, 2010). Part-whole can be shading a region ($\frac{1}{2}$ a desktop), part of a group of people ($\frac{3}{5}$ of the class went on the field trip), or part of a length (we walked $3\frac{1}{2}$ miles).

Division. As with whole numbers, division means sharing into equal-sized groups. Fraction instruction should build from the familiar experience of sharing as a way to introduce and support fraction understanding (Empson & Levi, 2011; Lamon, 2020; Siegler et al., 2010). Division is often not connected to fractions, which is unfortunate. Consider posing these two tasks to build a connection between division and fractions, the first resulting in a whole number, the second in a fraction:

Four friends are sharing 8 cheese sticks, making sure each friend gets the same amount. How many cheese sticks will each friend get?

Four friends are sharing 10 cheese sticks, making sure each friend gets the same amount. How many cheese sticks will each friend get?

Students can determine how many cheese sticks each friend will get and must feel comfortable being able to represent that quantity as division and as fractions (Flores, Samson, & Yanik, 2006). The second story can be represented in all of these ways: $\frac{10}{4}$, $4\overline{)10}$, $10 \div 4$, $2\frac{2}{4}$, and $2\frac{1}{2}$.

Measurement. Unlike the part-whole construct that focuses on parts of an object, the measurement construct focuses on cardinal size (e.g., how many fifths are in a given length). Measurement involves identifying a unit and using it to determine the length of an object. For example, $\frac{5}{8}$ is actually five iterations of the unit $\frac{1}{8}$. Stated another way, the length $\frac{5}{8}$ is 5 times $\frac{1}{8}$. The measurement interpretation is critical to student understanding of fractions and operations and, therefore, should be emphasized much more than is currently the case in U.S. schools (Fuchs et al., 2016; NMAP, 2008).

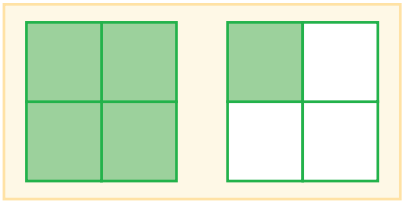
Operator. Fraction as operator builds on the concept of seeing a fraction as a multiple of a unit fraction. Fractions can be used to indicate an operation, as in $\frac{4}{5}$ of 20 square feet or $\frac{2}{3}$ of the audience.

Ratio. Discussed at length in Chapter 17, the concept of ratio is yet another context in which fractions are used. Ratios can be part-part or part-whole, and both can be written using fraction notation.

Fraction Language and Notation

The way that we write fractions with two numbers and a horizontal bar between is a convention, one that is complex and difficult for students to understand after working for so long with whole numbers. That is why fraction instruction should begin with use of the words (e.g., one-fourth) rather than the symbols (e.g., $\frac{1}{4}$). This allows students to focus first on making sense of fractions as *part*, without the complication of also trying to make sense of the symbols. A good time to introduce the vocabulary of fractional parts is *after* they have explored equal sharing tasks, discussed in the next section.

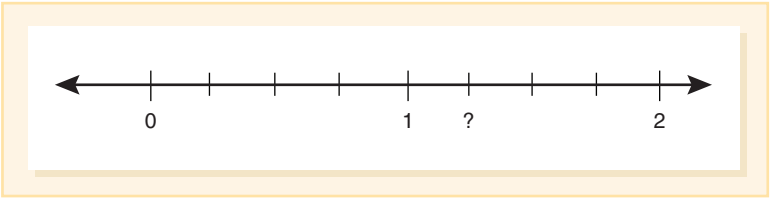
Students may understand the idea of halves and fourths, yet not understand the meaning of the symbols $\frac{1}{2}$ and $\frac{1}{4}$. Building meaning for the symbols can be done through counting by fractional parts (iterating) and representing fractions across models. For example, you can ask students to use fraction symbols to describe how many window panes are shaded green, with one window equaling one whole.



There are (at least) four ways to notate this quantity:

$$\frac{5}{4} \qquad 1\frac{1}{4} \qquad \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} \qquad 1+\frac{1}{4}$$

The number line is particularly important in helping students understand that a fraction is a number in and of itself (and not a number over another number). Consider repeating the preceding activity, with a number line representation:



Notice that this example ($\frac{5}{4}$) is a fraction greater than 1. It is important to explore fractions greater than, equal to, and less than 1. This helps students develop understanding of fractions as numbers between whole numbers as well as another way to represent whole numbers.

Standards for Mathematical Practice
MP6. Make sense of problems and persevere in solving them

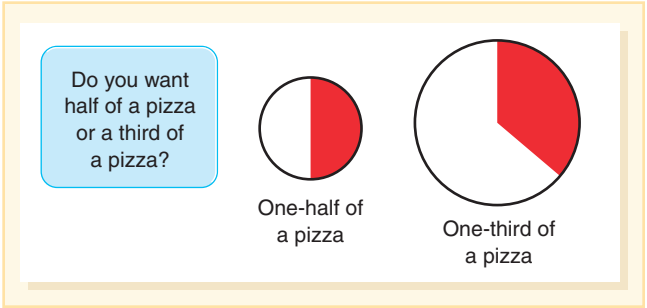
The term *improper fraction* is sometimes used to describe fractions greater than 1, for example, $\frac{7}{4}$ *or* $\frac{12}{3}$. But this term can be a source of confusion, because the word *improper* implies that this representation is not acceptable, which is not the case at all—in fact, this way to write the fraction is often the desired form in algebra.

Fraction Size Is Relative

A fraction by itself does not define the *size* of the whole. A fraction only tells us about the *relationship between* the part and the whole. Consider the following situation:

Pizza Fallacy: Mark is offered the choice of a third of a pizza or a half of a pizza. Because he is hungry and likes pizza, he chooses the half. His friend Jane gets a third of a pizza but ends up with more than Mark. How can that be?

Give students a chance to reason about this problem. This visual illustrates how Mark got misdirected in his choice.



The point is that whenever two or more fractions are discussed in the same context, one cannot assume (as Mark did) that the fractions are all parts of the same size whole. Teachers help students understand fractional parts when they regularly ask, "What is the whole?" or "What is the unit?" Comparing, combining, or otherwise computing with fractions can only be done if the fractions are parts of the same-sized whole. Worked examples are an effective way to help students focus on the size of the whole. For example, ask students to analyze Angelica's reasoning:

Angelica draws these pictures and concludes that two-fourths is greater than one-half. How might you respond to Angelica?

Although this is a standard in grade 3 (see Table 14.1), it is important to continue to attend to the whole when students are engaging in comparing or ordering fractions and doing operations with fractions. Monson, Cramer, and Ahrendt (2020, p. 117) use the context of an allowance to focus on different wholes, posing the following task:

Julia spent $\frac{1}{2}$ of her allowance on a movie, and Hamdi spent $\frac{1}{3}$ of her allowance on lunch. Is it possible that Hamdi spent more money than Julia?

Follow-up questions engage students in reasoning about the relationship of parts to the whole:

- Assume the allowances are the same. How do these fractions compare?
- If Hamdi's allowance is much larger than Julia's, what do you know about who spent more money?
- Is it possible that they spent the same amount of money? What allowances can make this situation true?

Pearson eText
Video Example 14.1

Pearson eText
Video Example 14.2
What do you notice about the students' understanding of fractions and fraction notations?

Fraction Representations

Learner Outcome 14.2 **Describe three types of fraction representations, and what situations and manipulatives fit within each.**

Making sense of fractions begins with concrete experiences, including familiar contexts, manipulatives, and visuals (Battista et al., 2017; Siegler et al., 2011). Recall the importance of teaching using a concrete-semi-concrete-abstract (CSA) approach, wherein you begin with concrete examples (contexts and manipulatives) and move to abstract, making connections among representations. Numerous studies have found that a CSA approach is effective with fraction concepts (Flores, Hinton, & Meyer, 2020). The contexts should address area, length, and set situations.

Manipulatives can help students clarify ideas that are often confused when presented in a purely symbolic model. Concrete tools lead to the use of mental models, which builds students' understanding of fractions (Empson & Levi, 2011; Lamon, 2020; Petit, Laird, Marsden, & Ebby, 2016). Table 14.2 provides an at-a-glance explanation of three types of representations, along with example contexts and manipulatives. A significant amount of research on fractions has found that using visuals improves student understanding and achievement with fractions, in particular for students with special needs in math (Flores, Hinton, & Meyer, 2020; Siegler et al., 2011; Tian & Siegler, 2017).

TABLE 14.2 REPRESENTATION TYPES FOR FRACTION CONCEPTS

Representation Type	Description	Sample Contexts	Sample Manipulatives and Visuals
Area	Fractions are determined based on how a part of a region or area relates to the whole area or region.	Quesadillas (circular food) Garden plot or playground Pan of brownies	Circular Fraction Pieces (Fraction Circles) Color Tiles Geoboards Pattern Blocks Rectangles (piece of paper or a drawing) Grid or Dot Paper Regions
Length	Fractions are represented as a subdivision of a length of a paper strip (representing a whole) or as a length/distance between 0 and a point on a number line, subdivided in relation to a given whole unit. (The number line can represent positive and negative fractions.)	Walking/distance traveled String lengths Music measures Measuring with inches or yards	Cuisenaire Rods Fraction Strips (paper strips folded into fractional parts) Number lines
Set	Fractions are determined based on how many discrete items are in the whole set, and how many items are in the part.	Students in the class Type of item in a collection Color of counter in a set	Objects (e.g., small toys) Counters (e.g., two color counters, color tiles, color cubes) People (e.g., students in the class)

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

And

Area Models

As the name implies, area models have an area that is partitioned. Area is a good place to begin fraction explorations because it lends itself to equal sharing and partitioning. Circular fraction pieces are the most commonly used area model. Circles readily show the part-whole concept of fractions and help students see the relative size of a part to the whole (Cramer, Wyberg, & Leavitt, 2008). Figure 14.1 illustrates a variety of area models. Grid paper or dot paper provides flexibility in selecting the size of the whole and the size of the parts (see Blackline Masters 5–11 for a selection). Activity 14.1 (adapted from Roddick & Silvas-Centeno, 2007) uses pattern blocks to help students develop concepts of partitioning and iterating.

Pearson eText Blackline Master: Circular Fraction Pieces
Pearson eText Blackline Master: 1-Centimeter Grid Paper
Pearson eText Blackline Master: 1-Centimeter Dot Paper

As students transition to drawing visuals, circles can have limitations, including that they are difficult to partition and show equal parts if students are drawing them. This can make comparing fractions, for example, difficult to do. Rectangles are easier for students to partition accurately.

Length Models

Length models showcase the measurement interpretation of fractions described earlier. Length models are very important in developing students' understanding of fractions

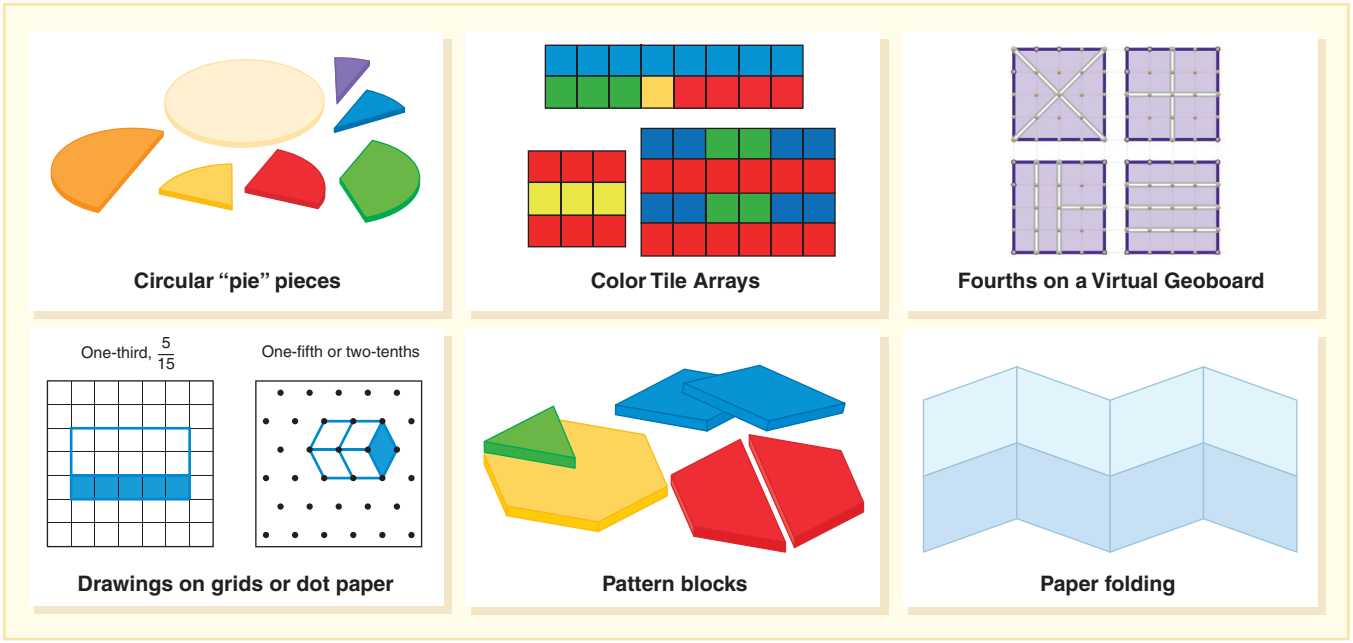
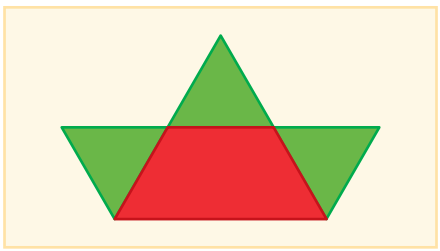


FIGURE 14.1 Area models for fractions

Activity 14.1 Playground Fractions

CCSS-M: 1.G.A.3; 2.G.A.3; 3.NF.A.1

Create this playground with your pattern blocks (see the Playground Fractions activity page).



It is the whole. For each of the following fractions, find the pieces of the playground and draw it on your paper. For use with grades 1 and 2, use words, not fraction symbols (e.g., half of, one-half, or four-thirds).

- $\frac{1}{2}$ playground
- $1\frac{1}{2}$ playgrounds
- 2 playgrounds
- $\frac{1}{3}$ playground
- $\frac{2}{3}$ playground
- $\frac{4}{3}$ playground

You can challenge students by making more complex shapes of pattern blocks and/or focusing on other fractional amounts, such as eighths, thirds, and sixths.

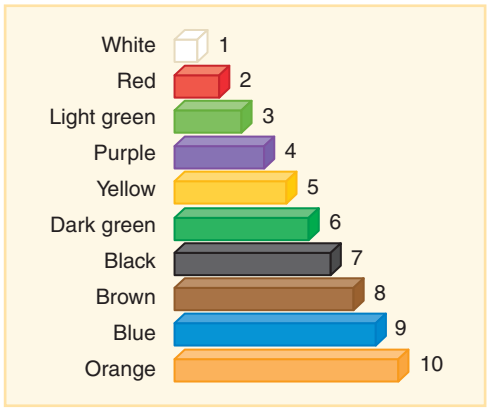
Pearson eText Activity Page: Playground Fractions

(Cramer et al., 2017; Flores et al., 2020; Fuchs et al. 2016; Morano et al., 2020). This may be because linear representations are one-dimensional, which may help illustrate fraction size. For example, students can see that a unit that fits into a paper strip two times is larger than a unit that fits into a paper strip of the same length four times (Saxe, Diakow, & Gearhart, 2012; Siegler et al., 2011).

Reviews of research on fractions (Petit et al., 2016; Siegler et al., 2010) report that the number line helps students understand a fraction as a number (rather than as one number over another number) and helps develop other fraction concepts. Researchers recommend that teachers use number lines as a central representational tool in teaching fraction concepts from the early grades on. However, the number line is a significantly more sophisticated model than area models. Therefore, it is important to begin the exploration of fraction lengths

by using tools such as Cuisenaire rods and fraction strips, along with using meaningful length contexts such as walking, cooking, heights, and musical measures. Music, for example, lends to exploring $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$ in the context of notes (Courey, Balogh, Siker, & Paik, 2012; Goral & Wiest, 2007).

Cuisenaire rods consist of the following colors and lengths:



Cuisenaire rods provide flexibility because any length can represent the whole. For example, if you wanted students to work with tenths, select the orange rod. The white rod, the unit, is $\frac{1}{10}$ of the whole, because it takes ten of those units to equal the whole. As illustrated in Figure 14.2, the red rod represents $\frac{2}{10}$ or $\frac{1}{5}$, and the yellow rod represents $\frac{1}{2}$ of the orange rod.

***TECHNOLOGY* Note.** Virtual Cuisenaire rods and accompanying activities can be found online at various free websites such as Toy Theater Teacher Tools (they call them Graph Color Bars) (https://toytheater.com/), MathsBot (https://mathsbot.com/), and the University of Cambridge's NRICH Project (https://nrich.maths.org/). ■

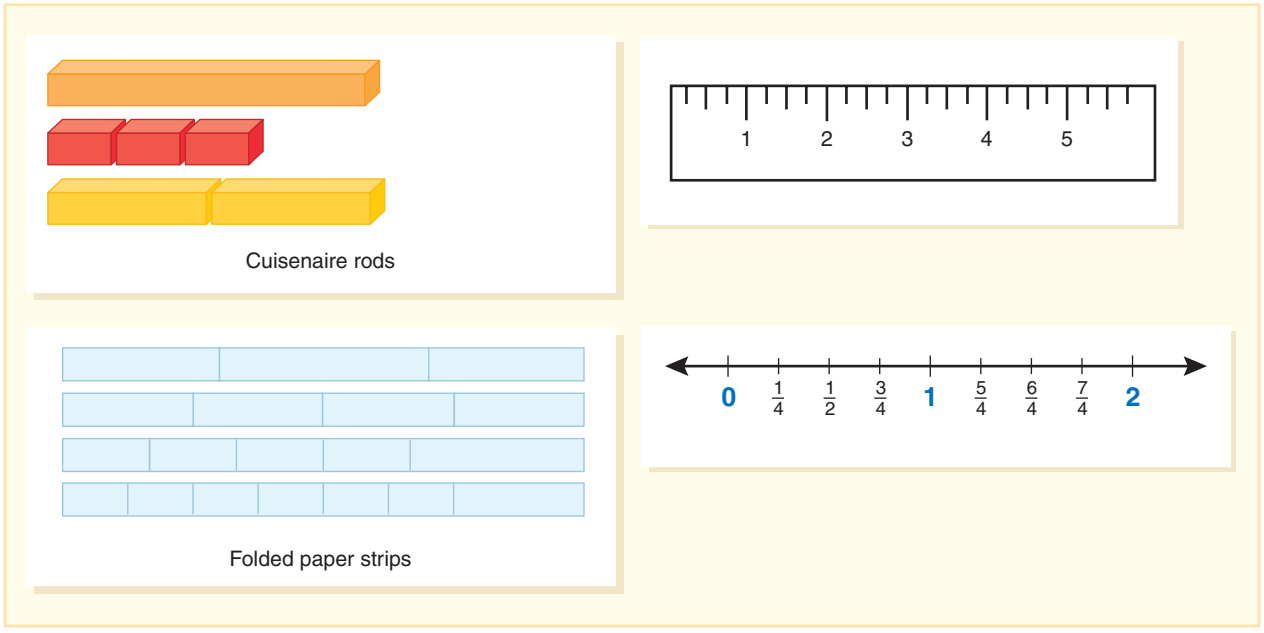


FIGURE 14.2 Length or measurement models for fractions

Folded inch-wide paper strips, called *fraction strips*, are an excellent tool for helping students bridge from area models to the number line. The lengths lend themselves to observable comparisons, thus encouraging an understanding of the relative size of fractions.

Pearson eText Blackline Master: Fraction Strips

***TECHNOLOGY* Note.** Visnos (visnos.com) has a virtual fraction wall, which is a visual of fraction strips. Fractions are represented by layers of bricks, which can be turned on or off. It illustrates, for example, that one half is equal to two-fourths or four-eighths and adds the total of the fractional parts that are turned on. ■

With sufficient experiences with fraction strips and connecting the strips to the number line, students can make sense of this more sophisticated representation. Being able to understand and use the number line is important! As it turns out, being able to make sense of the number line—for example, being able to locate a fraction on a number line—predicts student success in fraction operations and algebra (Clarke et al., 2008; Flores et al., 2006; Siegler et al., 2010; Usiskin, 2007; Watanabe, 2006). The following activity (based on Bay-Williams & Martinie, 2003) engages students in thinking about fractions on a number line, using a classic game as a context.

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Activity 14.2 Who Is Winning?

CCSS-M: 3.NF.A.2a, b; 3.NF.A.3a, b, d

Use the Who Is Winning Red Light, Green Light? activity page. Explain that six friends are playing Red Light, Green Light. The fractions tell the distance they have already moved from the start (0) toward the finish line (1). Begin by asking students to use reasoning simply to decide who is winning. (Who can they rule out?) To focus more on exact location, ask students to plot each person's position on a number line.

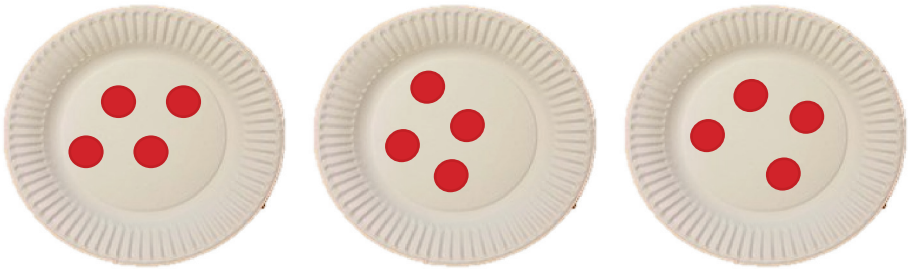
Mary: $\frac{3}{4}$	Larry: $\frac{1}{2}$	Carrie: $\frac{5}{6}$	Han: $\frac{5}{8}$	Shawn: $\frac{5}{9}$	Juan: $\frac{2}{3}$

This game can be differentiated by changing the value of the fractions or the number of friends (fractions). Acting out the game in the class is engaging and builds background to support multilingual learners and students with special needs in math.

Pearson eText Activity Page: Who Is Winning Red Light, Green Light?

Set Models

In set models, the whole is understood to be a set of objects, and subsets of the whole set make up fractional parts. For example, 3 objects are one-fourth of a set of 12 objects. The set of 12 is the whole, and each object is the fractional unit. The idea of referring to a collection of counters as a single entity may make set models difficult. One challenge is to focus on the number of equal-sized groups (not how many are in the group) to determine the fraction of the whole. For example, when the whole is 12 counters, a subset of 4 counters is one-*third*, not one-fourth, because *3* equal sets make the whole. To help students see the whole, you can place the counters on plates or draw a loop around the set:



Two-color counters are a versatile manipulative to represent the set model. Counters can be flipped to change their color to model various fractional parts of a whole set. Any set of countable objects (e.g., a box of crayons) can be a set model. Students are also an engaging set model. Activity 14.3 uses a group of students as the whole set. It can be done as an energizer, routine, warm-up, or full lesson.

Activity 14.3 Class Fractions

CCSS-M: 3.NF.A.1; 3.NF.A.3b

Invite a group of students to come to the front of the room—for example, six students if you want to work on halves, thirds, and sixths. Sort the students in some way (e.g., wearing tennis shoes/not wearing tennis shoes). Ask the class, "What fraction of our standing friends [are wearing tennis shoes]?" Have students record the fraction symbols and share their answers. Sort the group in ways that equivalencies emerge (e.g., 3 students in one group and 3 in the other). If students offer equivalencies (e.g., one says "three-sixths" and another says "one-half"), show how sixths refers to people and halves refer to two groups of equal size. Repeat with different-sized groups (e.g., 8 students).

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

FORMATIVE ASSESSMENT Notes. To determine whether students understand the meaning of a fraction, ask them to represent a given fraction across models. For example, fold a piece of paper into three parts labeled *area*, *length*, and *set* at the top of each section, or use the Fraction Assessment teacher resource. Give students a fractional value (e.g., $\frac{3}{4}$) and ask them to draw a representation in each section. ■

Pearson eText Teacher Resource: Fraction Assessment

***TECHNOLOGY* Note.** Virtual manipulatives are available for all three models. Virtual manipulatives have been found to affect student achievement positively, especially when they are paired with using the actual manipulatives (Moyer-Packenham, Ulmer, & Anderson, 2012). Recommended sites include:

Brainingcamp. This site offers area, set, and length models (including a versatile number line) (brainingcamp.com).

Cyberchase (PBS): *Cyberchase* (https://pbskids.org/cyberchase/) is an engaging show. The related website offers games, videos, and activities that model fractions with real-world connections, such as "Thirteen Ways of Looking at a Half".

Math Learning Center Free Apps (https://www.mathlearningcenter.org)

Toy Theater Teacher Tools (https://toytheater.com/) ■

Fractions as Numbers

Learner Outcome 14.3 **Explain foundational concepts of fractional parts, including equal sharing, iterating, and partitioning.**

One of the most significant ideas for students to develop about fractions is the sense that fractions are numbers—quantities that have values. You may not be familiar with the terms *partitioning* (splitting equally) and *iterating* (counting a repeated amount), but as you will see, they connect to whole-number concepts you will recognize. These two actions are critical to working with fractions (Petit et al., 2016; Siebert & Gaskin, 2006) meaningfully and emphasize the numerical nature of fractions. Contexts also build meaning of fractional parts.

Equal Sharing Problems

Given students' prior experiences in division with fairly sharing items among family and friends, sharing tasks are a good place to begin the development of fractions (Siegler et al., 2010). Sharing tasks allow students to develop concepts of fractions from an activity that makes sense to them, rather than having the structure imposed on them. In this approach, students do not begin with traditional part-whole tasks of identifying halves, thirds, fourths, and so on. These fractional parts and students' understanding of fractions emerge as a result of their fair sharing in meaningful contexts (Lewis, Gibbons, Kazemi, & Lind, 2015). Sharing stories have these three features:

1. The objects in the story are easy for students to draw and partition (e.g., crackers, brownies, bagels, cookies, pancakes, stick pretzels, etc.).
2. There are no fractions in the story (the fraction emerges from the partitioning).
3. They can be solved with no formal instruction on fractions or use of fraction symbols.

Researchers recommend following a progression of sharing tasks that builds on children's understanding of whole-number quantities and informal partitioning strategies (e.g., Empson & Levi, 2011; Lewis et al., 2015; Siegler et al., 2010). Table 14.3 provides such a progression, with examples of corresponding story problems.

TABLE 14.3 PROGRESSION FOR TEACHING EQUAL SHARING PROBLEMS

Progression	Example Problems
1. Problems whose solutions are whole numbers.	Three friends want to share 15 grapes fairly. How many grapes will each child get?
2. Problems with 2, 4, or 8 sharers and whose solutions are mixed numbers (greater than 1).	Two friends want to share 5 quesadillas so that everyone gets the same amount. How much will each friend get? Four friends want to share 7 oranges fairly. How much will each friend get?
3. Problems with 2, 4, or 8 sharers and whose solutions are less than 1.	Four friends want to share 3 cookies so that everyone gets the same amount. How much will each friend get? Eight friends want to share six quesadillas. How much will each friend get?
4. Problems with 3 or 6 sharers and whose solutions are mixed numbers (greater than 1).	Three friends want to share 10 sticks of clay to make clay animals. If everyone gets the same amount, how much does each friend get? Four cookies are shared fairly among three friends. How much will each friend get?
5. Problems with 3 or 6 sharers and whose solutions are less than 1.	Three friends want to share 2 pizzas equally. How much does each friend get? Six friends bake a batch of 20 cupcakes. How much is each friend's fair share?

Questioning can help students make connections between the size of a piece and its name. Compare these two questions:

A. "How many pieces is the [brownie] cut into?"
B. "How many of these parts fit into a whole brownie?"

At first glance, these may seem similar, but the latter question focuses on the size of the piece, which is really what matters in naming the fraction.

Play dough can make sharing tasks accessible to students as young as kindergarten and engaging to students of all ages (Cwikla, 2014).

Activity 14.4 Cut Me a Fair Share!

CCSS-M: 1.G.A.3; 2.G.A.3

Give students a ball of play dough and a plastic knife. Explain that they are going to be finding a way to share a group of brownies fairly with a group of friends. Look at the progression in Table 14.3 as you select your tasks. A level 2 example might be:

Four friends want to share six brownies so that each friend gets the same amount of brownie. How much will each friend get?

To ensure that the brownies are the same size, use square cookie cutters. Ask students to make brownies and then cut them so that each friend has a fair share. The Cut Me a Fair Share activity page provides a visual of the number of brownies and the number of children, providing helpful support to many students, in particular students with special needs in math. Encourage students to share their ways of thinking through a visual representation (see the example in Figure 14.3). Next, offer a selection of other sharing tasks with different numbers of brownies and different numbers of sharers (see the Brownie Sharing Cards activity page). For a full lesson on sharing, see the Equal Sharing Stories Expanded Lesson.

Pearson eText Activity Page: Cut Me a Fair Share!
Pearson eText Activity Page: Brownie Sharing Cards
Pearson eText Expanded Lesson: Equal Sharing Stories

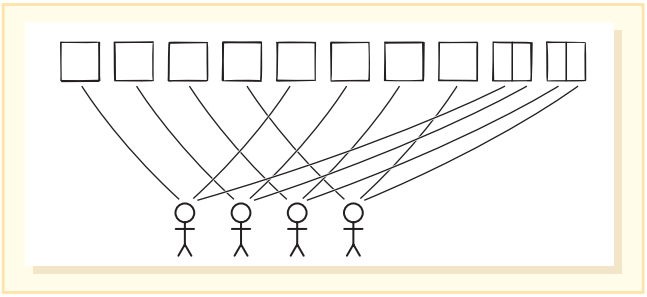


FIGURE 14.3 Ten brownies shared with four children.

Pearson eText
Video Example 14.3
What conceptual knowledge does Felisha bring to figuring out how much of a cookie is a fair share?

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

The way in which students approach sharing tasks increases in sophistication with experiences, and the insights they gain provide a strong foundation for understanding fractions. Initially, students will start sharing without relating the number of sharers to the shares. Then, they begin halving to share fairly (as in Figure 14.3). Eventually, they notice patterns, like how they partition a shape is related to the number of sharers. This pattern emerges from experience, as is illustrated by Elizabeth's work in Figure 14.4.

As you can see in the progression in Table 14.3, having 3 sharers (or 6) is more challenging because then students cannot rely on halving to get to the answer. Consider how you might solve these three examples, and then compare your thinking to the possible methods in Figure 14.5.

a. Four candy bars shared with six friends
b. Four pizzas shared with three friends
c. Five sandwiches shared with three friends

Eventually, students notice that if there are 5 sharers, they could split every cookie into fifths, and eventually notice the connection between sharing and division (i.e., that a things shared b ways gives $\frac{a}{b}$ to each sharer (Empson & Levi, 2011).

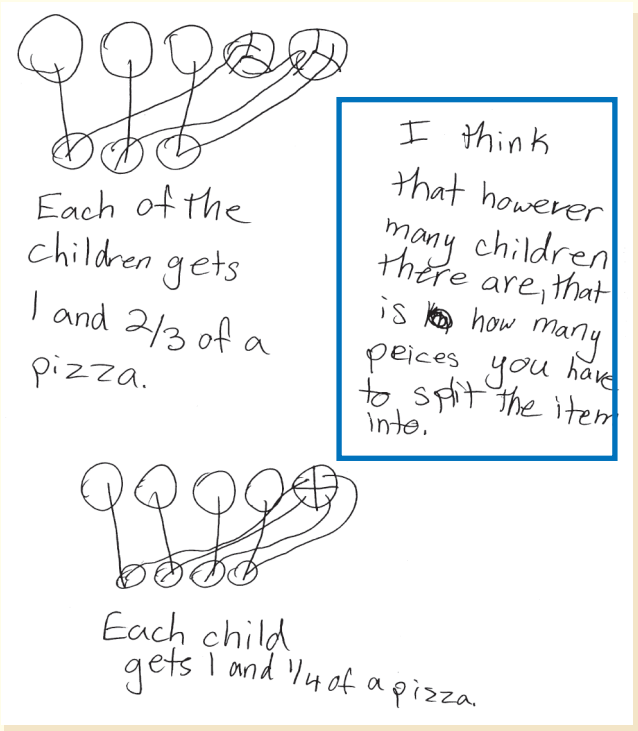

Each of the children gets 1 and 2/3 of a pizza.
I think that however many children there are, that is how many peices you have to split the item into.
Each child gets 1 and 1/4 of a pizza.

FIGURE 14.4 Elizabeth explains a pattern for finding equal shares (in box)

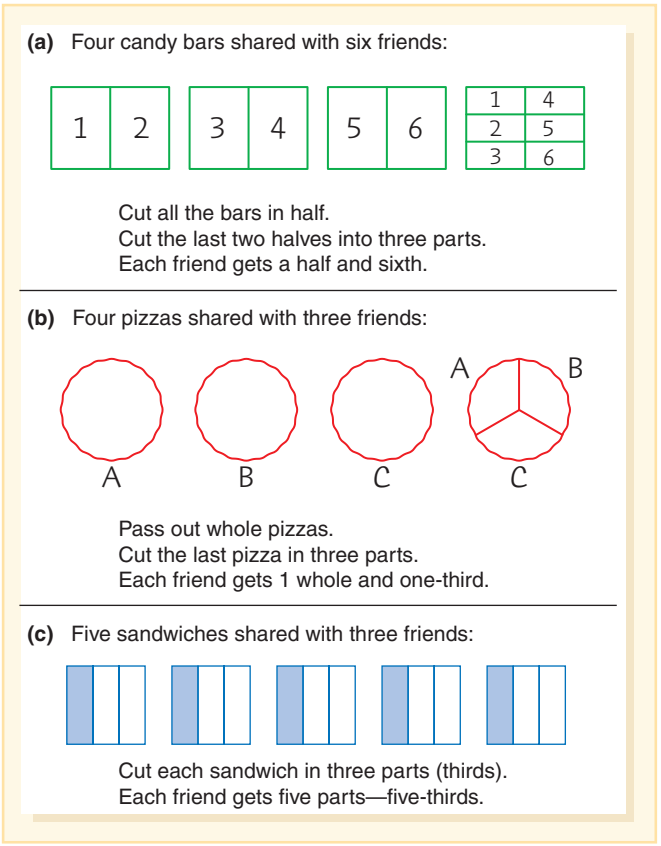
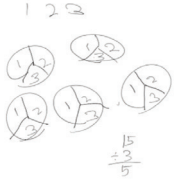

(a) Four candy bars shared with six friends:
1 2 3 4 5 6
1 4
2 5
3 6
Cut all the bars in half.
Cut the last two halves into three parts.
Each friend gets a half and sixth.
(b) Four pizzas shared with three friends:
A B C A B C
Pass out whole pizzas.
Cut the last pizza in three parts.
Each friend gets 1 whole and one-third.
(c) Five sandwiches shared with three friends:
Cut each sandwich in three parts (thirds).
Each friend gets five parts—five-thirds.

FIGURE 14.5 Three sharing strategies

Pearson eText Activity Page: Brownie Sharing Cards

Students may be at a different place in their progression of equal sharing tasks, so a way to support and challenge each student is to implement a differentiated lesson. Figure 14.6 shows how one teacher planned such a lesson (Williams, 2008). As students explore the series and then look for patterns, they discover an important pattern—the fair share (fraction) turns out to be in the form of $\frac{number\ of\ brownies}{number\ of\ people}$. Can you explain why that is the case?



Pearson eText
Video Example 14.4
In this video consider how the student solves the task of sharing five sandwiches with three friends.

Group A: Students who need more experience with halves and fourths	Group B: Students ready to solve tasks with 3 and 6 sharers, sharer gets more than 1	Group C: Students ready to solve tasks with 3 and 6 sharers, sharer gets less than 1
How can 2 people share 3?	How can 4 people share 3?	How can 3 people share 5?
How can 2 people share 5?	How can 3 people share 4?	How can 3 people share 2?
How can 4 people share 3?	How can 3 people share 5?	How can 6 people share 4?
How can 3 people share 4?	How can 6 people share 4?	How can 12 people share 4?

FIGURE 14.6 Differentiated groups for the equal-sharing tasks. Use a context such as brownies.

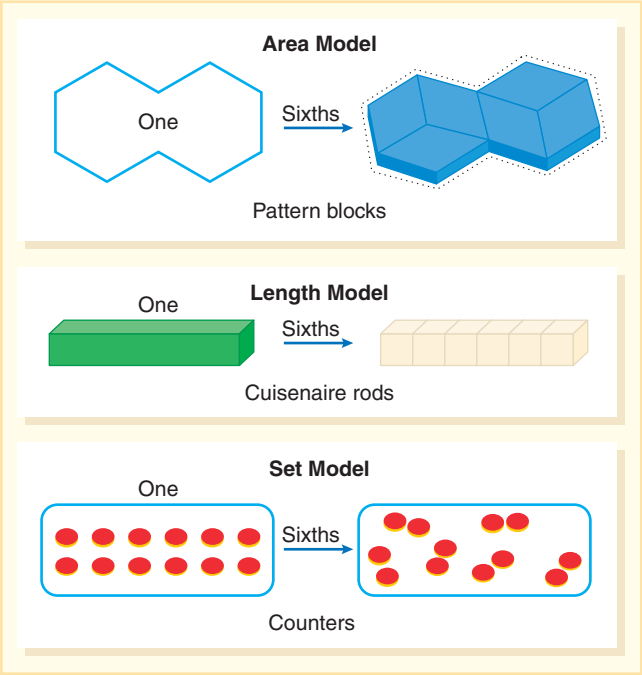
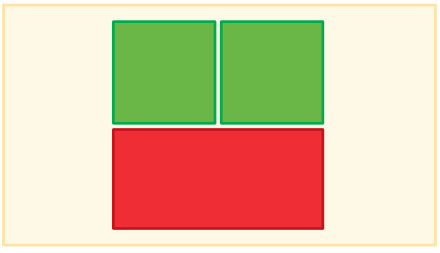


FIGURE 14.7 Showing partitioning with concrete area, length, and set models

Partitioning

Sectioning a shape or number line into equal-sized parts or lengths is called *partitioning*. In all the sharing tasks, students are partitioning quantities to find fair shares and naming the fair share. That process prepares students for general partitioning, in which they partition and can name the size of the parts. When an area is partitioned into four equal shares, the parts are called *fourths*. Too often, students see illustrations wherein representations are already partitioned into equal shares. Such pre-partitioning results in students not noticing the relationship between partitioning and the names of fractional parts (Watanabe, 2007). For area models, these pre-drawn shares also tend to be the same shape, which is not the only option. As mentioned earlier, using different models builds a stronger understanding for students. Figure 14.7 illustrates sixths across area, length, and set models.

Partitioning with Area Models. Area models are the first types of models to use in teaching fractional parts. When partitioning an area into fractional parts, students need to be aware that (1) the fractional parts must be the same size, though not necessarily the same shape; and (2) the number of equal-sized parts that can be partitioned within the unit determines the fractional amount (e.g., partitioning into 4 parts means each part is *one-fourth* of the unit). It is important for students to understand, however, that sometimes visuals do not *show* all the partitions. For example, consider the following picture:



Pearson eText
Video Example 14.5
Observe how these students reason about the size of the parts when the whole is not fully partitioned. What concept has the teacher targeted, and why is this challenging for students?

A student might think, "If I partitioned this region so that all pieces were the same size, then there will be four parts; therefore, the smaller partitioned region represents one-fourth"—not one-third, as many students without a conceptual understanding of fractional parts might suggest.

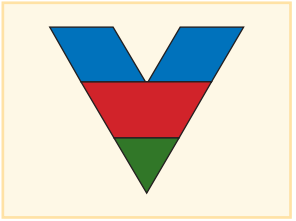
Some manipulatives, like fraction circles, can mislead students to believe inaccurately that fractional parts must be the same *shape* as well as the same size. Color tiles can be used to create rectangles that address this misconception, such as the one illustrated here:

Students who recognize that each color represents thirds understand that fractional parts must be the same size, and that the shape of each part may be different. Similarly, pattern blocks are a good tool to focus on equal shares, because each piece is *not* the same size. Recall that Activity 14.1 (Playground Fractions) focuses on equal-sized parts. Activity 14.5 is a similar task, but more challenging.

Activity 14.5 How Much Cake to Take?

CCSS-M: 1.G.A.3; 2.G.A.3; 3.NF.A.1

Using pattern blocks, create a cake with different-colored shapes. For example, explain that the cake pictured here was made for Valerie's birthday party and will be shared with four friends:



Ask, "Can we give each friend one of the four colored pieces (shapes) as their fair share?" Engage students in a discussion of why this would not be a fair share (they are different-sized pieces). Then, distribute pattern blocks to students and have them find fair shares, either by drawing partitions or trading out pieces.

In this case, the cake can be partitioned into fourths or eighths. In addition, or as a challenge, ask students to create a cake that can be shared fairly with 4 people, but using different shapes, and have them trade cakes with their peers to find the fourths.

Activity 14.6 offers an opportunity for students to justify their thinking about sketches of various shapes.

Activity 14.6 Partitioning: Fourths or Not Fourths?

CCSS-M: 1.G.A.3; 2.G.A.3; 3.NF.A.1

Use the Fourths or Not Fourths activity page showing examples and non-examples of fourths (which are very important to use with students with special needs in math) (see Figure 14.8).

Ask students to identify the wholes that are correctly divided into fourths (equal shares) and those that are not. For each response, have students explain their reasoning. Repeat with other fractional parts, such as thirds or eighths. To challenge students, ask them to draw shapes that fit each of the four categories listed on the handout (and below) for other fractional parts, such as sixths (see the Sixths or Not Sixths activity page). For multilingual students, emphasize the 'th' sound as you discuss fourths and sixths.

Pearson eText Activity Page: Fourths or Not Fourths
Pearson eText Activity Page: Sixths or Not Sixths

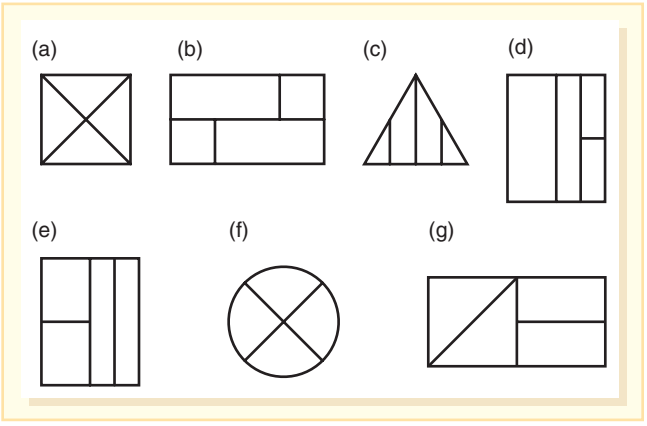


FIGURE 14.8 Which of these partitioned regions show fourths?

In Figure 14.8, the shapes fall in each of the following categories:

1. Same shape, same size [equivalent]: (a) and (f)
2. Different shape, same size [equivalent]: (e) and (g)
3. Different shape, different size [not equivalent]: (b) and (c)
4. Same shape, different size [not equivalent]: (d)

FORMATIVE ASSESSMENT Notes. Activity 14.6 is also a good diagnostic interview to assess whether students understand that it is the *size* that matters, not the shape. If students only select (a) and (f) as correct answers without considering (e) and (g), they hold the misconception that parts should be the same shape. Future tasks are needed that focus on fraction equivalence. ■

Importantly, students need many opportunities to actually partition. For example, you can ask students to partition a square into fourths themselves as many ways as they can.

Activity 14.7 Geoboard Finding Fourths

CCSS-M: 1.G.A.3; 2.G.A.3; 3.NF.A.1

Place students in groups of 3, each with their own geoboard. Explain that each geoboard in their group must show fourths in a different way (see ideas in Figure 14.1). When students have found three ways, bring them two new geoboards and ask them to find two more ways. Students may not initially think of examples where the fourths are different shapes (but still the same size). For students in grade 3, extend the discussion into naming the unit fractions. Have each student record each way on a Geoboard recording sheet.

Pearson eText Blackline Master: Geoboard Recording Sheet

Partitioning with Length Models. Partitioning with length models is also critical to understanding fractions as numbers. A good way to transition from area to length models is to use paper strips (because they still have an area) (Cramer et al., 2017). Making fraction strips helps students make sense of partitioning, and the strips themselves will be used to explore fraction iterating, equivalence, and comparison.

Activity 14.8 Make Fraction Strips

CCSS-M: 3.NF.A.1; 3.NF.A.2a, b

Cut blank paper long-wise into strips approximately one inch wide. Give each student 6 strips. Have them fold each strip to create their personal set of fraction strips:

Halves	Thirds
Fourths	Sixths
Eighths	Ninths

You may want to make other strips (fifths, tenths, twelfths). After they are folded, students use markers to draw lines where the folds are and label each section of the strip (e.g., $\frac{1}{2}$ on each section of the Halves strip). After the strips are finished, ask students to describe what they notice about their different fraction strips. To simplify the creation process, or for students who may have difficulty folding accurately, use the Fraction Strips teacher resource, cut out the strips, and have students fold on the dotted lines and label each section.

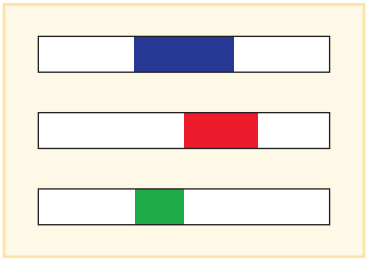
Pearson eText Blackline Master: Fraction Strips

A common partitioning activity is for students to identify a shaded section of an already partitioned area. As students explore shaded regions, provide examples where the shaded sections are in different positions, and where partitioning isn't already shown to strengthen students' understanding of equal parts.

Activity 14.9 What Fraction Is Colored?

CCSS-M: 3.NF.A.1; 3.NF.A.2a, b

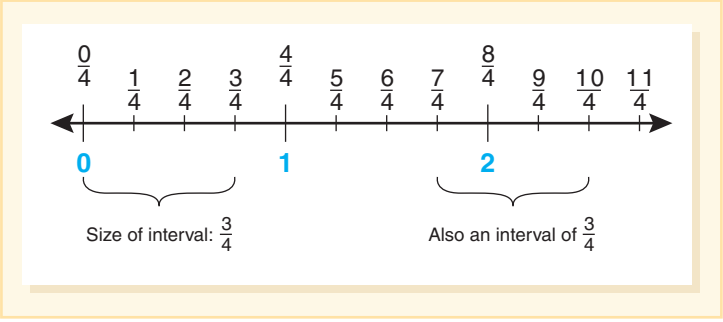
Prepare a set of paper strips prior to doing this activity. Color the strips so that they have a fractional amount shaded in various positions (not just left justified!) (Sarazen, 2012). Here are a few examples:



Give each student a paper strip and explain that it represents one whole. Ask students to explain what fraction is colored and explain how they know. A common misconception is for students to count the number of parts and call each of these colored parts one-third. If this happens, ask if the parts are the same size and, if not, to partition the strip to make same-sized parts. Use toothpicks or uncooked spaghetti to illustrate the partitions:

Students can also justify their reasoning by measuring the length of each partition.

Students must be able to partition a number line and realize that the tick mark on the number line represents the end of a unit length. For example, a student might partition a number line into fourths and label each a fourth. The marking $\frac{3}{4}$ means a length that is three units that are each one-fourth long:

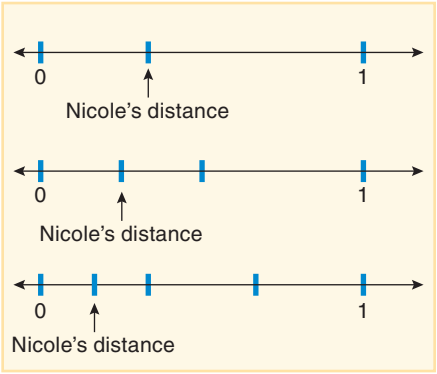


The number line is particularly useful in helping students see the relative size of a fraction, for example, that $\frac{3}{4}$ means a length of three-fourths, not a position. Therefore, an interval of $\frac{3}{4}$ can be found at various places on the number line. To help students understand that a fraction on a number line represents a length, add a length context like walking.

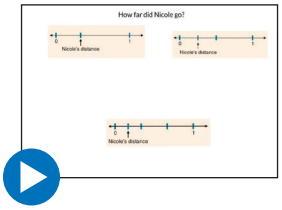
Activity 14.10 How Far Did Nicole Go?

CCSS-M: 3.NF.A.1; 3.NF.A.2a, b

Give students number lines partitioned such that only some of the partitions are showing. Use a context such as walking to school. For each number line, ask, "How far has Nicole gone? How do you know?"



Students can justify their reasoning by measuring the length of each partition.



Pearson eText

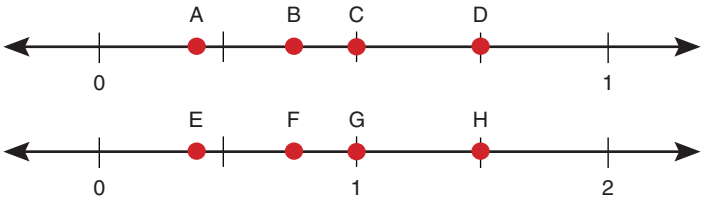
Video Example 14.6

In this video consider how the student solves the "How far did Nicole go?" problem. Pay attention to what the student does to determine each distance. What questions could you ask this student to follow up on their definition of denominator?

Locating a fractional value on a number line is particularly challenging but very important. Shaughnessy (2011) found four common errors students make in working with the number line: They (1) use incorrect notation, (2) change the unit (e.g., ignore the whole numbers on the number line), (3) count the tick marks rather than the space (units) between the marks, and (4) count only the tick marks that appear without noticing any missing ones. These challenges are evidence that we must use number lines more extensively in exploring fractions, along with formative assessment tools to monitor students' developing conceptions.

FORMATIVE ASSESSMENT Notes. Use diagnostic interviews or written prompts to see if students can locate fractions on a number line (see Figure 14.9). Ask students to locate a fraction and then ask why they placed it where they did. For more probes, see the EDC *Addressing Accessibility in Mathematics* website. ■

1. Choose the letters that show the location of $\frac{3}{8}$ on each number line



2. Place a star where the fraction $\frac{1}{2}$ goes on each number line.

3. Place a triangle where the fraction $\frac{9}{8}$ goes on each number line.

FIGURE 14.9 Diagnostic interview and/or written prompts for fractions on a number line

Partitioning with Set Models. Students can also partition with set models such as coins, counters, or baseball cards. Each share or part has two counters, but it is the number of shares that makes the partition show *sixths*, not the number of counters. As with the other models, when the equal parts are not already marked out, then students may not see how to partition effectively. Students seeing a picture of two cats and four dogs might think $\frac{2}{4}$ are cats (Bamberger, Oberdorf, & Schultz-Ferrell, 2010). Consider the following problem:

Standards for Mathematical Practice
MP7. Look for and make use of structure

Eloise has 6 trading cards, Andre has 4 trading cards, and Lu has 2 trading cards. What fraction of the trading cards does Lu have?

A student who answers "one-third" is not thinking about equal shares but about the number of people with trading cards.

Understanding that parts of a whole must be partitioned into equal-sized parts across different models is an important step in conceptualizing fractions and provides a foundation for exploring sharing and equivalence tasks, all of which are prerequisites to performing fraction operations (Cramer, Ahrendt, Monson, Wyberg, & Colum, 2017).

Iterating

Counting fractional parts, or *iterating*, helps students understand the relationship between the parts (the numerator) and the whole (the denominator) and to understand a fraction as a number (not two numbers). In whole-number learning, counting precedes and helps students to add and later subtract. This importance of the counting process is also true with fractions. And in the same way that students count by ones before they skip count, iterating with fractions begins with unit fractions (in grade 3).

Unit Fractions. Understanding unit fractions centers on the big idea that one whole can be seen both as one whole and as a grouping of same-size unit-fraction amounts (Empson et al., 2020). A unit fraction has a one in the numerator. The number in the denominator tells the number of partitions to make in the whole. Time spent developing the meaning of unit fractions is time well spent, because developing the concept of a unit fraction is quite challenging for students but essential for all other fraction concepts.

Iterating can help develop the meaning of unit fractions and fractions in general. The purpose of iterating activities is for students to develop the measurement interpretation of fractions, specifically that $\frac{3}{4}$, for example, can be thought of as a count of three parts called fourths (Post, Wachsmuth, Lesh, & Behr, 1985; Siebert & Gaskin, 2006; Tzur, 1999).

Iterating can be done concretely by using a single piece, representing one unit or one share, and repeating it a number of times to measure how many are in the whole. To develop a flexible understanding of unit fractions, children need multiple opportunities to reason about the relationship between unit fraction amounts and one whole in different ways (Empson et al., 2020). After experiences iterating, students should be able to answer the question, "How many fifths are in one whole?" just as they know how many ones are in ten. However, only 44% of fourth graders answered this question correctly on the 2008 National Assessment of Educational Progress (NAEP) math test (Rampey, Dion, & Donahue, 2009), hence the need to increase attention on iterating and engage students in iterating tasks across multiple models.

Iterating with Area Models. Iterating can be done with area models, for example counting quesadilla slices and using circular fractional pieces, as illustrated in Figure 14.10. Or students can make their own circular fraction pieces by cutting out colored paper circles and/or colored paper plates and labeling each set (McCoy, Barnett, & Stine, 2016; Monson et al., 2020).

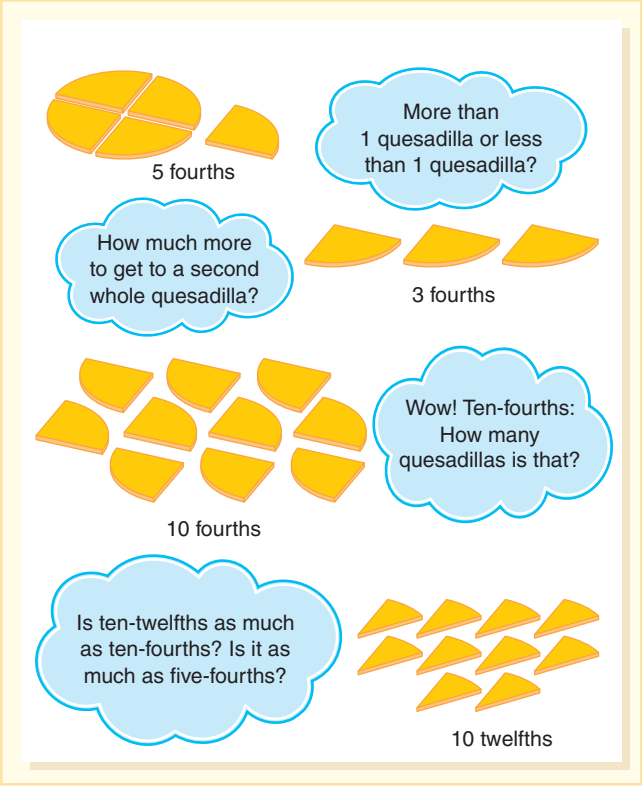

5 fourths
More than 1 quesadilla or less than 1 quesadilla?
How much more to get to a second whole quesadilla?
3 fourths
Wow! Ten-fourths: How many quesadillas is that?
10 fourths
Is ten-twelfths as much as ten-fourths? Is it as much as five-fourths?
10 twelfths

FIGURE 14.10 Iterating fractional parts in an area model

Pearson eText Activity Page: Circular Fractional Pieces

Activity 14.11 Paper Plate Pieces

CCSS-M: 3.NF.A.1; 3.NF.A.3c; 3.NF.A.3d

You will need different-colored paper plates (2 or 3 of each), a bag for each same-colored set, and magnets to glue on the backs of the pieces. To prepare, mark the backs of the plates with how they need to be cut. Give each group of students a bag of plates, magnets, and a card with a fraction with a denominator that matches the size of pieces they have (see the teacher resource). For example, if the group has eighths, their card might be $\frac{16}{8}$. Include fractions less than, equal to, and greater than 1.

Have each group (1) cut out the pieces, (2) glue on the magnets (or you can do this in advance), and (3) post on the board the number of pieces that match the fraction on their card (post in rows, not formed into circles). For example, the group with $\frac{16}{8}$, will post 16 eighths pieces.

As a class, count one group's set together. For example, if one group posted five-fourths, count "One-fourth, two-fourths,... five-fourths." After counting, ask questions about the quantities, like "How do we know if we have one-whole? More than a whole? Two wholes?" Figure 14.10 provides other questions.

Pearson eText Blackline Master: Fraction Cards

Activity 14.12 involves creativity. Students are given a unit fraction and asked to build a creature of a particular size, using pattern blocks. The challenge helps students focus on the size of the parts, not on the number of pieces or partitions (Champion & Wheeler, 2014; Ellington & Whitenack, 2010).

Activity 14.12 Pattern Block Creatures

CCSS-M: 3.NF.A.1; 3.NF.A.3c

Ask students to build a pattern block creature that fits a set of rules (a creature represents one-whole) (see the Pattern Block Creatures activity page). These rules begin with creating a same-color creature, for example:

- The red trapezoid is one-fourth of the creature.
- The blue parallelogram is one-sixth of the creature.

Then ask students to build mixed-colored creatures. For example:

- The yellow hexagon is one-half of the creature. Use three colors.
- The blue parallelogram is one-sixth of the creature. Use at least two colors.

After students create their creatures, pair them up to prove to their partner they have a whole creature by counting the parts (e.g., counting by fourths or sixths).

To challenge students, ask them to write their own rules for their own pattern block creature. Instruct them to trace the outline of their creature on a piece of paper and write their rule next to the creature, such as "The red trapezoid is _____ of my creature." Students can trade with other students to see whether they can figure out the fractional amounts for each other's creatures.

Pearson eText Activity Page: Pattern Block Creatures

Iterating with Length Models. Iterating makes sense with length models because iterating is like measuring. Cuisenaire rods are a good way to introduce iterating and exploring different-sized wholes and the related size of unit fractions.

Activity 14.13 Row of Rods: More, Less, or Equal to One Whole?

CCSS-M: 3.NF.A.1; 3.NF.A.2a, b

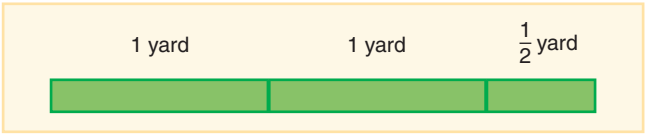
Give students the A Collection of Fractional Parts activity page and a collection of Cuisenaire rods. Indicate the kind of fractional part they have. For example, a collection might have seven light-green rods with a task card that says, "Each light-green rod is $\frac{1}{8}$." Students decide whether the collection of rods is less than one whole, equal to one whole, or greater than one whole. Ask students to draw pictures or use symbols to explain their answer. Ask, "Why does it take more fourths to make a whole than thirds do?" "Why did we get two wholes with four halves but just a little more than one whole with four thirds?" You can connect to equivalencies by asking, "What is another way to describe ten-fourths?" (Answers include: 2 wholes and 2 fourths, 1 whole and 6 fourths, or 2 wholes and one-half).

Pearson eText Activity Page: A Collection of Fractional Parts

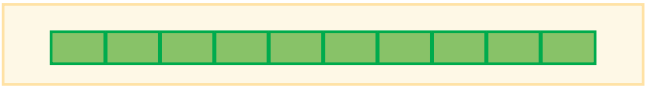
Fraction strips are also very useful in exploring iteration. Students can work together and combine their strips to show quantities greater than one whole and sketch various lengths (e.g., 5 fourths, 15 eighths, etc.). Measurement problems provide an authentic context for iterating with lengths (i.e., a fraction strip can represent a yard).

Drew has $2\frac{1}{2}$ yards of ribbon and needs pieces that are one-fourth of a yard for making bows. How many pieces can they cut?

To begin, students draw a strip (or line) representing $2\frac{1}{2}$ yards:



Next, the strip is partitioned into fourths (the unit) and counted—10 fourths:



The number line is an effective model for iterating with situations involving fractions greater than 1. As students count unit lengths, you can ask:

What does the numerator tell us?

What does the denominator tell us?

What might a fraction equivalent to 1 whole look like? Equivalent to 2?

Here are some likely explanations for the numerator and denominator from third graders:

- The numerator is the counting number. It tells how many parts we have counted.
- The denominator tells what size piece is being counted. For example, if there are four parts in a whole, then we are counting *fourths*.

Iterating with Set Models. Iteration can also be done with set models; the key is to focus on what quantity represents the whole.

Activity 14.14 Counting Counters

CCSS-M: 3.NF.A.1; 3.NF.A.3c

Give students 12 counters and distribute the Counting Counters: Find the Part activity page. Have students use their counters to answer the related problems:

1. How many counters are in one-fourth of a set? [partitioning]
2. How many counters are in three-fourths of the set? [iterating]

As students count by threes, also count by fractional parts: one-fourth (three counters), two-fourths (six counters), three-fourths (nine counters).

Then reverse: give the part and ask students to find the whole (using the other activity page). These problems are like puzzles for students. For example: "Two counters represent $\frac{1}{8}$ of my set; how many in my whole set?" For students with special needs in math or students who may need additional experiences, use more unit fractions than non-unit fractions.

Pearson eText Activity Page: Counting Counters: Find the Part
Pearson eText Activity Page: Counting Counters: Find the Whole

Iterating Abstractly. Finally, iterating can be done abstractly. Choral counting, often done as a routine with whole numbers, is an excellent routine to use with fractions (McMillan & Sagun, 2020). For example, students can count by three-fifths, starting at 0. These authors suggest preparing the list of counts in an array; the first row counts from 0 to $\frac{12}{5}$, the next row starts at $\frac{15}{5}$ and goes to $\frac{27}{5}$ and continues to count by units of 3 fifths for five rows. Students begin by rote counting and then look for patterns. These patterns help students make sense of equivalence and build foundations for fraction operations.

Calculators are also effective in exploring iterating.

Activity 14.15 Calculator Fraction Counting

CCSS-M: 3.NF.A.1; 3.NF.A.3a, c

Many calculators display fractions in correct fraction format and offer a choice of showing results as mixed numbers or simple fractions. Ask students to type in a fraction (e.g., $\frac{1}{4}$) and then + and the fraction again. To count, press 0 Op1, Op1, Op1, repeating to get the number of fourths wanted. The display will show the counts by fourths as well as the number of times that the Op1 key has been pressed. Ask students questions such as the following: "How many fourths to get to 3?" "How many fifths to get to 2?" These problems can get increasingly challenging: "How many fourths to get to 3?" "How many thirds to get to 6? Estimate and then count by two-thirds on the calculator." Students, particularly students with special needs in math, can coordinate their counts with a concrete representation (e.g., Cuisenaire rods) or a number line.

Partitioning and Iterating. In many problems, partitioning and iterating go hand in hand. You partition to find the unit fraction and then count by the unit fraction. That is the focus of Activity 14.16.

Activity 14.16 A Whole Lot of Fun

CCSS-M: 3.NF.A.1; 3.NF.A.2a, b

Use the A Whole Lot of Fun activity page and the Cutouts for Fraction Strips activity page like the one here:

Tell students that this strip is three-fourths of one whole. Ask students to sketch strips of the other lengths on their paper (e.g., $\frac{1}{2}$ of the whole, $\frac{5}{4}$ of the whole, etc.). You can repeat this activity by selecting other lengths for the starting amount (e.g., $\frac{2}{3}$) and selecting different fractional values to sketch. A context, such as height of a plant, is effective in helping students make sense of the situation. Be sure to use fractions less than and greater than 1, and mixed numbers.

Pearson eText Activity Page: A Whole Lot of Fun
Pearson eText Activity Page: Cutouts for A Whole Lot of Fun

An engaging way to help students make the connection between partitioning and iterating is to have them estimate an equal length (e.g., one-third of a pretzel stick) and then check to see if they are accurate. Estimating one share and seeing whether it is accurate builds understanding of the relative size of unit fractions (Tzur & Hunt, 2015).

Activity 14.17 Estimate and Iterate

CCSS-M: 3.NF.A.1; 3.NF.A.2a, b

Select a length context such as pretzel sticks, licorice sticks, or ribbon. Distribute 8–10 white paper strips to each pair of students, and one colored paper strip to represent the whole [pretzel stick]. Special rule: no paper folding! Hand out an Estimate and Iterate task card from the activity page. For example: Shares for 3 friends

Without using folding, the students cut a piece from one of their additional strips that they estimate to be one-third of the length. To check their work, they iterate it to see if three measures of that unit are the same length as the whole [pretzel]. If the cut piece is too short or too long, they cut another piece until they find a piece that is approximately accurate. They label the piece as one-third and/or $\frac{1}{3}$. As they work, ask, "What strategies are you using to estimate the length of your piece?" "Will your next try be a longer piece or a shorter piece? Why?" Challenge students with other fractional amounts such as sixths and eighths. For multilingual learners, emphasize the *th* sound to distinguish from counting whole numbers (e.g., fourths not fours For multilingual learners).

Pearson eText Activity Page: Estimate and Iterate Task Cards

FORMATIVE ASSESSMENT Notes. The tasks in Figure 14.10 can be used as performance assessments. If students are able to solve these types of tasks, they can partition and iterate. That means they are ready to do equivalence and comparison tasks. If they are not able to solve problems such as these, provide a range of similar tasks, using real-life contexts and involving area, length, and set models. ■

Magnitude of Fractions

Magnitude of fractions means knowing the relative size of the fractions, meaning how big it is compared to other numbers (e.g., on a number line). Increasingly, research is finding that understanding fraction magnitude is critical to student success. For example, the level of a student's understanding of the magnitude of fractions predicts their success in algebra (Booth, Newton, & Twiss-Garrity, 2014) and leads to better performance with overall mathematics achievement (Siegler, Thompson, & Schneider, 2011; Torbeyns, Schneider, Ziqiang, & Siegler, 2015). In fact, interventions focused on the magnitude of fractions show promise for improving student success with fraction operations (Tian & Siegler, 2017). The use of contexts, models, and mental imagery can help students build a strong understanding of the magnitude of fractions (Bray & Abreu-Sanchez, 2010; Lemonidis & Kaiafa, 2019; Petit et al., 2016; Tobias, 2014).

As with whole numbers, students are often less confident and less capable of estimating than they are at finding exact answers, yet a focus on estimation can strengthen their understanding of fractions (Clarke & Roche, 2009; Resnick et al., 2016). Have students approximate fractions for shaded regions and points on the number line, asking students, "About how much?" and having them justify their answers (see Figure 14.11).

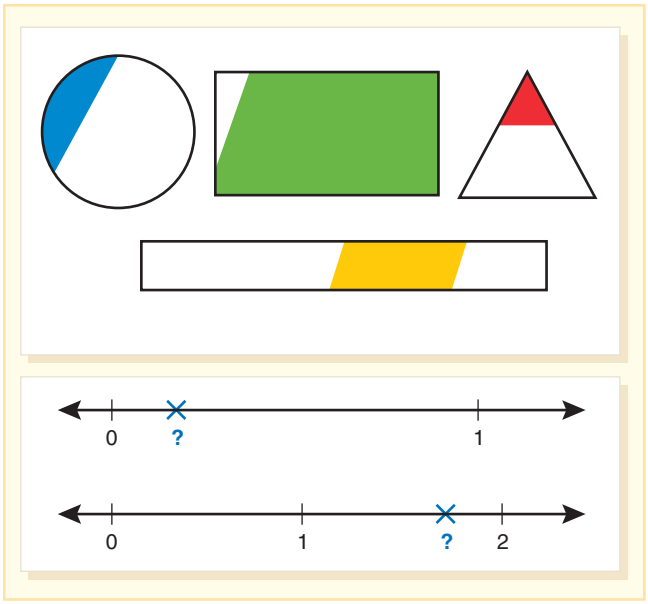


FIGURE 14.11 Example visuals for estimating fractional amounts

For set examples, ask questions like, "About what fraction of your classmates are wearing sweaters?" Or, after tallying survey data about a topic like favorite dinner meal, ask, "About what fraction of our class picked spaghetti?"

Students need many opportunities to explore magnitude across models, including significant attention to the number line. A progression to support students making sense of the number line is to ask students to represent fractions first with circles, then using paper strips, and finally with number lines (Monson et al., 2020). The number line is particularly important for helping students develop a better understanding for the relative size of a fraction (Petit et al., 2016). Students who are able to locate fractions on a number line accurately are likely to excel in mathematics throughout elementary and middle school; conversely, students who cannot accurately place fractions on a number line are likely to struggle in mathematics (Resnick et al., 2016). Activity 14.18 can be used as a routine or full class lesson to engage students in reasoning about the relative size of fractions on a number line.

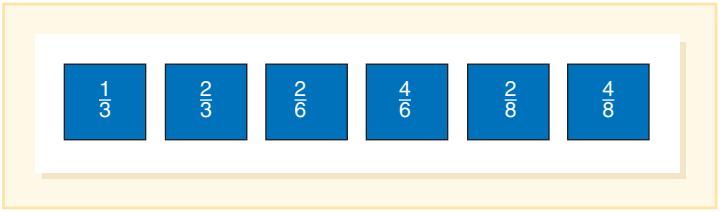
Activity 14.18 Fractions *on* the Number Line

CCSS-M: 3.NF.A.2a; 3.NF.A.2b

Use a clothesline or rope as long as the room. (This works well in the hall, gym, or outside, too!) Have two students stand and hold the end of the rope and number cards. For example, the student on the left end can hold 0 and the student on the right end can hold 1 (these end points can change to other values). Give out a variety of fraction cards (appropriate for your group of students) with clips or clothespins. Ask a volunteer to clip their fraction *on* the number line. Ask, "Is this fraction in the approximately correct location?" "How do you know?" Students can fold the rope to determine benchmark fractions such as $\frac{1}{4}$, $\frac{1}{2}$, or $\frac{3}{4}$ and use reasoning strategies to determine approximate locations on the number line. Continue to have students place their fractions on the number line. This activity can be used as a routine, each time varying the cards used, for example, including equivalent fractions and decimals and even variable expressions (see Bay, 2001).

Pearson eText Blackline Master: Fraction Cards

After experiences with number lines and other models, students need opportunities to reason about the relative size of fractions, using reasoning strategies and considering what benchmark (i.e., 0, $\frac{1}{2}$, 1) they are near. The 2019 NAEP math test asked students to sort these fractions as less than $\frac{1}{2}$, equal to $\frac{1}{2}$, and greater than $\frac{1}{2}$:



Less than 32% of fourth graders were able to place 4 or more of these correctly (https://www.nationsreportcard.gov/mathematics/sample-questions/?grade=4). Activities such as the one here, combined with visual tools and discussions, are ways we can better support student learning.

Activity 14.19 Zero, One-Half, or One

CCSS-M: 3.NF.A.3d; 4.NF.A.2

Make a set of fraction cards for each group of students. Have students sort the cards into close to 0, close to $\frac{1}{2}$, or close to 1. Ask students to explain how they are using the numerator and denominator to decide. For multilingual learners, be sure the term *benchmark* is understood and encourage illustrations as well as explanations. As an extension or alternative to differentiate this activity, ask students to create their own fractions close to each benchmark fraction.

Pearson eText Blackline Master: Fraction Cards

Notice that there have been 19 activities to help make sense of fractions. That is an indication of how very important it is that students understand what a fraction is before they begin comparing and computing with fractions.

Equivalent Fractions

Learner Outcome 14.4 **Illustrate the concept of equivalence across fraction models.**

As discussed in Chapter 13, equivalence is a critical but often poorly understood concept. This is particularly true with fraction equivalence. Fraction equivalence and comparisons are major topics in grades 3 and 4 (see Table 14.1) and must be well understood to compute fractions.

Pearson eText
Video Example 14.7
In this video late author John Van de Walle describes important ideas about equivalent fractions, which are also illustrated in a whole class lesson.

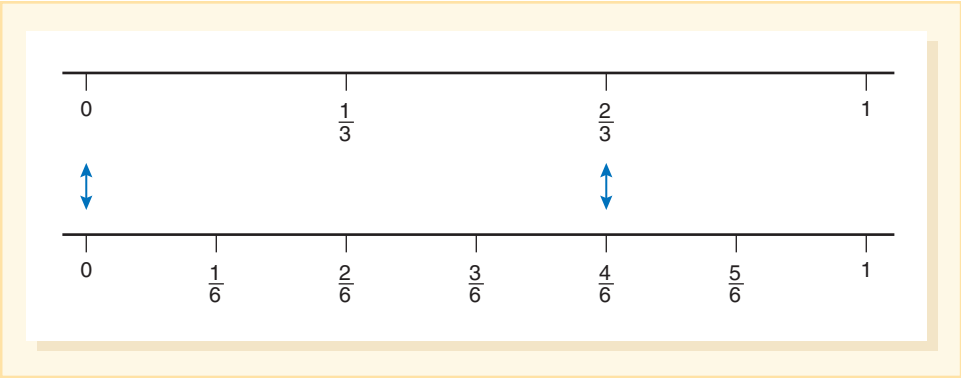
Conceptual Focus on Equivalence

Pause & Reflect

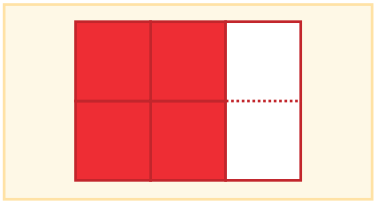
How do you know that $\frac{4}{6} = \frac{2}{3}$? Before reading further, think of at least two explanations. ●

Here are some possible answers:

1. They are equal because you can simplify $\frac{4}{6}$ and get $\frac{2}{3}$.
2. On a number line, $\frac{2}{3}$ and $\frac{4}{6}$ are the same length.



3. If you start with $\frac{2}{3}$, you can multiply the numerator and denominator by 2, and that will equal $\frac{4}{6}$.
4. If you cut a square into 3 parts and you shaded 2 parts, that would be $\frac{2}{3}$ shaded. If you cut all 3 of those parts in half, that would be 4 parts shaded and 6 parts in all. That's $\frac{4}{6}$, and it would be the same amount.



Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

5. If you have a set of 6 items and you take 4 of them, that would be $\frac{4}{6}$. But you can make the 6 into 3 groups, and the 4 would be 2 groups out of the 3 groups. That means it's $\frac{2}{3}$.

All of these answers are correct. But let's think about what they tell us. Responses 2, 4, and 5 are conceptual, and 1 and 3 are procedural:

> *Concept:* Two fractions are equivalent if they are representations for the same amount or quantity—if they are the same number.
>
> *Procedure:* To get an equivalent fraction, multiply (or divide) the numerator and denominator by the same nonzero number.

Students must understand equivalence (grade 3 focus) and understand why the procedure makes sense (grade 4 focus). Rushing too quickly to the algorithm can impede students' conceptual understanding of fractions and fraction equivalence. Students need to see the connections between the concept and the procedure, and this requires multiple experiences (and time). Be patient!

Equivalent Fraction Activities across Models

To help students build a strong and flexible understanding of equivalent fractions requires the use of many contexts and models in which the goal is to rename a fractional amount. This is the first time in students' experience that they are seeing that a fixed quantity can have more than one name (and fractions actually have an infinite number of names).

Equivalency with Area models. Area models are a good place to begin understanding equivalence (see examples in Figure 14.12). And many concrete manipulatives can support such reasoning.

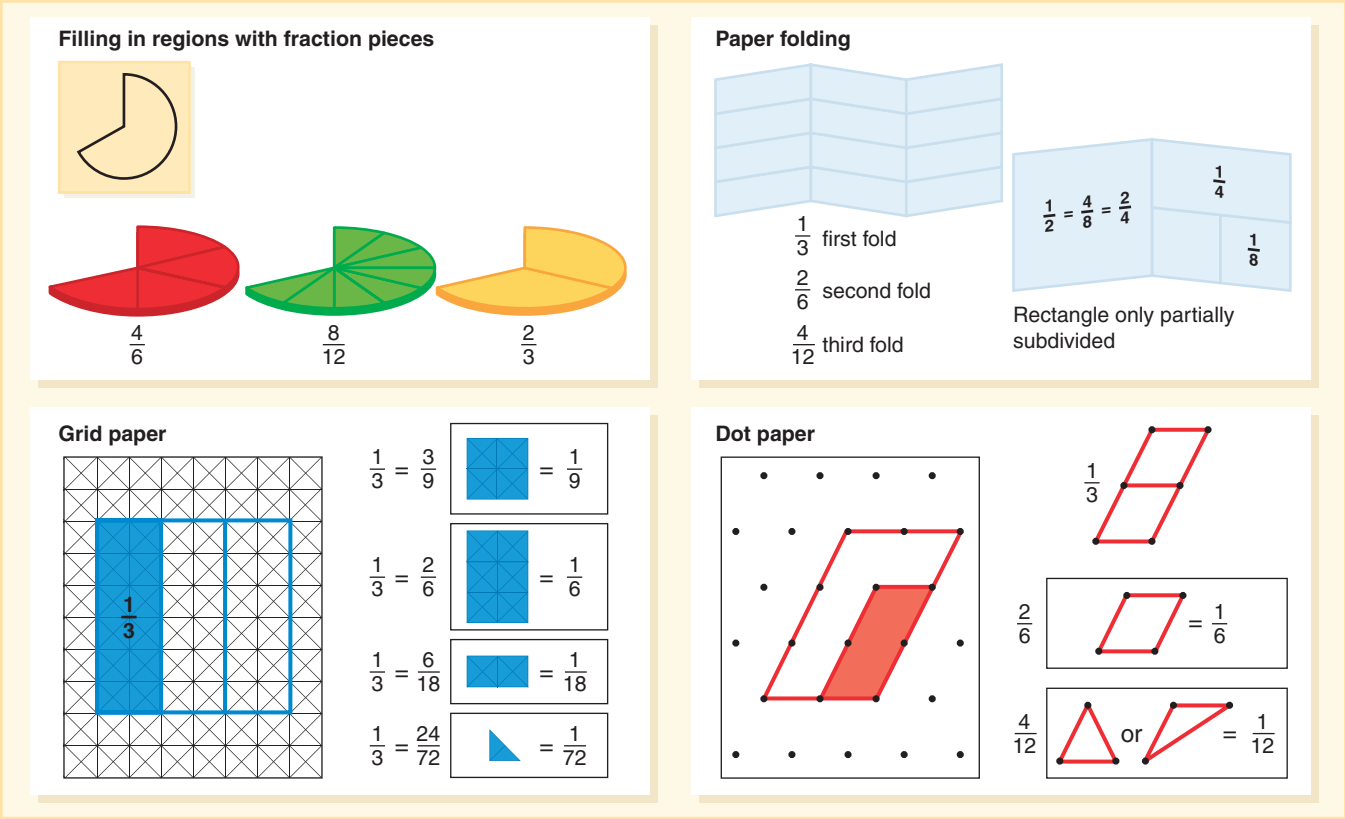


FIGURE 14.12 Area models for equivalent fractions

Activity 14.20 Making Stacks

CCSS-M: 3.NF.A.1; 3.NF.3a, b, c

Select a manipulative that lends itself to exploring fractions as an area (e.g., pattern blocks or fraction circles). Prepare fraction cards with different fractional amounts, such as $\frac{2}{3}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $\frac{3}{2}$, $\frac{4}{3}$, and 2 (see teacher resource). Students use their manipulative to (1) determine a whole, (2) build the fraction on their card with pieces by stacking it on top of the whole, and (3) find as many equivalent fractions as possible, stacking each new equivalency on top. For example, with fraction circles and a card of $\frac{3}{4}$, a student picks the whole circle as the whole, places 3 of the fourths on top of the whole, and then looks for other same-sized pieces to stack on top. In this case, 6 eighths works. Ask for students to tell a story to explain the equivalency. For example, if they ate three-fourths of a pizza, it is the same amount as six-eighths, just cut in different-sized pieces. Repeat with other fraction cards, including fractions that are greater than 1.

Pearson eText Teacher Resource: Fraction Cards

In a follow-up classroom discussion, ask students to consider what other equivalencies are possible (and justify their thinking) beyond what they could build with their manipulative. For example, ask, "What equivalent fractions could you find for $\frac{3}{4}$ if we had sixteenths in our fraction pieces? If you could have a piece of any size at all, what other fraction names are possible?" or "If the pizza was cut into 16 equal-sized pieces, what would your equivalent portion be?"

The next activity moves from using manipulatives to using sketches on paper, opening up more fraction equivalencies for each illustration.

Activity 14.21 Dot Paper Equivalencies

CCSS-M: 3.NF.A.1; 3.NF.3a, b, c

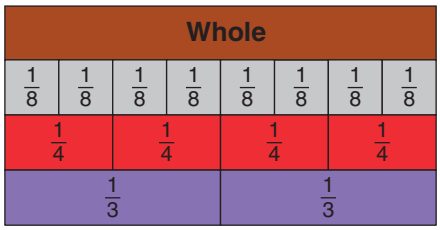
Use the Fraction Names activity page, which includes three different grids with a fraction shaded (each enclosed area represents one whole). Ask students how many fraction names they think the first problem has. Then ask them to see how many they can find (working individually or in partners). Invite students to share and explain the fraction names they found for problem 1. Repeat for the next two problems. Alternatively, cut this page into three task cards, laminate the cards, and place each at a station along with an overhead pen. Have students rotate in partners to a station and see how many fraction names they can find for that shape (using the pen as needed to show their ways). You or your students can create your own pictures, using your choice of grid paper or dot paper (see Blackline Masters 5–11). Figure 14.12 shows an example on isometric dot paper.

Pearson eText Activity Page: Fraction Names
Pearson eText Expanded Lesson: Dot Paper Equivalencies
Pearson eText Blackline Master: 1-Centimeter Square/Diagonal Grid Paper
Pearson eText Blackline Master: 1-Centimeter Isometric Dot Paper

The Dot Paper Equivalencies activity involves what Lamon (2020) calls *unitizing*—that is, given a quantity, finding different ways to chunk the quantity into parts to rename it. She points out that this is a key ability in understanding equivalent fractions.

Equivalency with Length Models. Length models must be used to develop understanding of equivalent fractions. Asking students to locate $\frac{2}{5}$ and $\frac{4}{10}$ on a number line, for example, can help them see that the two fractions are equivalent, but the line is partitioned differently (Siegler et al., 2010). Cuisenaire rods are a versatile, concrete model. For example, as illustrated in Figure 14.13(a), you can select the brown piece as the whole, construct same-color trains to find fraction parts, and then look for equivalencies. Similarly, you can have students pull out their set of fraction strips (Figure 14.13(b)) and find equivalencies such as one-fourth.

(a) Cuisenaire Rods to show equivalencies of halves, fourths, and eighths.



(b) Fraction strips to look for equivalencies

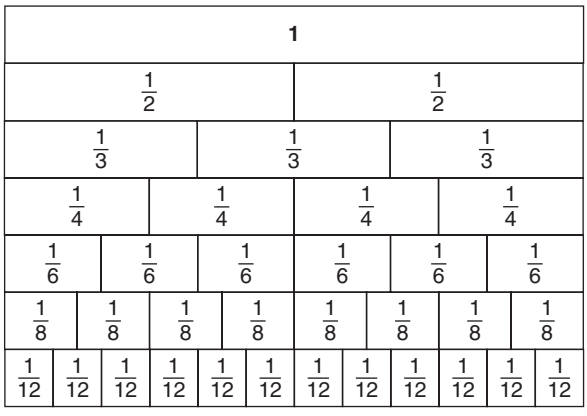


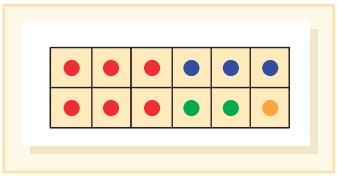
FIGURE 14.13 Length models for exploring equivalent fractions

Equivalency with Set Models. Legos, a highly motivating manipulative, can help students learn to write fraction equivalencies. (For an excellent elaboration on this idea, see Gould, 2011.) Lego bricks can be viewed as an area (array) or as a set. (Students can count the studs.)

Activity 14.22 LEGOLAND®: Building Options

CCSS-M: 2.G.A.3; 3.NF.A.1; 3.NF.3a, b, c; 4.NF.B.3a, b

Hand out one 2-by-6 LEGO brick to each student. Ask them to describe their brick (there are 12 studs, two rows of 6). This one piece is the foundation. Give students a collection of other, smaller bricks and ask them to select ones that exactly cover their brick. One possible solution is shown here:



After they have built their Legoland, ask students to tell the fraction of their land that is represented by each piece they used (e.g., the 2-by-6 is $\frac{6}{12}$ as well as $\frac{1}{2}$ and $\frac{2}{4}$).

To focus on iteration and build connections to addition (grade 4), students can write equations to describe their Legoland. In the one pictured here, that would be

$$\frac{6}{12} + \frac{3}{12} + \frac{2}{12} + \frac{1}{12} \text{ OR } \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}.$$

Note that students may have difficulty naming fractions' parts, naming the blue part as $\frac{1}{3}$ rather than $\frac{1}{4}$ because they see three pieces (Wilkerson, Bryan, & Curry, 2012). This becomes a good topic for a classroom discussion: What is the fractional value of the blue pieces (simplified)?

Two-color counters are an effective tool for fraction equivalencies, using sets. With set models, the goal is to create same-sized subsets of the whole set, the focus of Activity 14.23.

Activity 14.23 Apples and Bananas

CCSS-M: 3.NF.A.1; 3.NF.3a, b, c

Use the Apples and Bananas activity page or just have students set out a specific number of counters in two colors—for example, 24 counters—with 16 of them red (apples) and 8 of them yellow (bananas). The 24 counters make up the whole. Ask students to group the apples and bananas into equal-sized groups, such that a group has to have only apples or only bananas. Challenge them to find as many ways as they can. For each that they find, record the fraction that are apples and the fraction that are bananas. Counters can be arranged in different groups or in arrays (see illustrations in Figure 14.14) to see patterns. Recording the possibilities in a table helps students notice patterns.

Pearson eText Activity Page: Apples and Bananas

In the activities so far, there has only been a hint of a rule for finding equivalent fractions. For Apples and Bananas, the problem can be extended to help build this connection by showing the equivalencies. For example, the group of bananas was described with these fractions:

$$\frac{1}{3} \quad \frac{2}{6} \quad \frac{3}{9} \quad \frac{4}{12} \quad \frac{8}{24}$$

If you pick two of these fractions and the related illustrations, you can see that as the size of the group gets smaller (e.g., is half the size), the number of groups gets bigger (e.g., is doubled). Hence, when you double the number of total groups, you also double the groups of bananas (so $\frac{1}{3} = \frac{2}{6}$).

Activity 14.24 begins to build the connection between the concept and the procedure by having students choose a representation.

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

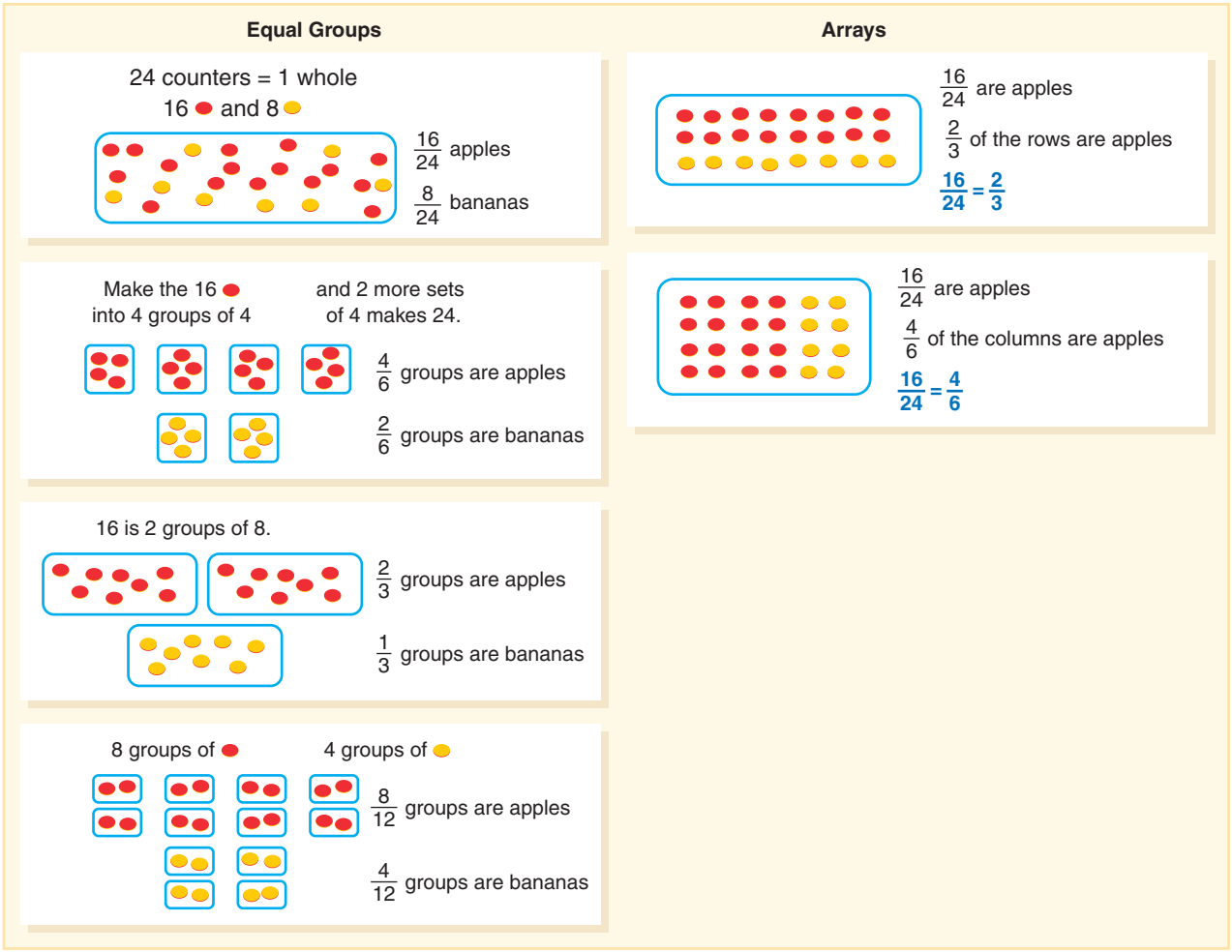

Equal Groups
Arrays
24 counters = 1 whole
16 and 8
16/24 apples
8/24 bananas
16/24 are apples
2/3 of the rows are apples
16/24 = 2/3
Make the 16 into 4 groups of 4
and 2 more sets of 4 makes 24.
4/6 groups are apples
2/6 groups are bananas
16/24 are apples
4/6 of the columns are apples
16/24 = 4/6
16 is 2 groups of 8.
2/3 groups are apples
1/3 groups are bananas
8 groups of
4 groups of
8/12 groups are apples
4/12 groups are bananas

FIGURE 14.14 Finding equivalent fractions with Apples and Bananas

Activity 14.24 Missing-Number Equivalencies

CCSS-M: 3.NF.3a, b, c; 4.NF.A.1

Use the Missing-Number Equivalencies activity page or give students an equation expressing an equivalence between two fractions, but with an unknown value. Ask students to use their fraction strips or counters, or draw pictures to find the equivalent fraction. Example equations:

$$\frac{5}{3} = \frac{\ }{6} \qquad \frac{2}{3} = \frac{6}{\ } \qquad \frac{8}{12} = \frac{\ }{3} \qquad \frac{9}{12} = \frac{3}{\ }$$

The missing value can be in a numerator or a denominator; the missing number can be either larger or smaller than the corresponding part of the equivalent fraction. Figure 14.15 illustrates how Zachary represented the equivalencies with equations and by partitioning rectangles. The examples shown above involve simple whole-number multiples between equivalent fractions. Next, consider pairs such as $\frac{6}{8} = \frac{\ }{12}$ or $\frac{9}{12} = \frac{6}{\ }$. In these equivalencies, one denominator or numerator is not a whole-number multiple of the other. Include equivalencies for whole numbers and fractions greater than one: $\frac{8}{x} = \frac{6}{6}$ $\frac{10}{3} = \frac{x}{9}$

Pearson eText Activity Page: Missing-Number Equivalencies

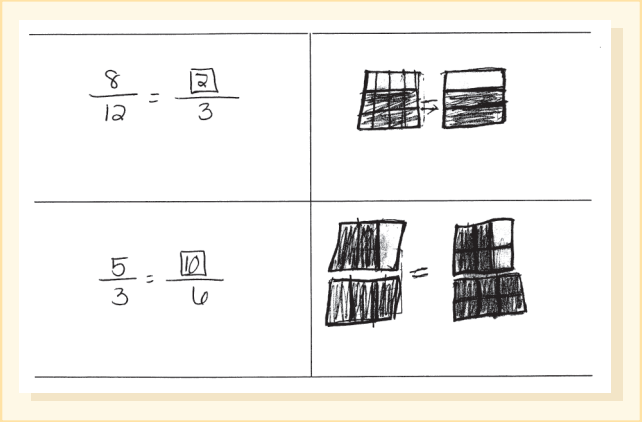


FIGURE 14.15 Zachary illustrates equivalence fractions by partitioning rectangles

Fractions Greater than 1

Students must be able to move flexibly between recording fractions greater than 1 (e.g., $\frac{13}{6}$) and mixed numbers (e.g., $2\frac{1}{6}$). If students have iterated (counted) by fractional parts beyond one whole, they are more likely to recognize this equivalence, without needing an algorithm to shift the notation. If they have not had these experiences with iterating, you must start there (this could be a daily routine). Connecting cubes is very effective in showing how fractions greater than one can be written as whole numbers or mixed numbers (Neumer, 2007) (see Figure 14.16). Students identify one cube as the unit fraction $(\frac{1}{5})$ for the problem $(\frac{12}{5})$. They count out 12 fifths and build wholes. Conversely, they could start with the mixed number, build it, and find out how many total cubes (or fifths) were used. Repeated experiences in building and naming the values both ways will help students notice a pattern that actually explains the algorithm for moving between notating quantities as mixed numbers and as fractions greater than 1.

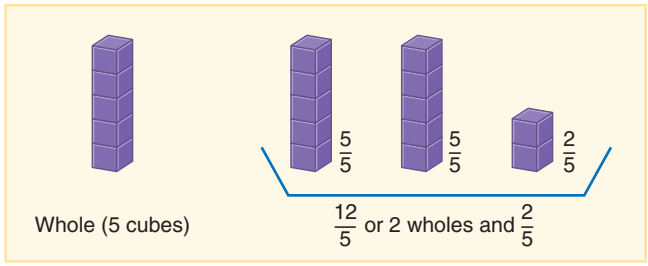


FIGURE 14.16 Connecting cubes are used to represent the equivalence of $\frac{12}{5}$ and $2\frac{2}{5}$

Challenge students to figure out the two equivalent forms by just picturing the stacks of connecting cubes in their heads. A good explanation for $3\frac{1}{4}$ might be that there are 4 fourths in one whole, so there are 8 fourths in two wholes and 12 fourths in three wholes. The extra fourth makes 13 fourths in all, or $\frac{13}{4}$. (Note the iteration concept playing a role.) Do not rush to an algorithm, because it can interfere with students making sense of the relationship between the two representations. It is not appropriate just to tell students how to convert a mixed number to a fraction without understanding. Allow students to make this generalization through multiple experiences and the identification of patterns.

Pearson eText
Video Example 14.8
In this video consider how the teacher moves from concrete experiences to mental images to support students' understanding of the relationship between mixed numbers and fractions.

Developing an Equivalent-Fraction Algorithm

When students understand that fractions can have different (but equivalent) names, they are ready to develop a method for finding equivalent names for a particular value. An area model is a good visual for connecting the concept of equivalence to the standard algorithm for finding equivalent fractions (multiply both the numerator and denominator by the same number to get an equivalent fraction). Ask students to look for a pattern in the way that the fractional parts in both the part and the whole are counted.

Activity 14.25 Garden Plots

CCSS-M: 3.NF.3b; 4.NF.A.1

Have students draw a square garden on blank paper or give each student a square of paper (like origami paper). Begin by explaining that the garden is divided into rows of various vegetables. In the first example, you might illustrate four rows (fourths) and designate $\frac{3}{4}$ as corn. Ask students to partition their square into four rows and shade three-fourths, as in Figure 14.17. Then explain that the garden is going to be shared with family and friends so that each person gets a harvest that is $\frac{3}{4}$ corn. Show how the garden can be partitioned horizontally to represent two people sharing (eighths). Ask what fraction of the newly divided garden is corn $(\frac{6}{8})$. Next, tell students to illustrate ways that more friends can share the garden (they can choose how many friends, or you can). For each newly divided garden, ask students to record what fraction of the garden is corn, using a table, equations, or both:

People sharing garden	1	2	3	4	...
Sections of corn	3	6			
Sections of garden	4	8			

Ask students to look for patterns in their diagrams and their table.

Pearson eText Activity Page: Garden Plots Recording Page

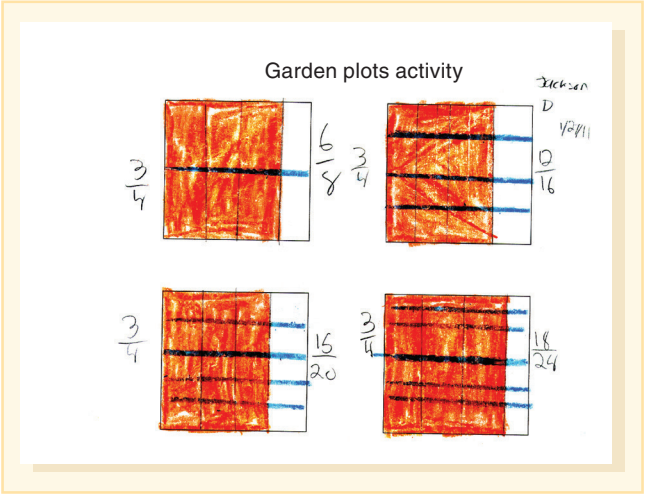


FIGURE 14.17 A third grader partitions a garden to model fraction equivalencies

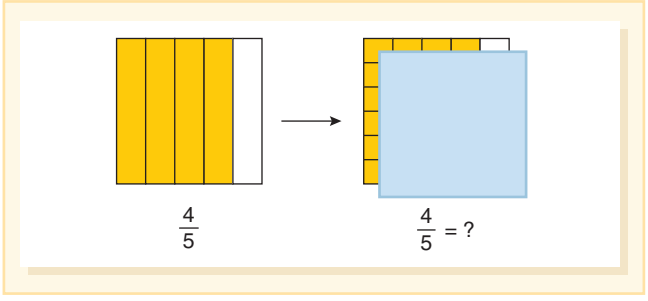


FIGURE 14.18 Connecting partitioning to the algorithm for generating equivalent fractions

The partitioning helps students see the connection to multiplication. With varied experiences, partitioning using the area model can help build the meaning of the algorithm. Figure 14.18, for example, shows an area (garden) of $\frac{4}{5}$ has been partitioned into six rows, but the total area is covered. You can ask students, "How can you count the fractional parts if you cannot see them all?"

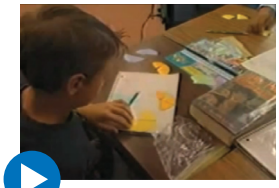
With the square covering part of the array, students can see that there are four columns and six rows to the shaded part, so there must be 4×6 parts shaded. Similarly, there must be 5×6 parts in the whole. Therefore, another name for $\frac{4}{5}$ is $\frac{4 \times 6}{5 \times 6}$, or $\frac{24}{30}$.

Writing Fractions in Simplest Terms. The multiplication scheme for equivalent fractions produces fractions with larger denominators. But creating equivalent forms also involves division, in particular as we often record quantities in their *simplest terms*. This means to write the fraction so that numerator and denominator have no common whole-number factors. Linear models like fraction strips and the number line are a good way to help students see that the simplest way is with the fewest partitions (compare $\frac{1}{2}$ to $\frac{4}{8}$). As you teach students to simplify fractions:

1. **Do not** use the phrase *reducing fractions*. Although this terminology used to be common, it has been removed from standards and textbooks because it is misleading. (The fractions are equivalent, not reduced!)
2. **Do not** mark answers wrong when the student does not simplify. For $\frac{1}{6} + \frac{1}{2}$, *both* $\frac{2}{3}$ and $\frac{4}{6}$ are correct answers.

Multiplying and Dividing by One. Mathematically, equivalence is based on the multiplicative identity (1). As we learn through iterating, any fraction of the form $\frac{n}{n}$ is equivalent to 1 and can be used as the identity. Therefore, $\frac{3}{4} = \frac{3}{4} \times 1 = \frac{3}{4} \times \frac{2}{2} = \frac{6}{8}$. Furthermore, the numerator and denominator of the identity element can also be fractions. In this way, $\frac{6}{12} = \frac{6}{12} \times \left(\frac{1/6}{1/6}\right) = \frac{1}{2}$. Understanding this idea is an expectation in the CCSS-M in grade 4.

***TECHNOLOGY* Note.** The DreamBox Learning (dreambox.com) teacher tool, "Equivalent Fractions on the Number Line," engages students in making sense of equivalent fractions by placing markers at set intervals along a race course. ■



Pearson eText
Video Example 14.9
In this video consider how the teacher uses manipulatives to support students' understanding of equivalent fractions.

Comparing and Ordering Fractions

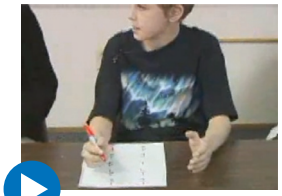
Learner Outcome 14.5 **Describe strategies for comparing fractions and ways to teach this topic conceptually.**

When students are looking to see whether two or more fractions are equivalent, they are comparing them. Smith (2002) suggests that the comparison question to ask is, "Which of the following two (or more) fractions is greater, or are they equal?" (p. 9). He points out that this question leaves open the possibility that two fractions that may look different can, in fact, be equal. Therefore, activities for equivalence can (and should) be adapted and used for comparing fractions. For example, with fraction strips, students can find fractional amounts and decide whether they are equal and, if not, identify the larger fraction.

Comparing Fractions, Using Number Sense

A limited understanding of fractions can make it difficult or impossible to compare the relative size of fractions. NAEP results found that only 21 percent of fourth-grade students could explain why one unit fraction was larger or smaller than another—for example, $\frac{1}{5}$ and $\frac{1}{4}$ (Kloosterman et al., 2004). For eighth graders, only 41 percent were able to put in order three fractions, given in simplified form (Sowder, Wearne, Martin, & Strutchens, 2004). This research builds a strong case for using *number sense* to compare fractions! That is the focus of the rest of this section.

Using Contexts. For the purpose of comparing, rectangles work better than circles, because rectangles are easier for students to draw more accurately, and they can be connected to the number line. Contexts might include filling a pitcher, a garden, a tray of brownies, and so on. But length models, in particular the number line, are an important next step in comparing fractions. A number line is one-dimensional, so it can make comparisons more visible. Contexts that fit linear comparisons include how far people have walked/run; length of hair/banner; and height of plants growing. Fraction strips are an excellent visual. Several earlier activities involve locating fractions on a number line. Activity 14.26 is more explicitly focused on comparing fractions.



Pearson eText
Video Example 14.10
In this video consider what strategies the student uses to compare fractions.

Activity 14.26 Stretching Number Lines

CCSS-M: 3.NF.A.3a, b, d; 4.NF.A.2

Cut strips of elastic (about 1 meter or yard in length). Hold the elastic taut and mark off ten partitions on each. Hand one elastic out to each pair of students. Ask students to use their stretching number line to find the place on a table that represents the fraction of the distance across the table. For each pair, ask: Ask "Which distance is greater, or are they equal?"

1a. $\frac{5}{10}$ of the distance across.
1b. $\frac{1}{2}$ of the distance across.
2a. $\frac{3}{10}$ of the distance across.
2b. $\frac{3}{8}$ of the distance across. (Note: they have to rethink the whole as 8 sections.)
3a. $\frac{3}{4}$ of the distance across. (Note: they have to rethink the whole as 8 sections.)
3b. $\frac{6}{8}$ of the distance across. (Note: they have to rethink the whole as 8 sections.)

After exploring a few examples, challenge students to come up with their own sets that include at least one pair that is equivalent and one pair that is not. Students can then trade with someone to determine which fraction is greater (or equivalent).

Comparing Unit Fractions. Students' whole-number knowledge does support their fraction understanding, but it can also interfere with comparing fractions. Students think, "Seven is more than four, so sevenths should be bigger than fourths." The inverse relationship between number of parts and size of parts cannot be told; it must be developed through many experiences, including ones described earlier related to partitioning, iterating, magnitude of fractions, and equivalence.

Activity 14.27 Visualizing and Ordering Unit Fractions

CCSS-M: 3.NF.A.3d; 4.NF.A.2

List a set of unit fractions such as $\frac{1}{3}, \frac{1}{8}, \frac{1}{5}$, and $\frac{1}{10}$ (assume same size whole for each fraction). Ask students to use reasoning to put the fractions in order from least to greatest. Challenge students to explain their reasoning with an area model (e.g., circles) *and* on a number line. Ask students to connect the two representations. ("What do you notice about $\frac{1}{3}$ of the circle and $\frac{1}{3}$ on the number line?")

Repeat with all numerators equal to some number other than 1.

Repeat with fractions that have different numerators and different denominators (grade 4). You can vary how many fractions are being compared to differentiate the task.

Standards for Mathematical Practice
MP7. Look for and make use of structure

Students may notice that larger denominators mean smaller fractions (this is an important pattern to notice), but it only holds true when the numerators are the same, and students can overgeneralize this idea. Therefore, it makes a good conjecture to explore as a class.

Comparing Any Fractions. You have probably learned specific algorithms for comparing two fractions, such as finding a common denominator or using cross-multiplication. Too often, these procedures are poorly understood and done with no thought about the size of the fractions. Additionally, they are often *not* the most efficient strategy. Comparing fractions must take advantage of students' understanding of fractions.

Pause & Reflect

Examine the pairs of fractions in Figure 14.19 and select the largest of each pair, using a reasoning approach that a fourth grader might use. ●

Activity 14.28 Which Fraction Is Greater?

CCSS-M: 4.a.NF.2

Use the Which Is Greater? activity page (which includes the problems illustrated in Figure 14.19). Ask students to use a reasoning strategy to determine which fraction is greater. For selected problems, ask students to explain how they determined their answer. Discuss the different reasoning strategies, asking students, "For what types of fractions does that strategy work?"

Pearson eText Activity Page: Which Is Greater

Which Is Greater?

Use an efficient strategy to compare.
Tell how you know which fraction is greater.

A.	$\frac{4}{5}$ or $\frac{4}{9}$	G.	$\frac{7}{12}$ or $\frac{5}{12}$
B.	$\frac{4}{7}$ or $\frac{5}{7}$	H.	$\frac{3}{5}$ or $\frac{3}{7}$
C.	$\frac{3}{8}$ or $\frac{4}{10}$	I.	$\frac{5}{8}$ or $\frac{6}{10}$
D.	$\frac{5}{3}$ or $\frac{5}{8}$	J.	$\frac{9}{8}$ or $\frac{4}{3}$
E.	$\frac{3}{4}$ or $\frac{9}{10}$	K.	$\frac{4}{6}$ or $\frac{7}{12}$
F.	$\frac{3}{8}$ or $\frac{4}{7}$	L.	$\frac{8}{9}$ or $\frac{7}{8}$

FIGURE 14.19 Comparing fractions, using reasoning strategies

Many comparisons, such as problems D and H, can be solved using several of the strategies listed, providing an excellent opportunity for students to share their thinking and evaluate which reasoning strategies work well for which types of fractions. Figure 14.20 illustrates one student's reasoning for A–F.

Here we share four ways to compare mentally. Students fluent in comparing fractions choose the best method, based on the fractional values being compared. The first two are the focus in grade 3, and the full set is a goal in grade 4.

1. *Same-size denominators.* To compare $\frac{3}{8}$ and $\frac{5}{8}$, think about having 3 of 8 parts versus 5 of the 8 parts.
2. *Same numerators.* Consider the case of $\frac{3}{4}$ and $\frac{3}{7}$. If a whole is partitioned into 7 parts, the parts will certainly be smaller than if partitioned into only 4 parts.
3. *More than/less than a benchmark.* To compare $\frac{3}{7}$ and $\frac{5}{8}$, notice that $\frac{3}{7}$ is less than $\frac{1}{2}$ and $\frac{5}{8}$ is more than $\frac{1}{2}$. Therefore, $\frac{5}{8}$ is the larger fraction.
4. *Closeness to a benchmark.* Why is $\frac{9}{10}$ greater than $\frac{3}{4}$? It is closer to 1: $\frac{9}{10}$ is only $\frac{1}{10}$ away from 1, whereas $\frac{3}{4}$ is $\frac{1}{4}$ away from 1.

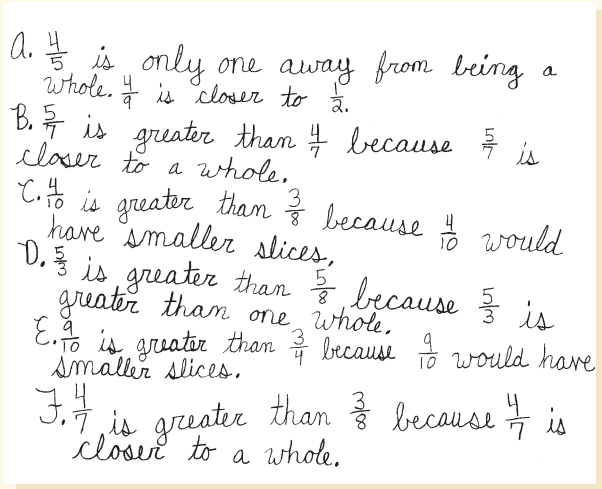

A. 4/5 is only one away from being a whole. 4/9 is closer to 1/2.
B. 5/7 is greater than 4/7 because 5/7 is closer to a whole.
C. 4/10 is greater than 3/8 because 4/10 would have smaller slices.
D. 5/3 is greater than 5/8 because 5/3 is greater than one whole.
E. 9/10 is greater than 3/4 because 9/10 would have smaller slices.
F. 4/7 is greater than 3/8 because 4/7 is closer to a whole.

FIGURE 14.20 One student explains how they compared the fractions in problems A through F from Figure 14.19

These strategies should emerge from your students' reasoning to tell students the four ways to compare fractions defeats the purpose of learning to choose efficient strategies. Instead, select pairs of fractions that will likely elicit desired comparison strategies. Use contexts such as distance from home to school (Barlow, 2018). Use number lines and other visuals to help students reason mentally. For example, you might pose two pairs of fractions with the same numerators. Ask students to tell which is greater, why they think so, and then justify their prediction, using a visual and/or a context.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Using Equivalent Fractions to Compare

Equivalent-fraction concepts can be used in making comparisons when the fractions don't lend themselves to the reasoning strategies described earlier. This, too, can be an activity involving reasoning strategies by asking students how they might change one or both of the fractions so they can compare them. Consider how you might compare $\frac{6}{8}$ to $\frac{4}{5}$. Beyond

changing both fractions to find a common denominator (and use the preceding strategy 1), they can change only the $\frac{4}{5}$ to $\frac{8}{10}$ so that both fractions are two parts away from the whole (and use strategy 4), or change both fractions to a common *numerator* of 12 (and use strategy 2). Imagine the fraction number sense students will develop when this is how fraction comparisons are approached.

Cross-multiplication (cross products) is a shortcut of the common denominator approach. The idea is that finding the common denominator results in the numerators being multiplied by the other denominator. This method, however, has several disadvantages: (1) it is hard for students to understand and (2) students overuse it when they could use a reasoning strategy and (3) it is confusing when there are more than two fractions to compare.

Standards for Mathematical Practice MP1.

Ultimately, students need an opportunity to compare and order fractions, choosing an efficient method. You can help them by helping them consider when they will use a strategy. For example, having them ask themselves:

- Are the numerators or denominators the same? If yes, I will use that information to compare.
- If not, can I use a benchmark to compare?
- If not, then I will find a common denominator.

Otherwise, we end up with students who are using cross-multiplication to compare fractions like $\frac{5}{13}$ and $\frac{5}{3}$, and that is a student who isn't using their number sense.

Addressing Challenges Related to Fraction Concepts

Because the teaching of fractions is so important, and because fractions are often not well understood even by adults, a recap of the big ideas, as well as challenges with fractions, is needed. Hopefully, you have recognized that one reason fractions are not well understood is that there is a lot to know about them, from part-whole relationships to division constructs, and understanding includes representing across area, length, and set models and includes contexts that fit these models. Many of these strategies may not have been part of your own learning experience, but they must be part of your teaching experience so that your students can fully understand fractions and be successful in algebra and beyond.

Teaching Recommendations

Iterating and partitioning must be a significant aspect of fraction instruction. Equivalence, including comparisons, is a central idea for which students must have sound understanding and skill. Connecting visuals with the procedure and not rushing the algorithm too soon are important aspects of the process. Clarke and colleagues (2008) and Cramer and Whitney (2010), researchers of fraction teaching and learning, offer research-based recommendations that provide an effective summary of this chapter:

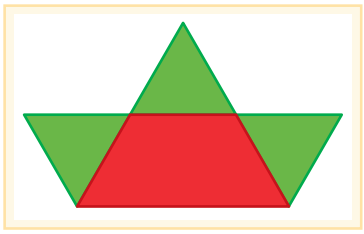
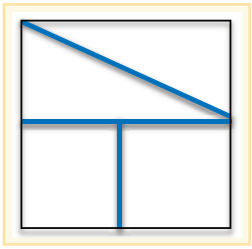
1. Give a greater emphasis to number sense and the meaning of fractions rather than to rote procedures for manipulating them.
2. Provide a variety of models and contexts to represent fractions.
3. Emphasize that fractions are numbers, making extensive use of number lines in representing fractions.
4. Spend whatever time is needed for students to understand equivalencies (concretely and symbolically), including flexible naming of fractions.
5. Link fractions to key benchmarks and encourage estimation.

Challenges and How to Help

Students build on their prior knowledge. Competence with whole numbers, in particular multiplication and division of whole numbers, supports fraction understanding (Namkung, Fuchs, & Koziol, 2018). When students encounter situations with fractions, they naturally

use what they know about whole numbers to solve the problems but can overgeneralize in ways that lead to errors in reasoning (Cramer & Whitney, 2010; Eichhorn, 2018; Lewis & Perry, 2017; Siegler et al., 2010). As you have read throughout this chapter, the most effective way to help students reach higher levels of understanding is to use multiple representations, including making sense of fractions on the number line. Common student misconceptions and challenges and how to help students navigate them are presented in Table 14.4.

TABLE 14.4 COMMON MISCONCEPTIONS AND CHALLENGES RELATED TO FRACTION CONCEPTS

misconceptions and Challenges	What It Looks Like	How to Help
1. Numerator and denominator are separate numbers (not seeing a fraction as a number)	$\frac{3}{4}$ is interpreted as a 3 over a 4. Student cannot locate a fraction such as $\frac{3}{4}$ on a number line because they don't think it is a number.	• Do sharing tasks! See Activity 14.4, for example. • Find fraction values on a number line (e.g., a warm-up activity each day, when students place values on a classroom number line). • Measure with inches to various levels of precision (e.g., to the nearest fourth, eighth, or sixteenth). • Avoid the phrase "three out of four" (unless referring to ratios or probability) or "three over four"; instead, say "three-fourths" (Siebert & Gaskin, 2006).
2. Fractional parts do not need to be equal-sized	Student says $\frac{3}{4}$ (three-fourths) of the figure below is green. 	• Have students create their own representations of fractions across various types of models (e.g., Activities 14.1, 14.12). • Provide problems, like the one illustrated here, in which all the partitions are not already drawn, and have students draw or show the equal-sized parts (e.g., Activity 14.9).
3. Fractional parts must be same shape	Student says this square is not showing fourths: 	• Provide examples and non-examples with partitioned shapes (e.g., Activity 14.6). • Ask students to generate as many ways as they can to show fourths (or eighths) (e.g., Activity 14.7).
4. Fractions with larger denominators are bigger	Student thinks $\frac{1}{5}$ is smaller than $\frac{1}{10}$ because the whole number 5 is less than 10.	*For misconceptions 4 and 5:* • Use contexts. For example, ask students whether they would rather go outside for $\frac{1}{2}$ of an hour, $\frac{1}{4}$ of an hour, or $\frac{1}{10}$ of an hour and to explain why. • Use visuals, such as paper strips or circles, to visualize the approximate size of each fraction. • Teach estimation and benchmark strategies for comparing fractions.
5. Fractions with larger denominators are smaller	Student thinks $\frac{1}{5}$ is more than $\frac{7}{10}$ because fifths are bigger than tenths.	
6. Representing fractions greater than 1	Student cannot represent a fraction such as $\frac{11}{8}$ because it is more than one whole.	• Infuse fractions of various magnitudes (less than, equal to, and greater than 1) from the beginning. • Iterate unit fractions on a number line and count beyond 1.
7. Determining a whole when given a part	Student is given an illustration and told this represents $\frac{3}{4}$, and they cannot create the one whole amount.	• Do more tasks involving this skill (see Activities 14.14 and 14.16). • Encourage students to draw a number line or use another tool. • Focus on unit fractions (what does *one-fourth* looks like?) and then figure out the whole.

misconceptions and Challenges	What It Looks Like	How to Help
8. Representing fractions on a number line: Ignoring the unit	Students uses the entire number line as the unit (rather than focusing on the distance between 0 and 1), for example, placing $\frac{3}{4}$ on a number line from 0 to 2 halfway between the 1 and the 2.	• Begin with an area model and then translate it to the number line. Rectangles or fraction strips are a good option. • Do more tasks involving the number line (many are shared in this chapter!).
9. Representing fractions on a number line: Focusing on tick marks	When locating $\frac{3}{4}$ between 0 and 1, student draws four tick marks and considers the third tick mark to be the correct location (though it is actually showing 3 fifths).	• Use Cuisenaire rods and paper strips to focus attention on the length of the interval or unit (not the tick marks). • Use measurement contexts such as feet, miles, etc.

RESOURCES FOR CHAPTER 14

WRITING TO LEARN

1. Describe the benefits of using equal sharing tasks.
2. Give examples of representations and contexts that fall into each of the three categories of fraction models (area, length, and set).
3. What does partitioning mean? Explain and illustrate.
4. What does iterating mean? Explain and illustrate.
5. What are two ways to build the conceptual relationship between $\frac{11}{4}$ and $2\frac{3}{4}$?
6. Explain why the number line is so important to developing fraction concepts.
7. Describe two ways to compare $\frac{5}{12}$ and $\frac{5}{8}$ mentally.

LITERATURE CONNECTIONS

Apple Fractions

Pallotta (2002)

Interesting facts about apples are connected to fractions as fair shares. Words for fractions are used and connected to fraction symbols, making it a good connection for fractions in grades 1–3.

Halfsy: A Math Adventure

Deitelbaum (2020)

Halfsy is on a journey from 0 to 1 (a length context for fractions), and on the way notices half of other things: the moon (area), birds (quantity).

The Doorbell Rang

Hutchins (1986)

The story is about sharing cookies, a changing situation as more and more children arrive. A great connection to sharing tasks.

The Lion's Share

McElligott (2009)

The story involves halving a rectangular cake over and over again, and provides a visual opportunity to explore unit fractions.

The Man Who Counted: A Collection of Mathematical Adventures

Tahan (1993)

This book contains a story, "Beasts of Burden," about a wise mathematician, Beremiz, and the narrator, who are traveling together on one camel. They are asked to figure out how to give each brother his share of 35 camels: *one-half to one*, *one-third to one*, *and one-ninth to one* (a set context).

RECOMMENDED READINGS

Articles

Clarke, D. M., Roche, A., & Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School*, *13*(7), 373–380.

Ten excellent tips for teaching fractions are discussed and favorite activities are shared. An excellent overview of teaching fractions.

Monson, D., Cramer, K., & Ahrendt, S. (2020). Using models to build fraction understanding. *Mathematics Teacher: Learning & Teaching PreK-12*, *113*(2), 117–123.

This article provides excellent tasks and visuals to develop a strong understanding of fractions, including fractions on the number line.

Freeman, D. W., & Jorgensen, T. A. (2015). Moving beyond brownies and pizzas. *Teaching Children Mathematics*, *21*(7), 412–420.

This article describes students' thinking as they compare fractions. In the more4U pages, they offer excellent sets of tasks with a range of contexts, each set focusing on a different reasoning strategy for comparing fractions.

Books

Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals.* Portsmouth, NH: Heinemann.

With many tasks and examples of student work, this book is a must-have for supporting students' emerging concepts of fractions and decimals.

Lamon, S. (2020). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies* (4th ed.). Oxford, UK: Taylor & Francis Group.

This book unpacks each of the fraction constructs with strong guidance to help students better understand fractions, and includes many excellent tasks.

Neagoy, M. (2017). *Unpacking Fractions*. Alexandria & Reston, VA: ASCD and NCTM.

This teacher-friendly book truly unpacks the ideas addressed in this chapter and includes many activities, ideas for using fraction strips and the number line, and useful teaching tips.

CHAPTER

15 Developing Fraction Operations

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you will be able to:

- **15.1** Describe a process for teaching fraction operations with understanding.
- **15.2** Use representations and strategies to teach adding and subtracting fractions effectively.
- **15.3** Design a developmental progression for teaching fraction multiplication conceptually.
- **15.4** Explain division of fractions, using both measurement and partitive examples.

A fifth-grade student asks, "Why is it that when we multiply 29 times two-ninths, the answer goes down?" (Taber, 2002, p. 67). Whole-number understandings support fraction understanding, and yet students need many conceptual experiences with fractions so that they can answer questions such as the one posed here. Also critical to student understanding of fraction operations is a firm understanding of the relative size of a fraction (magnitude) and equivalent fractions (discussed in Chapter 14) (Petit et al., 2010; Siegler et al., 2010).

Unfortunately, fraction computation is often presented in a rule-based manner, with little attention to conceptual understanding or reasoning (Alajmi, 2012). The result of such a rule-based approach is that far too many students achieve poorly on fraction computation and estimation assessments (e.g., Braithwaite, Tian, & Siegler, 2018; Siegler & Pyke, 2013; Lortie-Forgues, Tian, & Siegler, 2015; Newton, Willard, & Teufel, 2014). Yet, success with fraction operations predicts success in algebra and in mathematics in general through high school (Bailey, Hoard, Nugent, & Geary, 2012; Siegler et al., 2012; Torbeyns et al., 2015). This chapter presents a concept-based approach to teaching each of the fraction operations.

BIG IDEAS

- The meanings of each operation with fractions are the same as the meanings for the operations with whole numbers; therefore, effective instruction builds on students' prior knowledge.
- Visuals, including the number line, are critical to helping students understand the operations.
- Estimation must be an integral part of computation development to keep students' attention on the meanings of the operations and the expected sizes of the results.

- There are many ways to add and subtract fractions. The process assumes the fractions refer to the same size whole and then involves adding or subtracting like-sized parts (i.e., parts with the same denominator).
- Skip-counting and area models support development of concepts and algorithms for multiplication of fractions.
- Partitive (sharing) and measurement (equal subtraction) are two different thought processes for division of fractions.

Understanding Fraction Operations

Learning Outcome 15.1 **Describe a process for teaching fraction operations with understanding.**

Building an understanding of fraction operations takes time! State standards writers, such as the authors of the CCSSS Mathematics standards acknowledge this fact and include a multi-year developmental sequence (NGA Center & CCSSO, 2010). Table 15.1 picks up where Table 14.1 left off, describing the progression for fraction operations. Note that the term *fraction* refers to fractions less than, equal to, and greater than 1, and operations with fractions are a major focus at each of these grades.

Teaching fraction operations with understanding may be a significant point of departure from what you may have experienced as a student—learning only one designated (standard) algorithm. This is not effective! First, none of the algorithms helps students think conceptually about the operations. When students follow a procedure they do not understand, they have no way to know when to use it and no way to assess whether their answers make sense. Second, poorly understood algorithms can be forgotten or confused. Students ask, "Do I need a common denominator, or do I just add or multiply the bottom numbers?" "Which one do

TABLE 15.1 CCSS PROGRESSION OF FRACTION OPERATIONS IN GRADES 4–7

Grade 4	Grade 5	Grade 6	Grade 7
Understand **adding and subtracting** fractions (**same** denominators), including: • Showing a fraction, such as $\frac{5}{8}$, can be decomposed into parts, like $\frac{4}{8} + \frac{1}{8}$. • Using visual representations. • Solving story problems. • Using reasoning strategies.	Understand **adding and subtracting** fractions (**different** denominators), including: • Finding common denominators to add or subtract • Solving story problems • Using visuals • Estimating sums and differences		Understand adding and subtracting **rational numbers, both positive and negative**, including: • Using vertical and horizontal number lines • Using reasoning strategies
Understand multiplying **fractions by whole numbers** (e.g., $5 \times \frac{1}{4}$), including: • Using visuals to show that $8 \times \frac{1}{4}$ is the same as $4 \times \frac{2}{4}$, for example, (halve and double). • Solving story problems.	Understand multiplying fractions (**fractions as first or second factor or both**), including: • Using visuals • Creating stories • Using area models • Estimating (based on scaling) • Solving story problems		Extend **multiplying and dividing** to rational numbers, including: • Using strategies • Creating stories • Solving story problems
	Understand connection between fractions and **division** (equal sharing), for example, that $\frac{9}{4}$ means $9 \div 4$ (i.e., 9 parts shared equally with 4 people). Solve equal-sharing story problems.	Understand **division of fractions**, including: • Using visuals • Creating stories • Solving story problems	
	Understand **division involving unit fractions**, including: • Using visuals • Creating stories • Solving story problems		

you invert, the first or the second number?" Third, students can't adapt to slight changes in the fractions, such as decimals or mixed numbers. Commit to teaching fraction operations with understanding (and making sense of them yourself!).

Effective Teaching Process

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

A report summarizing what works for teaching fraction operations suggests that teachers "help students understand why procedures for computations with fractions make sense" (Siegler et al., 2010). The report suggests four steps to effective fraction computation instruction. Each is briefly described here (and then used as headers within each section of this chapter):

1. *Use contextual tasks.* This step should seem like déjà vu, because it applies to nearly every topic in this book. Contextual problems for fractions help students develop their own methods and build understanding (Cramer & Whitney, 2010; Lewis, Gibbons, Kazemi, & Lind, 2015). Problem contexts need not be elaborate, but need to be familiar and relevant to students and fit the type of fractions in the problem (e.g., using halves, fourths, and eighths for measuring in inches or cups, not thirds or fifths).
2. *Explore each operation with a variety of presentations (models).* Area, length, and set models give different insights into fractions, and understanding all of them provides a stronger conceptual understanding (Zhang, Clements, & Ellerton, 2015). There is strong evidence that using well-chosen concrete and semi-concrete representations, including number lines, supports student learning, in particular for students who struggle with mathematics (Jordan, et al., 2013; Fuchs et al., 2021). Importantly, the models must be connected to the symbolic operations. The visuals will help students make sense of the symbols and related operations, but only when they are explicitly connected through repeated experiences.
3. *Let estimation and reasoning strategies play a big role in the development of strategies.* "Should $2\frac{1}{2} \times \frac{1}{4}$ be more or less than 1? More or less than 2?" Estimation keeps the focus on the meanings of the numbers and the operations, encourages reflective thinking, and helps build number sense with fractions.

   Standards for Mathematical Practice
   MP2. Reason abstractly and quantitatively

   Just like with whole numbers, there are a *variety of ways* to solve fraction computation problems (Clarke et al., 2008). In many cases, a reasoning strategy can be applied, and a standard algorithm is not needed. Procedural fluency includes being *flexible* in how students approach fraction operations. Can you reason to get an exact answer without using the standard algorithm? One way is to apply the distributive property, splitting the mixed number and multiplying both parts by $\frac{1}{4}$: $(2 \times \frac{1}{4}) + (\frac{1}{2} \times \frac{1}{4})$. Two $\frac{1}{4}$s are $\frac{2}{4}$ or $\frac{1}{2}$, and a half of a fourth is $\frac{1}{8}$, so add the partial products of one-eighth to one-half and you have $\frac{5}{8}$.
4. *Address challenges, common errors, and misconceptions regarding computational procedures.* Students apply their prior knowledge—in this case, whole-number computation—to new knowledge. Using whole-number knowledge can be a support to learning. For example, ask, "What does 2×3 mean?" Follow this with, "What might $2 \times 3\frac{1}{2}$ mean?" The concepts of each operation are the same, but the procedures are different. This means that overgeneralizing whole-number knowledge also leads to errors (e.g., adding denominators when adding fractions). Teachers should present common misconceptions and discuss why some approaches lead to right answers and others do not (Siegler et al., 2010).

Ongoing connections must be made among the contexts, representations, and strategies. It takes multiple experiences and time for these relationships to be well understood. Although this chapter addresses operations in separate sections, students need opportunities to solve story problems to determine *which* operation fits the story. Too often, students are given subtraction story problems on a day they are learning how to subtract, which removes the central question of "Which operation is a match to this story situation?"

Pause & Reflect

Which operation can be used to solve the story situations in Figure 15.1? ●

Pearson eText Activity Page: Mixed Story Problems Involving Fractions

A. Jeremy's friends ate $\frac{3}{4}$ of his birthday cake. Jeremy ate $\frac{1}{8}$ of the remaining cake. How much of the cake did Jeremy eat?	D. Jessica walked $\frac{1}{8}$ of a mile. Her goal is $\frac{3}{4}$ of a mile. How much farther does she need to walk to reach her goal?
B. Jeremy's friends ate $\frac{3}{4}$ of one birthday cake and $\frac{1}{8}$ of another cake. How much cake was eaten?	E. Jessica walks $\frac{1}{8}$ of a mile to school. How many trips to and from school are needed for her to reach her goal of walking $\frac{3}{4}$ of a mile?
C. Jeremy's friends ate $\frac{3}{4}$ of his birthday cake and Jeremy ate $\frac{1}{8}$ of his cake. How much of the birthday cake did Jeremy eat?	F. Jessica is heading to a store that is $\frac{3}{4}$ of a mile away long. She is $\frac{1}{3}$ of the way there. How far has she walked? How much farther does she have to walk to get to the store?

FIGURE 15.1 Mixed story problems involving fractions

Let's look at Jeremy's situations in Figure 15.1 (area situation). In (a), you must figure out $\frac{1}{8}$ of the remaining cake ($\frac{1}{4}$), which is $\frac{1}{32}$ of the cake. In (b), it is not clear whether the cakes are the same-sized whole, so there is no way to solve this problem. If it is adapted to clarify that the cakes are the same size, then it could be solved using addition. In (c), the answer is in the problem—Jeremy ate $\frac{1}{8}$ of the cake. While this is the easiest to solve, without careful reading, students will calculate something.

In Jessica's first situation (length), the question asks for a difference (subtraction). The "How many trips?" question is actually asking, "How many eighths are in three-fourths?" which can be solved by counting eighths or by using division: $\frac{3}{4} \div \frac{1}{8}$. The final story is finding a part of a part of a mile and can be solved with partitioning or represented as $\frac{1}{3} \times \frac{3}{4}$. Story problems like these can be used as a full lesson and as warm-up tasks throughout the year, with the focus on representing the situation, determining which operation is appropriate, estimating the result, writing the problem symbolically, and solving it.

Addition and Subtraction

Learning Outcome 15.2 **Use representations and strategies to teach adding and subtracting fractions effectively.**

Here we describe how the four steps described earlier are used to develop a strong understanding of addition and subtraction. (Usually, both are taught together.)

Contextual Examples

Many contexts can be used for adding and subtracting fractions. Recall that there are four addition structures that are commonly associated with whole-number operations but must also be applied to fraction addition and subtraction (Chval, Lannin, & Jones, 2013). See Table 8.2 (p. 160) for a reminder of these problem types and structures.

Whereas sharing tasks (see Chapter 14) involve division, they are also an excellent opportunity to introduce adding fractions conceptually (Empson & Levi, 2011).

Two friends want to share $1\frac{3}{4}$ of a quesadilla so that each friend gets the same amount. How much will each friend get?

Pause & Reflect

Think of how students who have not yet learned fraction addition might solve this problem. ●

As students partition, they might first share the whole, giving each friend $\frac{1}{2}$. Then they might give each friend $\frac{1}{4}$, leaving one fourth, which they cut in half and then give each friend $\frac{1}{8}$. The equation is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$. As they reason about the size of the pieces, they intuitively recognize that one-half of a quesadilla is the same as four eighths and, from their partitioning, also know that one-fourth is two-eighths. That leads to $\frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$. Another student might reason that they could each get one whole quesadilla, but the person getting the one with $\frac{1}{4}$ eaten needs half of that fourth from the whole quesadilla, to be fair. So then both friends will have a whole minus one eighth: $1 - \frac{1}{8} = \frac{7}{8}$. This type of learning experience wherein students are using partitioning and reasoning, deepens students' understanding of the relationship among fractions and thus strengthens their fraction sense.

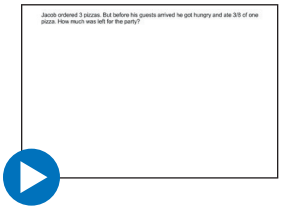
Read these examples and think about how these problems differ from each other (beyond the story context):

Jacob ordered 3 pizzas for a party. But before his guests arrived, he got hungry and ate $\frac{3}{8}$ of one pizza. How much was left for the party?

On Friday, Lydia ran $1\frac{1}{2}$ miles, on Saturday, she ran $2\frac{1}{8}$ miles and, on Sunday, she ran $2\frac{3}{4}$ miles. How many miles did she run over 3 days?

Sammy gathered $\frac{3}{4}$ of a pound of walnuts, and Chala gathered $\frac{7}{8}$ of a pound. Who gathered more? How much more?

In measuring the four pieces of wood needed for a picture frame, Elizabeth figured that she needed two pieces that were $5\frac{1}{4}$ inches and two pieces that were $7\frac{3}{4}$ inches. What length of wood does she need to buy to build her picture frame?



Pearson eText
Video Example 15.1
In this video compare the two ways the student solves the pizza problem.

Notice that these story problems (1) incorporate different addition problem types (join, compare, etc.); (2) use a mix of area and length contexts; (3) use a mix of whole numbers, mixed numbers, and fractions; and (4) sometimes include more than two addends. Meaningful contexts are embedded throughout the next sections, connected to representations and symbols. Traditional teaching has started with fractions less than 1 and eventually moved to mixed numbers, but using mixed numbers from the beginning helps students notice that a fraction is an actual location on a number line.

FORMATIVE ASSESSMENT Notes. Any of the preceding problems can be used as a formative assessment with an observation look-for tool. On the look-for tool would be such concepts as: (1) determines a reasonable estimate; (2) models the problem with manipulative or picture; (3) solves using an efficient method; (4) writes an accurate equation. Students may be able to illustrate but then not find fraction equivalences to solve the problem; or they may represent and solve but struggle to write an equation. A look-for tool provides guidance for what future instruction is needed. ■

Models

Recall that there are area, length, and set representations for illustrating fractions (see Chapter 14, Table 14.2). Instruction should initially focus on area and length models, because keeping track of the whole in the set model can be challenging.

Area Models. Circles are an effective visual for adding and subtracting fractions because they allow students to develop mental images of the sizes of different pieces (fractions) of the circle (Cramer, Wyberg, & Leavitt, 2008).

Jerome and Iliana ordered two medium pizzas, one cheese and one pepperoni. Jerome ate $\frac{5}{6}$ of a pizza, and Iliana ate $\frac{1}{2}$ of a pizza. How much pizza did they eat together?

Standards for Mathematical Practice
MP4. Model with mathematics

Figure 15.2 shows how students estimate first (including marking a number line) and then explain how they added the fraction, using fraction circles.

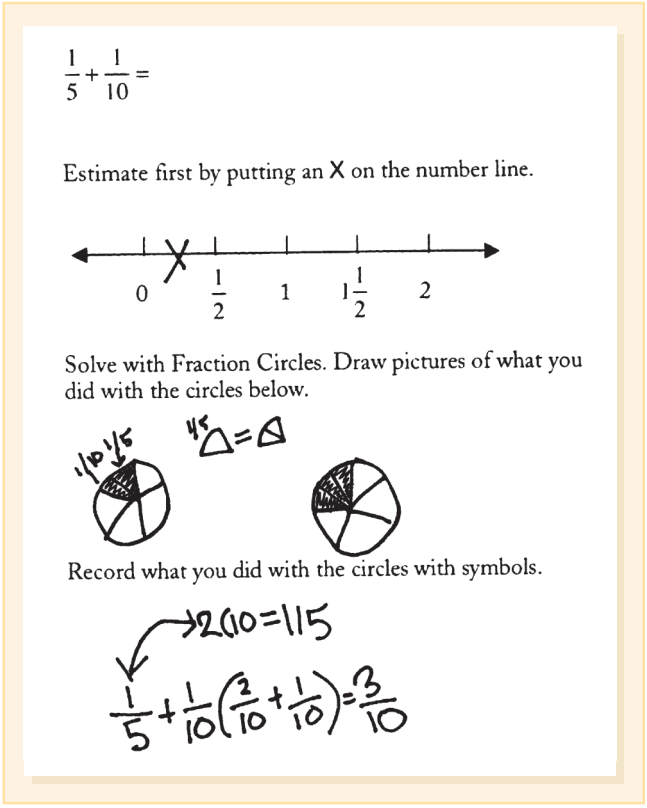

1/5 + 1/10 =
Estimate first by putting an X on the number line.
0
1/2
1
1 1/2
2
Solve with Fraction Circles. Draw pictures of what you did with the circles below.
Record what you did with the circles with symbols.
2/10=1/5
1/5 + 1/10(2/10 + 1/10)=3/10

FIGURE 15.2 A student estimates and then adds fractions, using a fraction circle

Source: K. Cramer, T. Wyberg, & S. Leavitt (2008). "The Role of Representations in Fraction Addition and Subtraction." *Mathematics Teaching in the Middle School, 13*(8), p. 495. Reprinted with permission. Copyright © 2008 by the National Council of Teachers of Mathematics. All rights reserved.

Consider this problem, in which the context is circular:

Jerome and Iliana ordered two medium pizzas, one cheese and one pepperoni. Jerome ate $\frac{5}{6}$ of a pizza, and Iliana ate $\frac{1}{2}$ of a pizza. How much pizza did they eat together?

Pause & Reflect

Think of two ways that students might solve this problem without using a common-denominator approach. ●

If students draw circles, some will try to fill in the $\frac{1}{6}$ gap in the pizza. Then they will need to figure out how to get $\frac{1}{6}$ from $\frac{1}{2}$ (i.e., how might they decompose $\frac{1}{2}$?). If they can think of $\frac{1}{2}$ as $\frac{3}{6}$, they can use one of the sixths to fill in the gap. Another approach, after drawing

Pearson eText
Video Example 15.2
In this video consider how Felicia solves a fraction addition problem using circles.

Pearson eText
Video Example 15.3
In this video consider how Felicia solves a fraction subtraction problem using circles.

the two pizzas, is to notice that there is a half ($\frac{3}{6}$) plus two more sixths in the $\frac{5}{6}$ pizza. Put the two halves together to equal one whole, and there are $\frac{2}{6}$ more—$1\frac{2}{6}$. These are certainly good solutions that represent the type of reasoning you want to encourage. Did you notice that this reasoning is like the Make Tens strategy with whole numbers? With fractions, the strategy becomes Make-a-Whole.

Rectangles, like circles, can be partitioned in whatever way is useful for the problem being solved. Activity 15.1 uses a rectangular context and representation. Geoboards or centimeter grid paper can be used to support reasoning.

Activity 15.1 Gardening Together

CCSS-M: 4.NF.B.3a, d; 5.NF.A.1; 5.NF.A.2

Distribute the Empty Garden activity page (or give each student a blank piece of paper or grid paper to represent the rectangular garden). Explain the situation and ask each student to design the garden to illustrate these quantities:

Al, Bill, Carrie, Danielle, Enrique, and Fabio are each given a portion of the school garden for spring planting. Here are their portions:

Al = $\frac{1}{4}$ Bill = $\frac{1}{8}$ Carrie = $\frac{3}{16}$

Danielle = $\frac{1}{16}$ Enrique = $\frac{1}{4}$ Fabio = $\frac{1}{8}$

They decide to pair up to share the work. What fraction of the garden will each of the following pairs or groups have if they combine their portions of the garden? Show your work.

Bill and Danielle — Al and Carrie

Fabio and Enrique — Carrie, Fabio, and Al

For multilingual learners, clarify the meaning of context words like *garden* and *mathematical language* and like *portion*. To challenge students, ask them to solve puzzle-type questions like: "Which two people could work together and have the least amount of the garden? The most? Which combinations of friends could combine to work on one-half of the garden?"

Pearson eText Activity Page: Empty Garden

Length Models. Suppose that instead of pizza, Jerome and Iliana were eating submarine sandwiches (length context). Students can use their fraction strips to represent and then reason about the problem. A student might notice that Jerome ate $\frac{1}{2}$ of a sandwich plus two more sixths, so the halves from each person make one whole, so altogether they have eaten $1\frac{2}{6}$ sandwiches. What if you instead asked students to compare the quantity that Jerome and Iliana ate? Laying the fraction strips side by side readily shows the difference of $\frac{2}{6}$.

Cuisenaire rods are also effective length models for thinking about adding and subtracting fractions. An additional first step is required: Which piece will you select as your whole (sandwich)? In the example given here, the 6 rod works well because the 3 rod represents halves and the white cube represents sixths. Figure 15.3 illustrates addition and subtraction, using Cuisenaire rods.

A critical model for adding or subtracting fractions is the number line (Deliyianni, Gagatsis, Elia, & Panaoura, 2016; Siegler et al., 2010). One advantage of the number line is that it can be connected to the ruler, which is a familiar context and an actual real-world opportunity to add and subtract fractions. The number line is also a more challenging model than an area model, because it requires that the student understand $\frac{3}{4}$ as 3 parts of 4 and as a value between 0 and 1 (Izsák, Tillema, & Tunc-Pekkam, 2008). Using the number line in addition to area representations can strengthen student understanding (Clarke et al., 2008; Cramer et al., 2008; Deliyianni et al., 2016; Fuchs et al., 2021; Petit et al., 2016).

Standards for Mathematical Practice
MP4. Model with mathematics

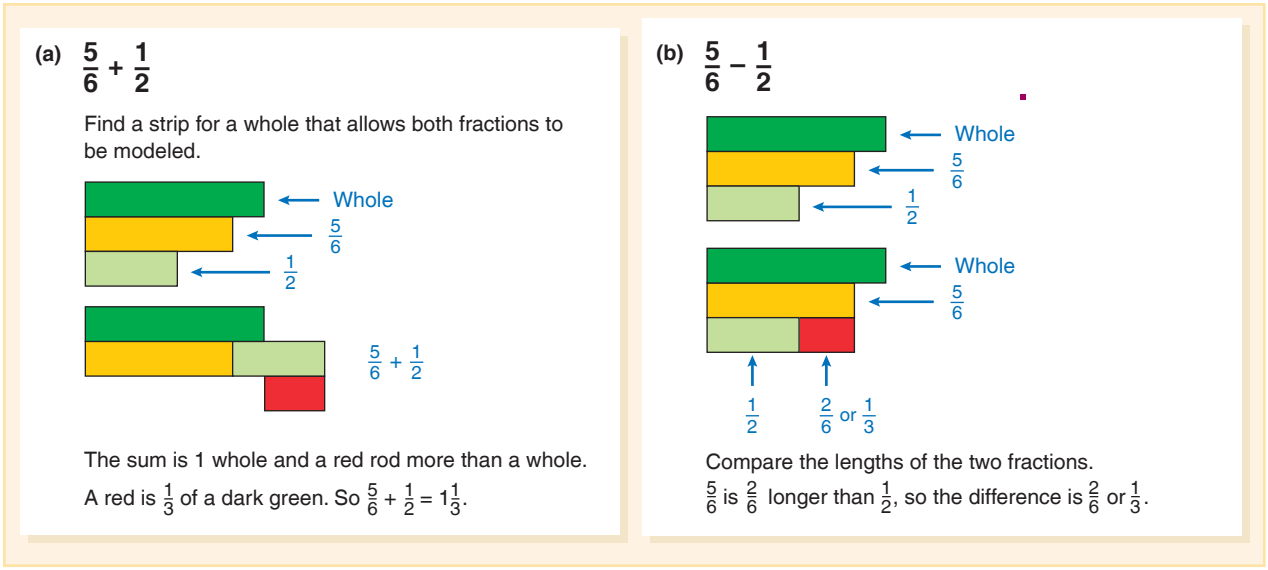

(a) 5/6 + 1/2
Find a strip for a whole that allows both fractions to be modeled.
Whole
5/6
1/2
5/6 + 1/2
The sum is 1 whole and a red rod more than a whole.
A red is 1/3 of a dark green. So 5/6 + 1/2 = 1 1/3.
(b) 5/6 − 1/2
Whole
5/6
1/2
Whole
5/6
1/2
2/6 or 1/3
Compare the lengths of the two fractions.
5/6 is 2/6 longer than 1/2, so the difference is 2/6 or 1/3.

FIGURE 15.3 Using Cuisenaire rods to add and subtract fractions

Activity 15.2 Jumps on the Ruler

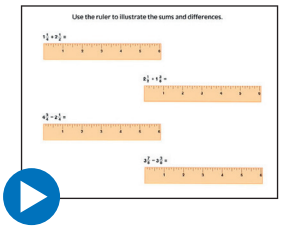
CCSS-M: 4.NF.B.3a, d; 5.NF.A.1; 5.NF.A.2

Use the Jumps on the Ruler activity page (or provide a ruler). Have students use jumps on the ruler to add or subtract (which does not require a common denominator, though students may naturally trade equivalent fractions to help them reason). Examples include:

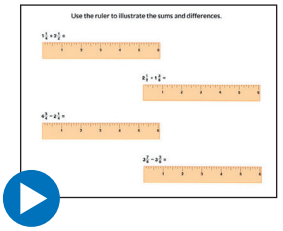
$$\frac{3}{4} + \frac{1}{2} \qquad 4\frac{1}{2} - 3\frac{3}{4} \qquad 4\frac{1}{8} - \frac{1}{2}$$

Either add a length context (length of grass in the yard, hair growing/getting cut) or ask students to create their own stories for each. These jumps encourage students to use reasoning strategies without having to find a common denominator (Taber, 2009). For example, for the first problem, students might mentally partition $\frac{1}{2}$ into $\frac{1}{4} + \frac{1}{4}$ to make a whole. To solve, they jump (add), going from $\frac{3}{4}$ to 1 and from 1 to $1\frac{1}{4}$. Students from other countries may not be as familiar with inches, because most countries use centimeters. In this case, begin the activity by discussing how the inch is partitioned. You can also label the tick marks as a reminder.

Pearson eText Activity Page: Jumps on the Ruler



Pearson eText
Video Example 15.4



Pearson eText
Video Example 15.5
Compare how the students in these videos engage in the Jumps on a Ruler activity. Do these students solve the problem correctly? Efficiently? In what ways are their approaches similar, and in what ways do they differ?

***TECHNOLOGY* Note.** Dreambox (www.dreambox.com) Teacher Tools offers a bar (length) model for exploring addition and subtraction. Students solve problems visually, trading out equivalent fractions and recording their answers. Other sites offer various virtual manipulatives by which a student can build a problem and solve it. Those sites include Toy Theater Teacher Tools (toytheatre.com), Math Learning Center free apps (mathlearningcenter.org), and Brainingcamp (brainingcamp.com). On each site, you or the student can select the manipulative you will use to represent the problems. ■

Estimating Sums and Differences

Estimation is one of the most effective ways to build understanding and procedural fluency with fraction operations (Braithwaite et al., 2018; Johanning, 2011). A frequently quoted result from the Second National Assessment of Educational Progress (Post, 1981) concerns the following item:

Estimate the answer to $\frac{12}{13} + \frac{7}{8}$. You will not have time to solve the problem, using paper and pencil.

Nearly two-thirds of middle school students could find the exact answer to this problem, but only one-fourth could correctly estimate (Reys, 1998). Notice that computing this answer requires finding the common denominator of thirteenths and eighths, but to estimate requires no computation whatsoever—only noticing that each fraction is close to 1, so the answer is close to 2. Fast-forward about 20 years, and we continue to struggle with this concept. Braithwaite and colleagues (2018) found that nearly half of fourth and fifth graders were unable to estimate sums of fractions. As part of their study, students estimated the values of the addends and the sums, and many students actually estimated sums at a value *less than* the value of the addends they estimated.

There are different ways to estimate fraction sums and differences (Siegler et al., 2010):

1. *Benchmarks.* Decide whether the fractions are closest to 0, $\frac{1}{2}$, or 1 (or to 3, $3\frac{1}{2}$, or 4—the closest whole numbers and the half in between them for mixed numbers). After making the determination for each fraction, mentally add or subtract.

Examples:

a. $\frac{7}{8} + \frac{1}{10}$. Think, "$\frac{7}{8}$ is close to 1, $\frac{1}{10}$ is close to 0, the sum is about 1 + 0 or close to 1."
b. $2\frac{1}{8} - \frac{9}{10}$. Think, "This is about 2 – 1, so the answer is about 1."

---

2. *Relative size of unit fractions.* Decide how big the fraction is, based on its unit (denominator), and apply this information to the computation.

Example: $\frac{7}{8} + \frac{1}{10}$. Think, "$\frac{7}{8}$ is just $\frac{1}{8}$ away from a whole (one), and $\frac{1}{8}$ is close to (but bigger than) $\frac{1}{10}$, so the sum will be close to, but less than, 1."

The following activity can be done frequently as a routine, warm-up, or a focus activity for a full lesson.

Activity 15.3 Over or Under 1

CCSS-M: 4.NF.B.3a; 5.NF.A.2

Use the Estimating Addition and Subtraction of Fractions Problems activity page, or select any fraction addition or subtraction worksheet (see Figure 15.4 for examples). You can cut the problems out, creating cards so that students can sort them on a placemat. Explain that these problems are not going to be solved, but estimated. To begin, project a problem for about 10 seconds and then hide or remove it. Discuss how students decided on their estimates (including connecting to mental models of number lines or other models to reason). Students with special needs in math benefit from visuals (e.g., drawings of circles and/or number lines). They may also need more think time. To adapt this activity to a learning center or independent

activity, use the Estimate, Write, Explain activity page. Over time, include tasks that are more challenging. Consider the following variations:

- Use a target other than 1. For example, estimate more or less than $\frac{1}{2}$, 2, or 3.
- Choose fraction pairs in which the fractions are both less than 1 and both greater than 1. Estimate sums or differences to the nearest half.
- Ask students to *create* equations that are slightly less than or slightly greater than 1 (or other values). They can trade equations with other students, who in turn decide whether the answer is greater than 1 or less than 1 (or other values).

Pearson eText Activity Page: Estimating Addition and Subtraction of Fractions Problems
Pearson eText Activity Page: Less Than One, Greater Than One Placemat
Pearson eText Activity Page: Estimate, Write, Explain

Pause & Reflect

Test your own estimation skills with the sample problems in Figure 15.4. Look at each computation for a few seconds and write down an estimate of the answer. After writing down all six of your estimates, look at the problems and decide whether your estimate is higher or lower than the actual computation. ●

A relevant and familiar context for fractions is cups, such as cups of water or milk (Fung & Latulippe, 2010; Zhang et al., 2015). Activity 15.4 uses cups to engage students in estimating sums and differences.

Estimate

1. $\frac{1}{8} + \frac{4}{5}$
2. $\frac{9}{10} + \frac{7}{8}$
3. $\frac{3}{5} + \frac{3}{4} + \frac{1}{8}$
4. $\frac{3}{4} - \frac{1}{3}$
5. $\frac{11}{12} - \frac{3}{4}$
6. $1\frac{1}{2} - \frac{9}{10}$

Number your papers 1 to 6.
Write only answers.

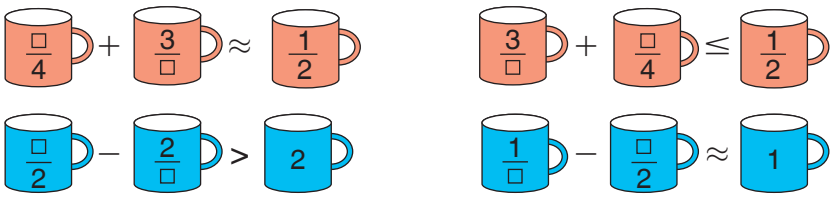
Estimate!
Use whole numbers and benchmark fractions.

FIGURE 15.4 Example of fraction estimation expressions

Activity 15.4 Cups of Milk

CCSS-M: 4.NF.B.3a; 5.NF.A.2

Ask students to fill in the missing numerator and denominator with a whole number that makes the equation approximately true. They cannot use any of the numbers already in the problem. The context of milk (or water) can help students reason about the quantities. Four examples are provided here:



Keep in mind that these are estimates, so many values are possible, and some of the answers include fractions greater than 1. Use this activity as a routine. Use the Cups of Milk teacher resource page and fill in what values you would like to use. Ask students to find at least one set of numbers to make the equation true.

Pearson eText Activity Page: Estimating Addition and Subtraction of Fractions Problems
Pearson eText Activity Page: Template for Cups of Milk Problems

Pearson eText
Video Example 15.6
In this video consider how the student uses pattern blocks to create visuals of adding fractions.

Standards for Mathematical Practice
MP7. Look for and make use of structure

Developing Strategies and Algorithms

Strategies and algorithms develop side by side with the visuals and situations. Without these concrete experiences, students overgeneralize whole-number ideas. (Examples are provided in the Table at the end of this section.)

Like Denominators. Fraction addition and subtraction begin with situations using like denominators. When working on adding with like denominators, however, it is important to be sure that students are focusing on the key idea—that the units are the same, so they can be combined (Mack, 2004). In other words, the problem $\frac{3}{4} + \frac{2}{4}$ is asking, "How many fourths altogether?" The expression $3\frac{7}{8} - 1\frac{3}{8}$ is taking away eighths (separate) or finding the number of eighths between the two quantities (comparing). Iterating (i.e., counting on or counting back by unit fractions) connects fraction operations to whole-number operations and explains why the denominator stays the same.

FORMATIVE ASSESSMENT Notes. A diagnostic interview can ask students to (1) explain the meaning of the numerators and denominators, (2) connect the addition problems to a situation, and (3) illustrate with a model, which can help you determine whether they have a deep understanding of the numerator and denominator. If student responses are rule-oriented and not grounded in understanding parts and wholes, then encourage students to focus on the meaning of the fractions by emphasizing the unit: "Three *fifths* plus one *fifth* is how many *fifths*?" This combining of like units must be well understood before moving to adding with unlike denominators. ■

Unlike Denominators. As discussed in Chapter 14, having a strong conceptual foundation of equivalence is critical to fraction operations. Students who have a level of automaticity in moving between commonly used fractions, such as $\frac{1}{2}$, $\frac{2}{4}$, $\frac{4}{8}$, and $\frac{8}{16}$ or $\frac{3}{4}$, $\frac{6}{8}$, and $\frac{12}{16}$, can adjust the fractions as needed to add or subtract fractions flexibly (Bay-Williams & SanGiovanni, 2021). For example, ask students to think about how far a person has walked, if the following equation represents their walking: $4\frac{1}{4} + 6\frac{7}{8}$. Given sufficient concrete experiences, a student should be able to trade out one-fourth for two-eighths readily to solve, without going through a "find a common denominator" procedure. Then continue to provide engaging reasoning tasks in which they have to consider the size of the parts (denominators), as in Activity 15.5.

Activity 15.5 Can You Make It True?

CCSS-M: 4.NF.B.3a; 5.NF.A.2

Use the Can You Make It True? activity page, which includes a list of equations with missing values, such as the ones here:

$$\frac{\square}{6} + \frac{\square}{3} = 1 \text{ and } \frac{4}{\square} - \frac{\square}{2} = \frac{1}{2}$$

To begin, post one example and invite students to share possible whole numbers that make the equation true. If they have found one way, ask, "Is there more than one way to make that equation true?" Include examples that are impossible (when a whole number is missing values), such as:

$$\frac{1}{\square} - \frac{1}{\square} = 1$$

As students work, ask them to explain the reasoning they are using. Encourage students to use visuals (e.g., number line or fraction pieces) to support their thinking. Challenge students by giving them a 3-by-5 card and having them create a missing-values fraction equation of their own (with an answer recorded on the back or on a sticky note). Students can exchange cards and challenge each other.

Pearson eText Activity Page: Can You Make It True?

As you begin to have students add fractions with unlike denominators, start with tasks where only one fraction needs to be changed to an equivalent fraction, such as $5\frac{3}{4} - 1\frac{1}{2}$ or $\frac{5}{8} + \frac{1}{4}$. Remember to ask students to estimate, draw, or use tools (e.g., fraction circles, fraction strips, or number lines) to support their thinking, and use a method that makes sense to them. Continue

with examples in which both fractions need to be changed to equivalent versions with a common denominator—for example, $\frac{2}{3} + \frac{1}{4}$ (or $\frac{2}{3} - \frac{1}{4}$), still using estimation, visuals, and student reasoning strategies. The three examples in Figure 15.5 show how thinking tools might be used. Note that each tool requires students to think about the size of a whole that can be partitioned into the units of both fractions (e.g., fifths and halves require a whole partitioned into tenths).

In the discussion of student solutions, focus attention on the idea of *rewriting an equivalent problem* to make it possible to add or subtract *equal-sized parts*. If students express doubt about the equivalence of the two problems ("Is $\frac{8}{12} + \frac{3}{12}$ really the answer to $\frac{2}{3} + \frac{1}{4}$?"), that should be a cue that the concept of equivalent fractions is not well understood, and more experience using examples, visuals, or concrete tools is needed.

Least Common Multiples. If students struggle to find a common denominator, spending time focusing on multiples and common multiples can help them more readily determine common denominators.

Activity 15.6 Game: How Low Can You Go?

CCSS-M: 4.OA.B.4; 5.NF.A.1

Use common multiple cards with pairs of numbers (see Figure 15.6). Place students in pairs. On a student's turn, they turn over a common multiple card and state a common multiple (e.g., for the 6, 9 card, a student might suggest 54). The partner gets a chance to suggest a smaller common multiple (e.g., 36). The original student then realizes that 18 is also a common multiple. The student suggesting the least common multiple (LCM) gets to keep the card. Be sure to include pairs that are prime, such as 9 and 5; pairs in which one is a multiple of the other, such as 2 and 8; and pairs that have a common divisor, such as 8 and 12.

For a variation on this activity, students could use a deck of cards in place of the common multiple cards. In this case, remove cards you don't want (i.e., numbers that are rarely denominators, such as 7). Two players flip up a card, and the rest of the activity plays out the same way.

Pearson eText Activity Page: Common Multiple Cards

Reasoning Strategies. Table 15.2 describes reasoning strategies for addition and subtraction with fractions. Hopefully, these names look familiar, because they are similar to strategies used with whole-number addition and subtraction. Three important points:

1. Although some of the examples have visual representations and others have symbolic representations, students may use visuals or numbers to reason on any of them.
2. Reasoning strategies are flexible, so the example given is just *one option* of how to enact that strategy.

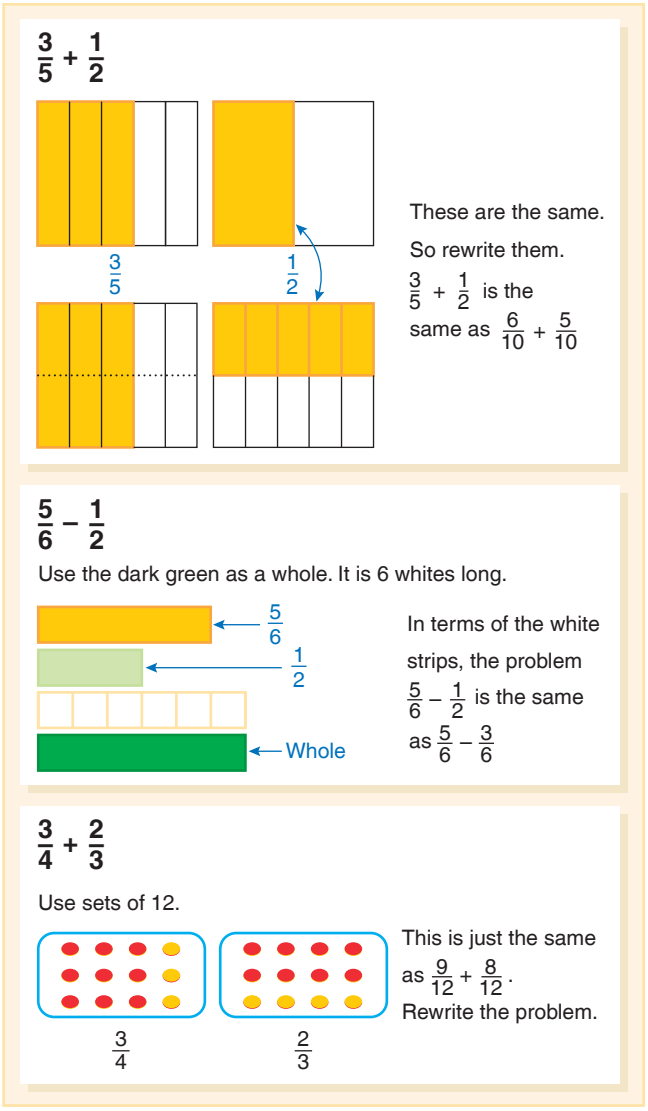

3/5 + 1/2
3/5
1/2
These are the same.
So rewrite them.
3/5 + 1/2 is the same as 6/10 + 5/10
5/6 – 1/2
Use the dark green as a whole. It is 6 whites long.
5/6
1/2
Whole
In terms of the white strips, the problem 5/6 – 1/2 is the same as 5/6 – 3/6
3/4 + 2/3
Use sets of 12.
3/4
2/3
This is just the same as 9/12 + 8/12. Rewrite the problem.

FIGURE 15.5 Adding fractions with unlike denominators with area, length, and set representations

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

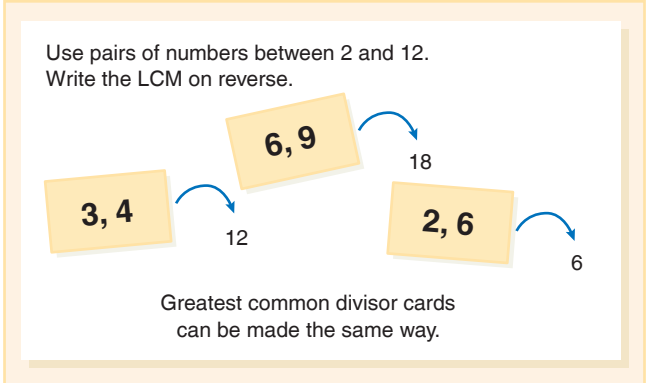


FIGURE 15.6 Common multiple cards

Pearson eText
Video Example 15.7
Listen as this student uses a Think Addition strategy to subtract.

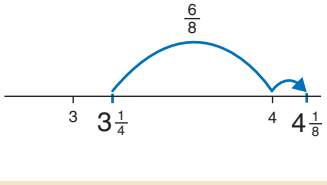
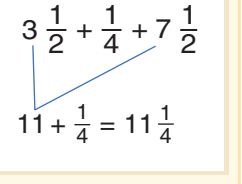
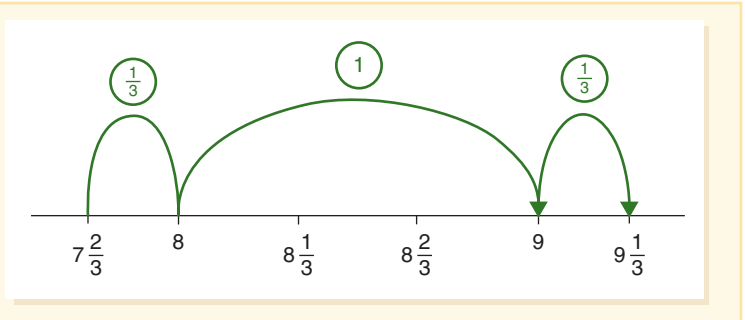
3. Students may record their thinking or reason mentally (e.g., visualizing a number line), and both are appropriate unless a student is struggling to work mentally, in which case it is helpful to ask them to record their thinking.

Fractions Greater than One. As you can see from the examples so far, a separate algorithm for mixed numbers in addition and subtraction is not necessary. Table 15.2 includes many examples with fractions greater than one. The Use Partials strategy is more or less the standard algorithm but approached in a more conceptual manner than the traditional regrouping explanation. Students tend to make many errors when regrouping fractions.

An excellent option for adding or subtracting mixed numbers is to change the mixed numbers to fractions greater than 1. You may think that changing the fractions to improper fractions is a step for *multiplying* fractions, but it works for addition and subtraction, too. Have a look:

$$5\tfrac{1}{8} - 3\tfrac{5}{8} = \tfrac{41}{8} - \tfrac{29}{8}$$
$$= \tfrac{12}{8} \text{ or } 1\tfrac{1}{2}$$

TABLE 15.2 REASONING STRATEGIES FOR FRACTION ADDITION AND SUBTRACTION

Reasoning Strategy	Example Problem	Implementation of the Strategy
1. Jumps on the Number Line (addition [count on] or subtraction as take-away [count back])	$\frac{7}{8} + 3\frac{1}{4}$	
2. Make a Whole (addition)	$3\frac{3}{4} + 7\frac{1}{2}$	
3. Compensation (addition and subtraction)	$7\frac{1}{6} - \frac{11}{12}$	$7\frac{2}{12} - \frac{11}{12}$: Adjust the subtrahend ($\frac{11}{12}$) to 1. Subtract: $7\frac{2}{12} - 1 = 6\frac{2}{12}$ Compensate ($\frac{1}{12}$ extra was taken away and must be added back in): $6\frac{2}{12} + \frac{1}{12} = 6\frac{3}{12}$ *or* $6\frac{1}{4}$
4. Think-Addition (subtraction as compare or find the difference)	$9\frac{1}{3} - 7\frac{2}{3}$	 Numerically: Jump up one-third, then one whole, then one third more. Total length, $1\frac{2}{3}$.
5. Use Partials (addition or subtraction)	$1\frac{3}{5} + 3\frac{4}{5}$	Add the wholes: 1 + 3 = 4 Add the fractions: $\frac{3}{5} + \frac{4}{5} = \frac{7}{5}$ or $1\frac{2}{5}$ Combine the parts: $5\frac{2}{5}$

This method is efficient and connects to the adding of fractions less than 1. Therefore, it should be included as a viable option for adding and subtracting fractions.

Challenges with Adding and Subtracting Fractions

It is important to talk about common misconceptions with fraction operations explicitly, because students may overgeneralize what they know about whole-number operations. Activity 15.7 provides students the opportunity to estimate the magnitude of fractions *and* fraction sums while encountering a common error students make (adding denominators) (Enlow & Özgün-Koca, 2020).

Activity 15.7 Least-to-Greatest Fraction Sort

CCSS-M: 4.NF.B.3a; 5.NF.A.2

Give students a set of four values: two fractions by themselves, adding the two fractions, and adding the numerators and denominators of those fractions. Two examples are shown here, and the Teacher Resource page offers more. Have students organize them from least to greatest, solving the addition problems. This is particularly powerful if they locate each on the number line. Discuss what they notice within each problem and across problems. In particular, ask them to compare the two types of addition problems.

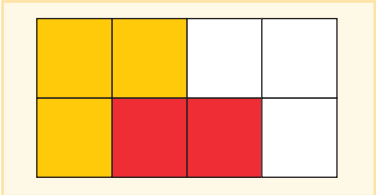
Example Sets:

$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4} + \frac{1}{2}$	$\frac{1+1}{4+2}$
$\frac{3}{4}$	$\frac{5}{6}$	$\frac{3}{4} + \frac{5}{6}$	$\frac{3+5}{4+6}$

Pearson eText Activity Page: Least-to-Greatest Fraction Sort Cards

Table 15.3 offers some common challenges with adding and subtracting fractions, with suggestions for helping students overcome them.

TABLE 15.3 COMMON CHALLENGES WITH ADDING AND SUBTRACTING FRACTIONS

Common Challenge	What It Looks Like	How to Help
1. Apply whole-number operation rules (i.e., overgeneralize)	$\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$	• Ask students to tell what the numerator and denominator mean. • Use estimation (see Activity 15.7). • Ask students to create a story, using a familiar context to make sense of the operation. • Use area and length visuals and be sure students connect the visual actions to symbolic actions.
2. Get different answers with models than with symbols (and aren't bothered that the two answers are different) (Bamberger et al., 2010)	For a task such as: Ani baked a pan of brownies for the bake sale and cut them into 8 equal-sized parts. In the morning, three brownies were sold; in the afternoon, two were sold. What fractional part of the brownies was sold? Student correctly illustrates:  But then writes $\frac{3}{8} + \frac{2}{8} = \frac{5}{16}$.	• Ask students to decide whether both answers could be right. • Have students justify which is right and prove why the other answer is not right. • Encourage students to do the problem on a number line.

(*continued*)

Common Challenge	What It Looks Like	How to Help
3. Do not find a common denominator, but mistakenly apply standard algorithm (Siegler et al., 2010)	$\frac{4}{5} + \frac{4}{10} = \frac{8}{10}$	• Ask students to model the problem on a number line or with fraction strips. • Have students estimate each fraction and the fraction sum.
4. Struggle to find common denominator (lack of fluency with basic facts and with common multiples)	Student is given $\frac{5}{6} - \frac{1}{4} =$ _____ and is unsure what to use for a denominator.	• Offer games and activities to reinforce multiplication facts (see Chapter 9). • Do common multiple activities, such as Activity 15.6. • Discuss that while finding a smaller common denominator (e.g., 12) has advantages, any common denominator (e.g., 24) will work. • Ask students to discuss strategies they use to find common denominators.
5. Mixed numbers: Subtract the smaller fraction from the larger (Petit et al., 2010; Siegler et al., 2010; Spangler, 2011)	When given a problem like $3\frac{1}{4} - 1\frac{3}{8}$, student subtracts the smaller fraction from the larger $(\frac{3}{8} - \frac{1}{4})$.	*For all of these mixed-number challenges, strategies to help include:* • Include more mixed numbers throughout addition of fraction units (rather than saving them for end of unit). • Use models and contexts, in particular the number line. Think about how straightforward $4 - \frac{7}{8}$ is on the number line or ruler. • Make these errors part of public discussions, for example, using worked examples with such errors.
6. Mixed numbers: Don't know what to do when one number is a whole number	For a problem like $4 - \frac{7}{8}$, student places an 8 under the 4 to get $\frac{4}{8} - \frac{7}{8}$. See Video Example 15.6 of Jacky's thinking.	
7. Mixed numbers: Focus only on the whole numbers in the problem	When given a problem like $14\frac{1}{2} - 3\frac{1}{8}$, student writes $11\frac{1}{2}$ or $11\frac{1}{8}$.	

Multiplication

Learning Outcome 15.3 **Design a developmental progression for teaching fraction multiplication conceptually.**

As you will see in the sections that follow, the ideas of iterating (counting) fractional parts and partitioning are foundational to understanding multiplication of fractions. Additionally, a focus on the unit is at the heart of multiplying fractions (I, Dougherty, & Berkaliev, 2015). If students have not had sufficient experiences with these two ideas, you must find ways to engage them in iterating and partitioning so that they are ready to apply these ideas to multiplication. After revisiting and expanding on the progression in Table 15.1 on p. 381, the progression for fraction multiplication problem types looks like this:

Fraction multiplied by a whole number	Whole number multiplied by a fraction	Fraction multiplied by a fraction—no subdividing	Fraction multiplied by a fraction—with subdividing
$4 \times \frac{1}{3}$	$\frac{4}{5} \times 20$	$\frac{1}{3} \times \frac{9}{10}$	$\frac{1}{3} \times \frac{1}{2}$
$6 \times 2\frac{1}{3}$	$\frac{5}{3} \times 6$	$\frac{3}{5} \times \frac{10}{12}$	$2\frac{1}{4} \times \frac{3}{4}$

Note: Fractions include fractions less than and greater than one, including mixed numbers.

This section is organized in this way, but it is important to continue to integrate the different types so that students don't find them confusing later on.

Multiplication of fractions is really about scaling. If you scale something by a factor of 2, you multiply it by 2. When you scale by 1 (1 times the size), the amount is unchanged (identity property of multiplication). Similarly, multiplying by $\frac{1}{2}$ means taking half of the original size, whereas multiplying by $1\frac{1}{2}$ means the original size plus half the original size. This scaling concept can enhance students' ability to decide whether their answers are reasonable.

Contextual Examples and Models

When working with whole numbers, we would say that 3×5 means "3 sets of 5" (groups) or "3 rows of 5" (area or array) or "5 three times" (lengths on a number line). Notice the set, area, and length models fitting with the multiplication structures. Different models must be used and aligned with contexts so that students get a comprehensive understanding of multiplication of fractions. The story problems that you use to pose multiplication tasks to students need not be elaborate, but it is important to think about the numbers and contexts that you use in the problems.

Multiply a Fraction by a Whole Number. Students' first experiences should be of this type because, conceptually, it is a close fit to multiplication of whole numbers, using the idea of equal groups. You can support reasoning by starting with a whole-number example and then using a similar example, using a fraction for the group size:

Marvin ate 3 pounds of meat every day. How many pounds of meat did Marvin eat in one week?
Murphy ate $\frac{1}{3}$ pound of meat every day. How many pounds of meat did Murphy eat in one week?

Pause & Reflect

What expressions represent each situation? What reasoning strategies would you use to solve each? ●

For Marvin, the expression is 7 groups of 3 pounds, or 7×3. You can solve this by skip-counting $3 + 3 + 3 + 3 + 3 + 3 + 3$ to get the answer of 21 pounds (but if you know your facts, you just *multiplied*). Murphy similarly ate 7 groups, but his group size was $\frac{1}{3}$, so $7 \times \frac{1}{3}$. It can also be solved by skip-counting, this time by thirds: $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$, which in total is seven-thirds $\left(\frac{7}{3}\right)$. Attending to the unit is important in making sense of this situation (I et al., 2015). For Murphy, an intermediate unit is his serving size of $\frac{1}{3}$. This unit is then skip-counted, or *iterated*, by the multiplier (in this case, 7). Manipulatives are an excellent way to explore this idea.

Activity 15.8 Hexagon Wholes

CCSS-M: 4.NF.B.4a, b

Distribute a set of yellow pattern block hexagons to students. To start, designate the hexagon as the whole. Ask students the fractional value of the green, blue, and red pieces. Ask students to find how many wholes for different quantities:

5 blue pieces? 10 green pieces?

Ask students to write a fraction equation to represent their problem. For the two examples here:

$5 \times \frac{1}{3} = 1\frac{2}{3}$ or $\frac{5}{3}$ $10 \times \frac{1}{6} = \frac{10}{6}$ or $1\frac{4}{6}$ or $1\frac{2}{3}$

Continue to pose similar problems and ask students what patterns they notice. Repeat the activity, using two hexagons as the whole. Repeat the activity with other manipulatives, like fraction circles, Cuisenaire rods, and fraction strips. Multilingual learners and students with special needs in math may benefit from having the shapes labeled with their names and the fraction symbols written with fraction words (e.g., $\frac{1}{6}$ is "one-sixth").

Pearson eText Activity Page: Circular Fraction Pieces

Notice that in the answers given, the fraction shows a pattern that can be generalized as $a \times \frac{1}{b} = \frac{a}{b}$, an important pattern for students to discover from having explored many examples that follow this pattern. Length examples and visuals are important to include, because the relationship is particularly noticeable. Activity 15.9 adapts the earlier Jumps on the Ruler activity with this purpose in mind.

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

Activity 15.9 Jumps on the Ruler—Multiplication

CCSS-M: 4.NF.B.4a, b

Use Jumps on the Ruler—Multiplication activity page to explore jumps of equal length. Ask students to illustrate their jumps on the rulers and then record their equation and answer. After students have completed the page, ask them to talk to a partner about what patterns they noticed. Then ask students to use the ruler (or number line) mentally to solve similar problems, such as $9 \times \frac{1}{4}$. As a challenge to students, pose the open-ended question:

What multiplication equations can you write that have an answer of 3 inches?

Pearson eText Activity Page: Jumps on the Ruler—Multiplication

Pearson eText
Video Example 15.8
What do you notice about Felicia's solution to $\frac{2}{3} \times 12$?

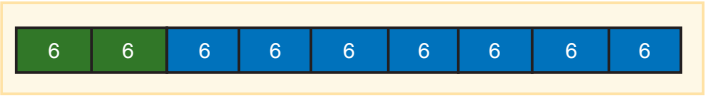
Standards for Mathematical Practice
MP4. Model with mathematics

Multiply a Whole Number by a Fraction. Students' second experiences with multiplication involve finding fractions of whole numbers, such as $\frac{2}{3} \times 12$.

Although multiplication is commutative, the thinking involved in this type of fraction multiplication involves partitioning (not iterating). Consider the following story problem (Englard, 2010):

A nurse has 54 bandages. Of those, $\frac{2}{9}$ are green and the rest are blue. How many of them are blue?

To solve, a student first partitions a strip into nine parts and then figures out the equal shares of bandages for each partition (6) to figure out how many are green (12):



Finding the number of blue bandages can be represented by the equation $\frac{7}{9} \times 54$. In essence, you are finding seven-ninths of a quantity.

Remember that multiplication is scaling. Consider how to use a similar visual representation strip (or bar) diagram to reason through these examples:

1. The walk from school to the public library takes 20 minutes. When Anna asked her mom how far they had gone, her mom said that they had gone $\frac{3}{4}$ of the way. How many minutes have they walked? (Assume a constant walking rate.)
2. There are 15 cars in Michael's Matchbox car collection. Two-thirds of the cars are red. How many red cars does Michael have?

How might students think through each problem conceptually? For problem 2, students might partition 15 into three groups (or partition a line into three parts) and then see how many are in two parts. Recording this in symbols (of 15) gives the following result: $15 \div 3 \times 2$.

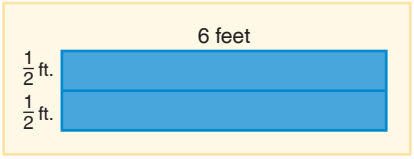
An area model, such as a rectangle, provides an excellent visual tool for illustrating and generalizing (Witherspoon, 2014), as can be seen in Activity 15.10.

Activity 15.10 How Big Is the Banner?

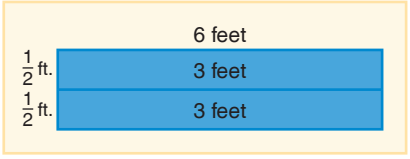
CCSS-M: 5.NF.B.4a, b

Show or explain that you have a roll of paper you will be using to make banners. The roll is one foot wide (you can also use one yard or one meter), and you are going to roll out several feet. The first banner you cut is 1 ft. by 6 ft.:

Ask students, "What is the area of this banner?" (1 × 6 = 6 square feet). You can ask additional questions about banners of other lengths (with a width of 1 foot). Then explain that you want to cut the banners lengthwise to make thinner banners. Ask students to use this rectangle to show banners that are $\frac{1}{2}$ ft. × 6 ft. Ask them to tell you the square feet of the new banner:



Students fill in the strips to show that there are 3 feet in each half-strip:



Repeat the process by asking the area of banners that are 6 feet long and one-third of the width and one-fourth of the width. After exploring this 6-foot banner, use a variety of lengths (e.g., 12 feet, 15 feet) and various widths (e.g., halves, fourths, thirds). Encourage students to notice patterns that help them determine the area of the banners, considering how the banner is scaled based on the values involved in the problem. For students with special needs in math, you can cut out grid paper and have them fold or draw on the paper to show the partitions.

Pearson eText Blackline Master: One-Inch Grid Paper

Counters (a set model) are an effective tool for finding parts of a whole. Activity 15.11 is focused on this idea.

Activity 15.11 Counting Counters: Fraction of a Whole

CCSS-M: 5.NF.B.4a

Recall that in Chapter 14, two-color counters were used to develop the concept of partitioning and iterating with prompts such as, "If the whole is 24, what is $\frac{3}{8}$ of the whole?" These can be slightly adapted to make the connection to multiplication more explicit by having students write the equations that match the question. Distribute the activity page along with enough counters for each student to explore. After enacting the problems with the counters, have students record the related equation. Ask students to explain what patterns they noticed as they solved the various problems.

Pearson eText Activity Page: Counting Counters: Fraction of a Whole

Multiply a Fraction by a Fraction—No Subdividing. Once students have had experiences with wholes of fractions (15 groups of $\frac{2}{3}$) and fractions of a whole ($\frac{2}{3}$ of 15), a next step is to introduce finding a fraction of a fraction but to carefully select tasks in which no additional partitioning is required. For example, to find one-half of two-fifths:

$$\frac{1}{2} \times \frac{2}{5} =$$

Notice that it is possible to find one-half of the numerator, without any partitioning. The answer is $\frac{1}{5}$.

Pearson eText Video Example 15.9

In this video consider how the student uses a visual to find fractions of fractions.

Activity 15.12 Solving Fraction Stories

CCSS-M: 5.NF.B.4a,b; 5.NF.B.6

Distribute the activity page or cut out each story and distribute just the first story to each student or small group. All students, in particular multilingual learners, benefit from multiple reads of the story before beginning to solve the problem. Use the Three Reads process (see Figure 4.5 in Chapter 4, p. 68) with the first story and ask students to use this protocol in small groups as they solve the other stories. Instruct students to create an illustration or use a manipulative to model the story and then write a multiplication equation that represents the situation.

Example story:

You have $\frac{3}{4}$ of a pizza left. If you give $\frac{1}{3}$ of the leftover pizza to your brother, how much of a whole pizza will your brother get?

Figure 15.7 shows how problems of this type might be modeled. However, there are other ways to partition these problems, and it is important for students to model and solve these problems in their own way.

Pearson eText Activity Page: Multiplication-of-Fractions Stories
Pearson eText Expanded Lesson: Multiplication-of-Fractions Stories

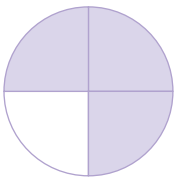
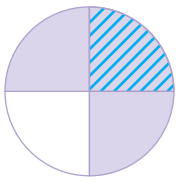
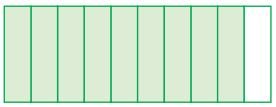
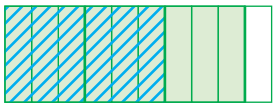
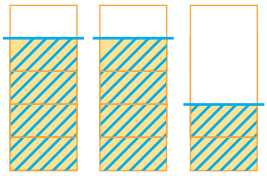
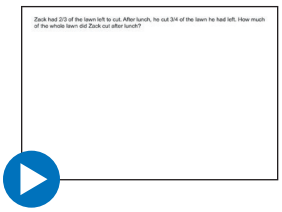
Task	Finding the starting amount	Showing the fraction of the starting amount	Solution
Pizza Find $\frac{1}{3}$ of $\frac{3}{4}$ (of a pizza) or $\frac{1}{3} \times \frac{3}{4}$			$\frac{1}{3}$ of the $\frac{3}{4}$ is $\frac{1}{4}$ of the original pizza. $\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$
Bread Find $\frac{2}{3}$ of $\frac{9}{10}$ (of a loaf of bread) or $\frac{2}{3} \times \frac{9}{10}$			$\frac{2}{3}$ of the $\frac{9}{10}$ is 6 slices of the loaf or $\frac{6}{10}$ of the whole. $\frac{2}{3} \times \frac{9}{10} = \frac{6}{10}$
Paint Find $2\frac{1}{2}$ of $\frac{4}{5}$ (ounces of paint) or $2\frac{1}{2} \times \frac{4}{5}$			$2\frac{1}{2}$ of the $\frac{4}{5}$ is $\frac{4}{5} + \frac{4}{5} + \frac{2}{5} = \frac{10}{5}$

FIGURE 15.7 Connecting representation to the procedure for three problems involving multiplication



Pearson eText
Video Example 15.10

Multiply a Fraction by a Fraction—with Subdividing. When the pieces must be subdivided into smaller unit parts, the problems become more challenging. Let's look at two situations.

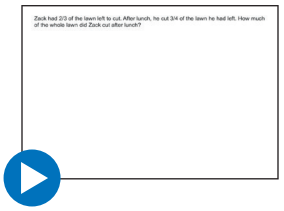
1. **Zack mowed $\frac{1}{3}$ of a field before lunch. After lunch, he mowed $\frac{3}{4}$ of what was left to mow. What fraction of the whole field still needs to be mowed?**
2. **Milo and Mazie loved their kitten treats. Milo ate $\frac{1}{3}$ of the treats. Mazie ate $\frac{3}{5}$ of what was left. What fraction was left for the other kittens?**

Pause & Reflect

Pause for a moment and figure out how you could illustrate the solution to these problems. ●

In Zack's mowing problem, you are finding $\frac{3}{4}$ of $\frac{2}{3}$ of the yard, so $\frac{3}{4} \times \frac{2}{3}$. You cannot find fourths of 2 without more partitioning. Figure 15.8 shows a way to use fraction strips to partition the thirds into sixths and solve the problem. The kitten problem leads to the dilemma of figuring out what might be the starting quantity that can be partitioned into thirds and then partitioned into fifths. Both thirds and fifths? 15. Figure 15.9 illustrates one way to reason through the problem.

Using Rectangles. Like with whole numbers, rectangles are an effective visual to show multiplication (while connecting to area). Because rectangles are used with area, students can make connections between whole numbers and fractions (Lischka & Stephens, 2020; Webel, Krupa, & McManus, 2016). Quilting is a good context, because quilts are rectangles, and the individual rectangles (or squares) within the quilt are a fraction of the whole quilt.



Pearson eText
Video Example 15.11
In these videos, the students solve the Lawn Mowing Problem. In what ways are their approaches similar, and in what ways do they differ?

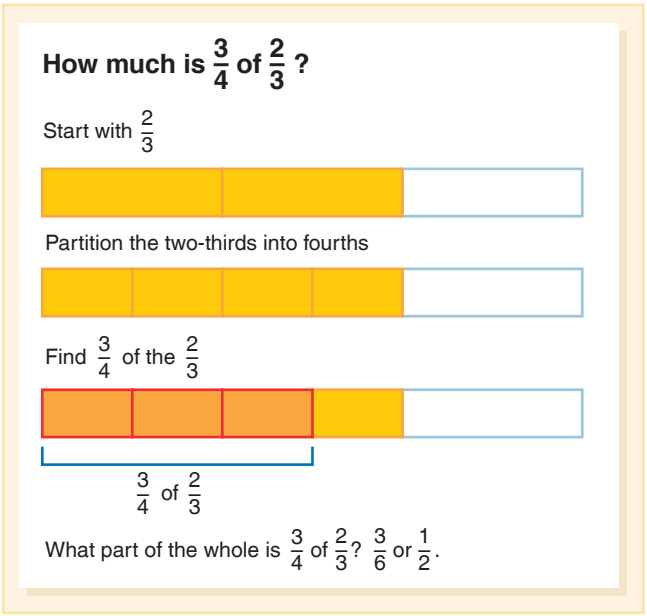

How much is 3/4 of 2/3 ?
Start with 2/3
Partition the two-thirds into fourths
Find 3/4 of the 2/3
3/4 of 2/3
What part of the whole is 3/4 of 2/3? 3/6 or 1/2.

FIGURE 15.8 Solutions to a multiplication problem when the parts must be subdivided

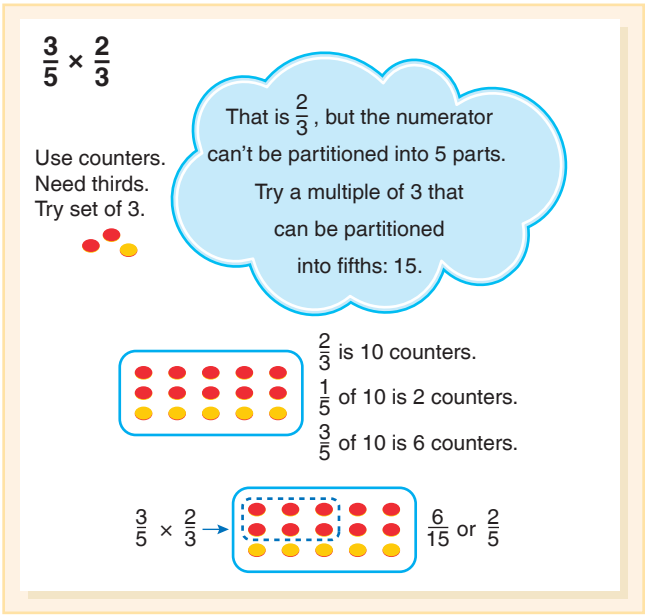

3/5 × 2/3
Use counters.
Need thirds.
Try set of 3.
That is 2/3, but the numerator can't be partitioned into 5 parts. Try a multiple of 3 that can be partitioned into fifths: 15.
2/3 is 10 counters.
1/5 of 10 is 2 counters.
3/5 of 10 is 6 counters.
3/5 × 2/3 → 6/15 or 2/5

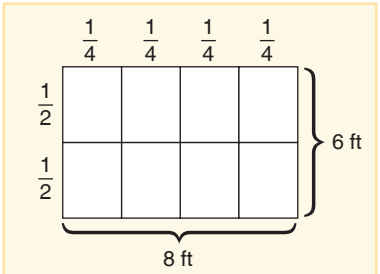
FIGURE 15.9 Modeling multiplication of fractions with a set model (two color counters)

Activity 15.13 Quilting Pieces

CCSS-M: 5.NF.B.4b; 5.NF.B.5b

Have students use grid paper to sketch a drawing of a quilt that will be 8 feet by 6 feet—or create a full-sized one for your class! Explain that each group will prepare a quilt piece that is 3 feet by 2 feet. Ask students, "What fraction of the quilt is one quilt piece?"

Repeat with a focus on the whole quilt (see diagram). Now each quilt piece is $\frac{1}{4}$ of $\frac{1}{2}$ of the quilt. Again, ask students, "What fraction of the quilt is one quilt piece?" Help students make the connection that 6 square feet out of 48 square feet (first exploration) is one-eighth of the whole quilt, and this can be represented as $\frac{1}{4}$ of $\frac{1}{2}$ of the whole quilt $(\frac{1}{4} \times \frac{1}{2} = \frac{1}{8})$.



Source: Adapted from J. K. Tsankova, & K. Pjanic, (2009/2010). The area model of multiplication of fractions. *Mathematics Teaching in the Middle School, 15*(5), 281–285.

The following activity, based on Imm, Stylianou, & Chae (2008) provides a context that engages students in exploring multiplication of fractions and the commutative property.

Activity 15.14 Part of a Part of a Lot

CCSS-M: 5.NF.B.4b; 5.NF.B.6

Post or distribute the Part of a Part of a Lot activity page. This task asks students to compare two community plans (A and B) for using the same-sized lot. Students can illustrate the situation by using rectangles and reason to answer the two questions:

Which community plan has the bigger playground?

Which community has the bigger rubber surface on its playground?

After reading the task, ask students to predict the answers to these questions. Place students in partners and ask one to solve the problem for community A and the other to solve for community B. Once they have completed their illustrations and solutions, ask students to compare their responses and be ready to report to the class what they decided.

Pearson eText Activity Page: Part of a Part of a Lot

As students continue to use visuals and explore contexts, they are ready to generalize the meaning of finding a fraction of a fraction.

Provide students with a square and ask them to partition and shade an area to illustrate the fraction that is the initial value (see Figure 15.10). For example, in $\frac{3}{5} \times \frac{3}{4}$, you are finding $\frac{3}{5}$ of $\frac{3}{4}$, so you start with an area of $\frac{3}{4}$ of the square, shown in vertical pieces (step [a] in Figure 15.10). To find fifths of the $\frac{3}{4}$, draw four horizontal lines through the $\frac{3}{4}$ (step [b]) or all the way across the square so that the whole is in same-size partitions, and shade three of the fifths (step [c]). The overlap of the shading illustrates what is $\frac{3}{5}$ of $\frac{3}{4}$ of the original whole. Notice how this model effectively shows that the answer is a part of a part and therefore less than the original starting value.

Standards for Mathematical Practice
MP7. Look for and make use of structure

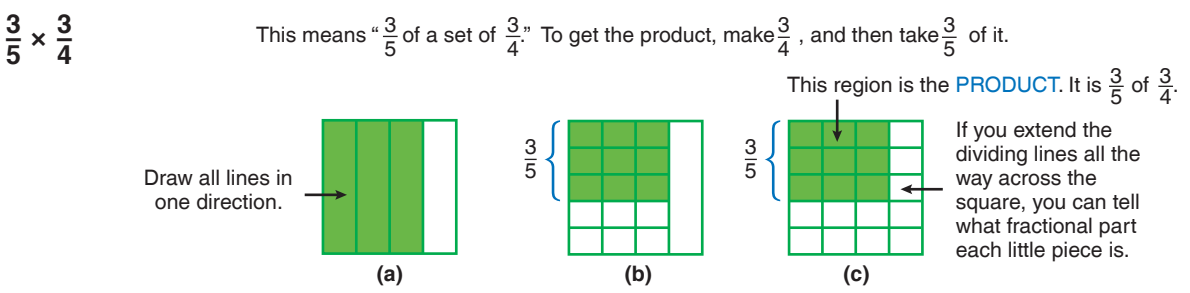

3/5 × 3/4
This means "3/5 of a set of 3/4." To get the product, make 3/4, and then take 3/5 of it.
This region is the PRODUCT. It is 3/5 of 3/4.
Draw all lines in one direction.
3/5
3/5
If you extend the dividing lines all the way across the square, you can tell what fractional part each little piece is.
(a)
(b)
(c)
There are three rows and three columns in the PRODUCT, or 3 × 3 parts.
The WHOLE is now five rows and four columns, so there are 5 × 4 parts in the whole.
PRODUCT = 3/5 × 3/4 = Number of parts in product / Kind of parts = 3 × 3 / 5 × 4 = 9/20.

FIGURE 15.10 Developing understanding of the algorithm for multiplying fractions

Estimating Products

Helping students learn to estimate products involving fractions serves (at least) two important purposes. First, there are many real-life instances when whole numbers and fractions must be multiplied, and mental estimates are quite useful. For example, sale items are frequently listed as half off, or we read of a one-third increase in the number of registered voters. Also, fractions are excellent substitutes for percentages, as you will see in the next chapter. To get an estimate

of 75 percent of $36.69, it is useful to think of 75 percent as $\frac{3}{4}$, finding one-fourth (about $9) and then three-fourths ($27). Second, estimating with fractions supports student understanding of multiplication. Remember, multiplication is scaling. So, when you multiply by 2, the product is twice the size of the other factor; when you multiply by one-third, the product is one-third of the size of the other factor. Estimating helps students attend to this notion of scaling.

Here are three strategies for estimating products:

- *Benchmarks:* $\frac{7}{8} \times \frac{5}{12} =$ _____. Think, "This expression is about one times one half, but a little less, so the answer will be less than the benchmark one-half."
- *Compatible numbers:* To estimate $\frac{3}{5}$ of $36.69, for example, a useful compatible number is $35. One-fifth of 35 is $7, so three-fifths is 3×7, or $21.
- *Relative size of unit fractions:* $\frac{1}{3} \times 3\frac{4}{5} =$ _____. Think, "I need a third of this value. A third of 3 is 1, and $\frac{1}{3}$ of $\frac{4}{5}$ is close to $\frac{1}{5}$, so a reasonable estimate is about _____."

The strategy students select depends on the size of the fractions given. That means they need many opportunities to estimate and think about and discuss which strategy they want to use in each case. Post this menu of choices for students so they have a reminder of their options (something particularly important for students with special needs in math). A final note: Be sure students are not computing the exact answer and then rounding to determine an estimate.

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

***TECHNOLOGY* Note.** Math Learning Center (mathlearningcenter.org) and Brainingcamp (braingingcamp.com) allow you to select a manipulative to solve fraction operation problems (any operation). For example, pattern blocks or number lines can be used to multiply or divide fractions. Dreambox (www.dreambox.com) Teacher Tools offers an activity in which students use an area model to multiply two fractions. ■

Developing Strategies and Algorithms

As with all operations, fluency means that students first ask whether they can solve the problem mentally before enacting an algorithm. With multiplication of fractions, that can include skip-counting or break-apart strategies. Activity 15.15 is an excellent activity that focuses on developing flexible thinkers. Researchers found that strong performance on arithmetic sentence production tasks with rational numbers predicted success through algebra (McMullen, Hannula-Sormunen, Lehtinen, & Siegler, 2020).

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Activity 15.15 Fraction Sentence Creation

CCSS-M: 5.NF.B.4b; 5.NF.B.6

Give students a set of numbers (fractions and/or decimals) and a target. Ask them to find as many mathematically correct arithmetic sentences as possible, using as many values as they want from the list (and they can use a number on the list more than once). They can use any of the operations.

Begin with a whole-number example so that they understand the options:

Target Number	Set of Numbers
8	1 2 4 5
1	$\frac{1}{4}$ $\frac{1}{2}$ 2 4
$\frac{1}{2}$	$\frac{1}{8}$ $\frac{1}{4}$ $\frac{3}{4}$ 2 4

After students have had a few minutes to generate their number sentences, record all the different equations. You can have rich discussions about what is considered different. Here are some equations students might write for the middle example:

$\frac{1}{2} + \frac{1}{2} = 1$ $\quad \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$ $\quad \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$ $\quad 2 \times \frac{1}{2} = 1$

$\frac{1}{2} \times 2 = 1$ $\quad 4 \times \frac{1}{4} = 1$ $\quad 2 \times \frac{1}{4} + 2 = 1$ $\quad 4 \times \frac{1}{2} \div 2 = 1$

This activity can be a routine, a learning center activity, or a partner activity within a lesson. Additionally, you can integrate decimals into the sets of numbers.

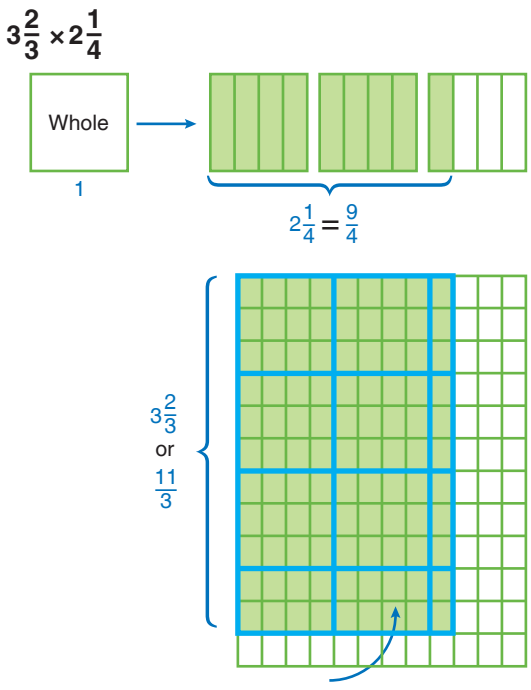
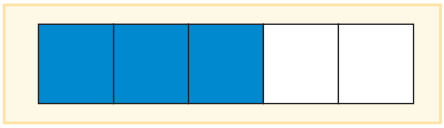

$3\frac{2}{3} \times 2\frac{1}{4}$
Whole
1
$2\frac{1}{4} = \frac{9}{4}$
$3\frac{2}{3}$ or $\frac{11}{3}$
The PRODUCT is $3\frac{2}{3}$ sets of $2\frac{1}{4}$.
There are 11 rows and 9 columns, or 11 × 9 parts, in the PRODUCT.
The WHOLE now has three rows and four columns, or 3 × 4 parts.
$3\frac{2}{3} \times 2\frac{1}{4} = \frac{11}{3} \times \frac{9}{4}$ = PRODUCT =
Number of parts / Kind of parts $= \frac{11 \times 9}{3 \times 4} = \frac{99}{12} = 8\frac{1}{4}$

FIGURE 15.11 The approach used to develop the algorithm for fractions less than 1 can be expanded to mixed numbers

Activity 15.16 Can You See It?

CCSS-M: 5.NF.B.3; 5.NF.B.5a, b

Post a partially shaded rectangle like the one shown here.



Ask students the following questions and have them explain how they see it.

Can you see $\frac{3}{5}$ of something? Can you see $\frac{5}{3}$ of ____?
Can you see $\frac{5}{3}$ of something? Can you see $\frac{2}{3}$ of ____?

Rectangles are a good way to illustrate multiplication of mixed numbers, as illustrated in Figure 15.11, demonstrating a break-apart strategy similar to finding partial products with whole numbers. The standard algorithm suggests that students change mixed numbers to fractions. That method also works (and is often most efficient when both numbers are mixed numbers).

With enough experiences using visuals, students will notice a pattern. Remember that "enough" is probably a lot more than is usually provided—in other words, this does not mean two or three examples, but several weeks working with different examples and representations. These exercises will lead students to focus on how the denominators relate to how the grid (or line) is partitioned (the total parts, or the denominator) and how the numerator determines how much of that total region is the solution to the problem.

When students are ready to start using the standard algorithm for fraction multiplication, continue to have students connect to the visuals. Ask, "Why are you multiplying the denominators?" "What does it mean to multiply the numerators?" Continue to ask students to estimate the answer, and then to check the reasonableness of their answers.

Factors Greater than One. Fractions greater than 1 may be written as fractions (e.g., $\frac{5}{3}$) or as mixed numbers (e.g., $1\frac{2}{3}$). Multiplying fractions in which at least one of the factors is greater than 1—for example, $\frac{3}{4} \times 2\frac{1}{2}$—should be integrated into multiplication with fractions less than 1. When fractions greater than 1 are mixed in, students can see the impact of multiplying by a number less than 1 and a number greater than 1. Activity 15.16 is a way to focus on this reasoning (Thompson, 1995).

Pause & Reflect

Find the four partial products in this multiplication: ●

Figure 15.12 shows how this product might be worked out by multiplying the individual parts. When multiplying two mixed numbers, there will be four partial products. These are added to get the product, although an estimate may be enough. Notice that the same four partial products of Figure 15.12 can be found in the rectangle in Figure 15.11. The

partial-products process is more conceptual than the traditional algorithm. It is important for students to understand different methods and then be allowed to choose one that they understand and that works best for them.

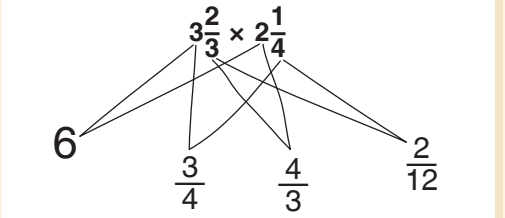


FIGURE 15.12 Using partial products to multiply mixed numbers

Challenges with Multiplying Fractions

When students begin working with fraction multiplication, they already have internalized concepts of whole-number multiplication and fraction addition and subtraction that can sometimes lead to confusion. This overgeneralizing is exacerbated when students are not given adequate time to explore multiplication of fractions conceptually and when they are too quickly pressed to memorize procedures, such as "multiply both the bottom and the top." The result of memorizing rules that don't make sense is errors and confusion, as described in Table 15.4.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

TABLE 15.4 COMMON CHALLENGES FOR MULTIPLYING FRACTIONS

Common Challenges	What It Looks Like	How to Help
1. Treat the denominator the same as in fraction addition/subtraction problems	$\frac{1}{4} \times \frac{3}{8} = \frac{2}{8} \times \frac{3}{8} = \frac{6}{8}$	• Engage students in comparing addition and multiplication and *why* they are different procedures—for example, comparing rectangle illustrations for both operations and comparing on a number line. • Be sure that visuals are connected to procedures. • Reinforce estimation before solving and compare the answer to the estimate after solving.
2. Inability to estimate approximate size of the answer	For a problem such as $\frac{8}{9} \times \frac{3}{5}$, the student multiplies and then rounds to find the answer.	• Explicitly teach and practice each of the preceding estimation strategies suggested. • Encourage and discuss estimation for all computations, using estimates to make sure answers are reasonable.
3. Think answer is wrong because it is less than the factors	Student multiplies $\frac{1}{4} \times 4\frac{1}{2}$ and gets the product $1\frac{1}{8}$, but thinks it must be wrong.	• Use visuals to illustrate. • Add contexts or ask students to create a context to fit the problem.
4. Confuse whether to multiply or divide by the given fraction	For "What is $\frac{1}{3}$ of \$24?" rather than divide by 3 or multiply by $\frac{1}{3}$ (both of which are correct), student divides by $\frac{1}{3}$.	• Estimate. • Ask, "Should the result be more or less than the initial amount?" • Ask students to interpret the situation and the answer, and see if they make sense.
5. Interpreting story situations into an appropriate expression	In Zack's mowing problem (p. 398), student is not sure if they need to multiply, subtract, or divide.	• Avoid the key word approach with story problems—it not only doesn't work; it makes things harder on students as problems get more complex! • Focus students' attention on the situation qualitatively (without a focus on actual numbers). • Encourage students to create a visual representation and then translate it to symbols.

Division

Learning Outcome 15.4 **Explain division of fractions, using both measurement and partitive examples.**

Can you think of a real-life example for dividing by a fraction? Few people can, even though we conceptually use division involving fractions in many real-life situations. Do you know the invert-and-multiply algorithm? Division of fractions has one of the most mysterious algorithms in K–8 mathematics. Avoid this cryptic approach and help students really understand when and how to divide with situations involving fractions.

Teaching of division involving fractions developmentally follows this progression of problem types:

1. A whole number divided by a whole number: $14 \div 5$ or $22 \div 7$
2. A fraction divided by a whole number: $\frac{1}{2} \div 4$ or $\frac{7}{8} \div 2$
3. A whole number divided by a fraction: $4 \div \frac{1}{3}$ or $4 \div \frac{2}{3}$
4. A fraction divided by a fraction: $\frac{7}{8} \div \frac{1}{8}$ or $2\frac{1}{4} \div \frac{1}{2}$

Within each problem type (except the first type), first explore (a) unit fractions, (b) non-unit fractions in which no additional partitioning is required, and, finally, (c) situations wherein additional partitioning is needed. This section is organized in this manner.

***TECHNOLOGY* Note.** PBS Learning Media (https://www.pbslearningmedia.org) is an excellent place to find instructional videos (for any topic, not just fractions!). For example, a search for fraction division leads to this option: *Division of Fractions: Using Fraction Strips*. ■

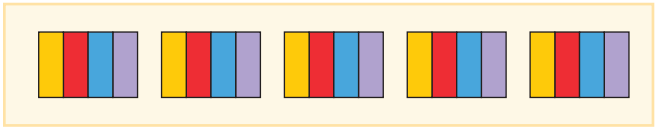
Contextual Examples and Models

Pearson eText
Video Example 15.12
In this video author Jennifer Bay-Williams provides an explanation and overview of division of fractions.

As with whole numbers, there are two meanings of division: partitive (sharing) and measurement (equal subtraction) (Gregg & Gregg, 2007; Kribs-Zaleta, 2008; Tirosh, 2000). These meanings are both needed to develop the different types of fraction problems. As you explore this division progression, notice that the contexts and the representations fit each other. When students are asked to share something like a brownie, they will draw rectangles, creating a representation to match the situation (Wahyu et al., 2020).

Divide a Whole Number by a Whole Number. A partitive or sharing context is helpful in interpreting division of a whole number by a whole number (Empson & Levi, 2011; Lamon, 2020). Very young children can understand sharing (e.g., 3 cookies shared with 2 people). Because it is a foundational concept closely tied to partitioning, sharing tasks are a major topic in Chapter 14. Here we focus on the connection to division.

Context: Four friends share 5 sandwiches.
One way to solve this is to give each friend 1 sandwich and $\frac{1}{4}$ of the fifth sandwich. Another approach is to cut each sandwich into fourths, and give each friend a piece from each sandwich:



Equation: $5 \div 4 = \frac{5}{4}$ *or* $1\frac{1}{4}$

Now, imagine two more friends arrived. How will six friends share 5 sandwiches? Now the latter approach works well. Each person will get one-sixth of each sandwich, so they will each get five sixths:

$$5 \div 6 = \frac{5}{6}$$

Standards for Mathematical Practice
MP7. Look for and make use of structure

This concept applies, no matter how messy the numbers. Think of 92 sandwiches shared with 11 people ($92 \div 11$). Still, each person would receive $\frac{1}{11}$ of each of 92 sandwiches and so have $\frac{92}{11}$ of the sandwiches. Understanding fraction division requires many fair sharing tasks! This section is only brief because of the many examples provided in the previous chapter. As you begin planning for fraction division, start with the progression of sharing tasks outlined in that chapter and connect them to the symbols, as illustrated here. This foundation is critical to making sense of the other forms of division involving fractions!

Divide a Fraction by a Whole Number. As indicated in Table 15.1, these problem types are typically introduced in grade 5 as unit fractions $\left(\frac{1}{4} \div 3\right)$ and in sixth grade as non-unit fractions $\left(\frac{9}{10} \div 3 \text{ or } 2\frac{1}{2} \div 6\right)$. Partitive division (sharing) is a good way to develop this type of division conceptually.

Unit Fractions. As noted earlier, unit fractions are a good place to begin making sense of the problem type.

Pause & Reflect

What story situation might you tell for $\frac{1}{4} \div 3$? ●

Many situations can fit this expression. Garden plots are one and the focus of Activity 15.17.

Activity 15.17 Plotting Your Garden Plots

CCSS-M: 5.NF.B.7a, c

Distribute the Garden Plots activity page (or post one of the stories). Here is an example story:

> Three gardeners are equally sharing $\frac{1}{4}$ of a garden plot. What fraction of the plot does each gardener have?

Have students illustrate their solution on the activity page and then record an equation to represent the situation and solution. After students finish their four tasks, ask them to explain what they notice across problems and what they notice between the situation and the equation. Emphasize the notion of "How much for *one*?" Students with special needs in math may benefit from having garden plots (rectangles) to partition (see Teacher Resource).

Pearson eText Activity Page: Plotting Your Garden Plots
Pearson eText Activity Page: Garden Plots

The garden context is an area model. Here are two story contexts that are length and set, respectively:

- *Water Bottles.* There is $\frac{1}{4}$ of a gallon of water that is poured equally into three water bottles. How much water is poured into each?
- *Cheese Sticks.* Arlo buys a bag of cheese sticks. He takes $\frac{1}{4}$ of the cheese sticks to a picnic. At the picnic, he decides to share the cheese sticks equally with 2 friends (and himself). What fraction of the bag does each person get?

Across these sharing situations, you are asking, "How much is the share for *one* friend?" And, beyond friends, the questions might be, "How many miles are walked in *one* hour?" or "How much ribbon for *one* bow?"

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

Non-Unit Fractions, No Additional Partitioning. After students have had adequate experiences and can interpret the meaning of dividing a unit fraction by a whole number, they are ready to divide any fraction (or mixed number) by a whole number. Begin with contexts. Ribbon provides a good length context:

Miguel has $5\frac{1}{3}$ yards of ribbon to make four bows for birthday packages. How much ribbon should he use for each bow if he wants to use the same length of ribbon for each?

When $5\frac{1}{3}$ is thought of as fractional parts, there are 16 thirds to share, or 4 thirds for each bow. Alternatively, one might think of allotting 1 yard per bow, leaving $1\frac{1}{3}$ yards. These 4 thirds are then shared, one per bow, for a total of $1\frac{1}{3}$ yards for each bow. Importantly, the unit parts (thirds) required no further partitioning to do the sharing. Students need numerous experiences with sharing tasks such as these before exploring tasks that do require additional partitioning.

Non-Unit Fractions, More Partitioning Required. Consider how Mark might equally distribute his time:

Mark has $1\frac{1}{4}$ hours to finish his three chores. If he divides his time evenly, how many hours can he give to each chore?

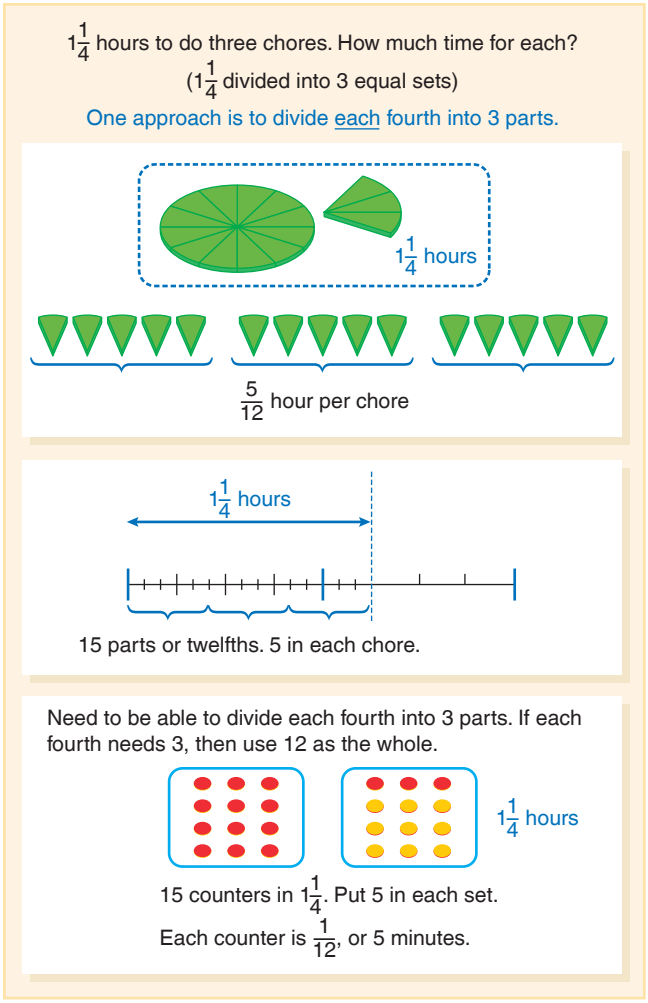

$1\frac{1}{4}$ hours to do three chores. How much time for each?
($1\frac{1}{4}$ divided into 3 equal sets)
One approach is to divide each fourth into 3 parts.
$1\frac{1}{4}$ hours
$\frac{5}{12}$ hour per chore
$1\frac{1}{4}$ hours
15 parts or twelfths. 5 in each chore.
Need to be able to divide each fourth into 3 parts. If each fourth needs 3, then use 12 as the whole.
$1\frac{1}{4}$ hours
15 counters in $1\frac{1}{4}$. Put 5 in each set.
Each counter is $\frac{1}{12}$, or 5 minutes.

FIGURE 15.13 Partitive division (sharing) with a whole-number divisor illustrated across three models

The question asks, "How many hours for one chore?" The 5 fourths of an hour that Mark has do not split neatly into three parts. So some or all of the parts must be partitioned. Figure 15.13 shows how to model these with each type of model (area, length, and set). In each case, all of the fourths are subdivided into three equal parts, producing twelfths. There are a total of 15 twelfths, or $\frac{5}{12}$ hour for each chore. (Test this answer against the solution in minutes: $1\frac{1}{4}$ hours is 75 minutes, which divided among 3 chores is 25 minutes per chore: $\frac{25}{60} = \frac{5}{12}$.). Expanded Lesson: Division-of-Fraction Stories offers a complete lesson, using and comparing Miguel and Mark's stories.

Pearson eText Expanded Lesson: Division-of-Fraction Stories

Divide a Whole Number by a Fraction. This problem type (a fraction as the divisor) lends to a measurement interpretation (also called *repeated subtraction*). In a measurement interpretation, an equal group is taken away from the total repeatedly. For example, "If you have 12 quarts of lemonade, how many canteens holding 3 quarts each can you fill?" Notice that this is not a sharing situation but, rather, a repeated subtraction situation, taking away 3 quarts repeatedly. In this case, the question we ask is, "*How many 3s are in 12*?"

The measurement interpretation is a good way to explore division by a fraction, because students can draw illustrations to show the measures (Cramer et al., 2010). Bar diagrams, fraction strips, and number lines support student reasoning. A general context for a measurement interpretation is counting servings of a particular size. As we look ahead, it is the measurement interpretation that connects to making sense of the algorithm for dividing fractions.

Unit Fractions. Again, we start with unit fractions. Activity 15.18 uses a context of serving sandwiches to build a foundation for division by unit fractions.

Activity 15.18 Sandwich Servings

CCSS-M: 5.NF.B.7b, c

Distribute the activity page, which engages students in this task:

At Super Sub Sandwich Shop, they make long sandwiches and slice them into the following servings:

Child-size: $\frac{1}{6}$ Small: $\frac{1}{3}$ Medium: $\frac{1}{2}$ Large: 1

Engage students in solving questions about how many servings can be made with, for example, 6 sub sandwiches. What is the most number of servings they can make? If they only made small servings, how many servings could they make? Then, have students write division expressions that fit each situation.

Next, reverse the process: Give students an expression and have them tell a sub sandwich story to match. For example:

$1 \div \frac{1}{3}$ $8 \div \frac{1}{3}$ $5 \div \frac{1}{2}$ $12 \div \frac{1}{6}$

After students explore these tasks, ask them what they notice about dividing by unit fractions. As a challenge, students can brainstorm combinations of different sandwich servings that can be made with 6 sandwiches. Fraction strips are a useful tool in supporting students with special needs in math.

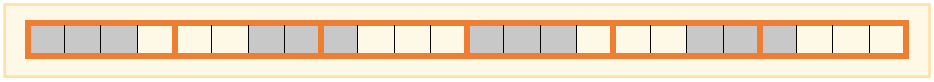
Pearson eText Activity Page: Sandwich Servings

This activity provides an opportunity to compare the impact of unit fractions as divisors and discuss *why* you get more servings when you have smaller fractions as divisors. This helps build the relationship between multiplication and division, and students may notice the general case: $1 \div \frac{1}{n} = n$ and therefore, $a \div \frac{1}{n} = a \times n$ (Cavey & Kinzel, 2014). In other words, if you get 3 servings from 1 sandwich, you will get 24 servings from 8 sandwiches.

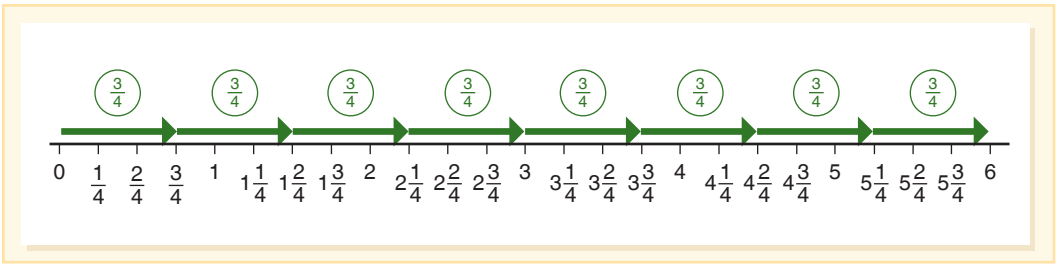
Non-Unit Fractions, No Additional Partitioning. Let's explore a measurement situation with non-unit fractions as divisors:

Les & Colin's Smoothies Shop has just bought a machine that blends 6 pints of smoothies for one batch. Their Smoothie cups hold $\frac{3}{4}$ of a pint. How many smoothies can be served from one batch?

Bar diagrams (or rectangles) can illustrate this situation (the thick border shows each pint, and the shaded/not shaded segments show each serving):

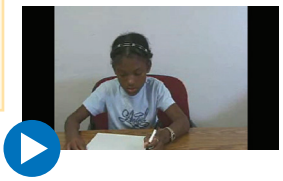


Number lines are also an important visual:


$\frac{3}{4}$
0 $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ 1 $1\frac{1}{4}$ $1\frac{2}{4}$ $1\frac{3}{4}$ 2 $2\frac{1}{4}$ $2\frac{2}{4}$ $2\frac{3}{4}$ 3 $3\frac{1}{4}$ $3\frac{2}{4}$ $3\frac{3}{4}$ 4 $4\frac{1}{4}$ $4\frac{2}{4}$ $4\frac{3}{4}$ 5 $5\frac{1}{4}$ $5\frac{2}{4}$ $5\frac{3}{4}$ 6

Non-unit fraction situations are more difficult for students than unit fractions are. A way to scaffold the reasoning *and* make sense of non-unit fractions is first to pose the unit fraction question: If the Smoothie cup only held *one-fourth*, how many servings could you make? Then students begin to see that if the serving is three times as big (three-fourths, not one-fourth), the number of servings is one-third as many.

Let's go back to the other division interpretation—partitioning (sharing). This interpretation can also be used to make sense of whole numbers divided by fractions. Remember that the focus question in sharing is, "How much for one (e.g., person)?" or "How many for one?" You may recall that in Chapter 14, you explored tasks that gave you a part and asked you to find the whole (see Activity 14.14). Essentially, these tasks are saying, "How much for one?" and are therefore a good way to build meaning for division by a fraction. Activity 15.19 adapts that activity to connect to division more explicitly.



Pearson eText
Video Example 15.13
In this video consider the type of problem and how the student solves a fraction by fraction division problem.

Standards for Mathematical Practice
MP1. Reason abstractly and quantitatively

Activity 15.19 Counting Counters: How Much in One Whole Set?

CCSS-M: 5.NF.B.7a, c

Use the Counting Counters: How Much for One? activity page and give students 40–50 two-color counters. Ask students to solve tasks that tell how many are in a fraction of a set and ask students to use reasoning to figure out how many are in the whole set.

Set 1: Unit fractions as divisors:
Example: If 8 counters represent $\frac{1}{4}$ of the whole, how many counters are in one whole set?
Expression:

Set 2: Non-unit fractions, but fractions less than 1:
Example: If 15 counters represent $\frac{3}{5}$ of the whole, how many counters are in one whole set?
Expression:

Set 3: Non-unit fractions, mixed numbers, and fractions greater than 1:
Example: If 18 counters represent $2\frac{1}{4}$ sets, how many counters are in one whole set?
Expression:

As students are working, ask them to describe their reasoning strategies. Help them notice that they are finding how many in *one part* (e.g., one-fourth or one-fifth) and then iterating (multiplying) to find out how many in *one whole* (e.g., four-fourths or five-fifths).

Pearson eText Activity Page: **Counting Counters: How Much for One?**

Divide a Fraction by a Fraction. Over time, using various contexts and numbers that vary in difficulty, students will be able to tackle problems that are more complex both in the context and in the numbers involved.

Unit Fractions. Let's start with fractions with like denominators and a unit fraction for the divisor.

Pause & Reflect

Illustrate and solve each expression in a way that makes sense to you. ●

$$\frac{7}{8} \div \frac{1}{8} \qquad \frac{3}{5} \div \frac{1}{5} \qquad \frac{11}{12} \div \frac{1}{12}$$

As you solve these, you might notice that you only need to attend to the numerator. You are figuring out how many eighths in seven eighths—there are 7! We will return to this idea later, but this is a great aha to begin working with when you start to teach fractions divided by fractions.

Next, explore two unit fractions, but with different denominators (where the smaller fraction is the divisor, and you can still use counting to reach the dividend). For example: $\frac{1}{2} \div \frac{1}{4}$. Using the measurement interpretation by thinking of the serving context, you are asking, "How many fourths in one-half?" The answer is 2. You can get two servings of one-fourth if you start with one-half of a serving. Figure 15.14 illustrates this idea with an area model and a length model.

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

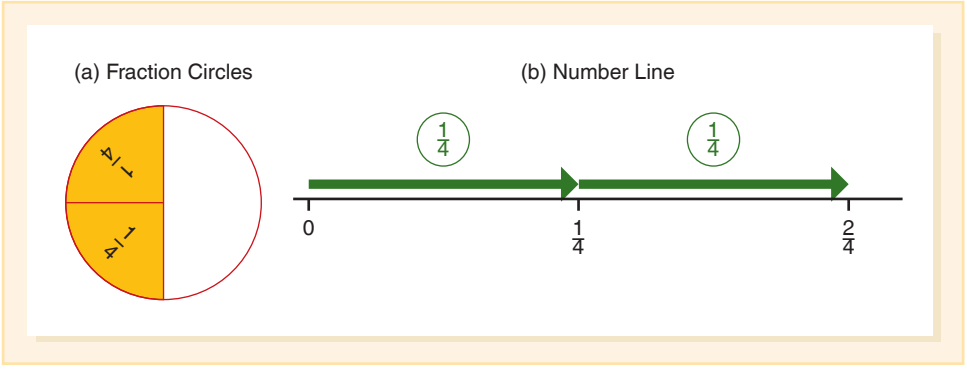


FIGURE 15.14 Models for division, involving unit fractions

Continue with unit-fraction divisors but vary the dividend (making sure it still turns out to be a whole number answer). For example:

$\frac{3}{4} \div \frac{1}{8}$ $\frac{2}{3} \div \frac{1}{6}$ $\frac{3}{4} \div \frac{1}{16}$

Non-Unit Fractions, No Additional Partitioning. At long last, students have the experiences to explore any kind of fraction. Although mixed numbers have had the reputation of being more difficult, using mixed numbers early can help students make sense of division of a fraction by a fraction:

Fabi had $2\frac{1}{4}$ cups of brown sugar. Each batch of cookies uses $\frac{3}{4}$ cup of brown sugar. How many batches of cookies can Fabi make?

Try solving this problem yourself. Use any model or drawing you wish to help explain what you are doing. The cookie problem is a classic measurement situation and familiar to many students. Partitioning (sharing) situations can also be used to make sense of fractions divided by fractions. Remember the big idea with sharing is, "How much for one [person]?"

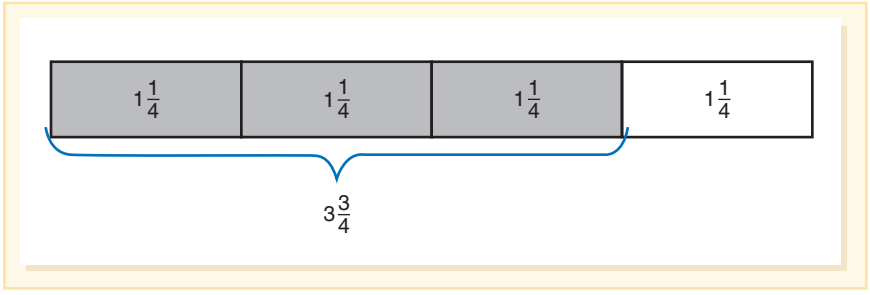
Activity 15.20 How Much for One?

CCSS-M: 6.NS.A.1

Pose contextual problems where the focus question is, "How much for one _____?" Then ask students to create bar diagrams, fraction strips, or number lines to fit the story. An example is provided here:

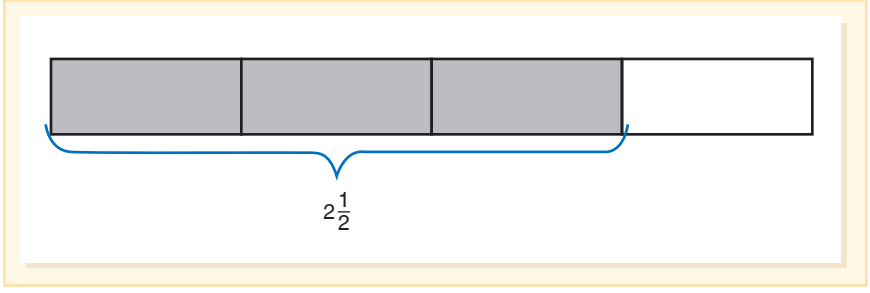
Story: Dan paid $\$3\frac{3}{4}$ for a $\frac{3}{4}$ pound box of cereal. How much money is that per pound?

Bar Diagram:



Story: Andrea found that if she walks quickly during her morning exercise, she can cover $2\frac{1}{2}$ miles in $\frac{3}{4}$ of an hour. She wonders how fast she is walking in miles per hour.

Bar Diagram:



After students diagram the problem, ask them to reflect on their process. They likely figured out how much for one-fourth and then continued to iterate that quantity to figure out how much for one whole.

Answers That Are Not Whole Numbers

Many problems are not going to come out evenly, and it becomes very important to make sense of what is left over. If Cassie has 5 yards of ribbon to make bows, and each bow needs $1\frac{1}{6}$ yards, she can make only four bows, because a part of a bow does not make sense. But if Farmer Brown begins with 4 gallons of concentrate, after making five tanks of mix, he will have used $\frac{15}{4}$, or $3\frac{3}{4}$ gallons, of the concentrate. With the $\frac{1}{4}$ gallon of remaining concentrate, he can make a *partial* tank of mix. He can make $\frac{1}{3}$ of a tank of mix, because it takes *3* fourths to equal a whole, and he has *1* fourth of a gallon (he has one of the three parts he needs for a full tank).

Here is another problem to try:

Johannah (Jo) is building a patio. Each patio section requires $\frac{1}{3}$ of a cubic yard of concrete. The concrete truck holds $2\frac{1}{2}$ cubic yards of concrete. If there is not enough for a full section at the end, Jo can put in a divider and make a partial section. How many patio sections can Jo make with the concrete in the truck?

Pause & Reflect

Try to solve this problem in some way that makes sense to you. ●

One way to do this is by counting how many thirds there are in _____.

Here you can see that you get 3 patio sections from the yellow whole and 3 more from the orange whole, and then you get 1 more full section and $\frac{1}{2}$ of what you need for another patio section. The answer is $7\frac{1}{2}$. Students will want to write the remainder as $\frac{1}{3}$ because they were measuring in thirds, but the question is how many sections can be made: $7\frac{1}{2}$ sections.

FORMATIVE ASSESSMENT Notes. For all operations with fractions, using a mathematical adaptation of the Frayer Model such as the Procedure—Illustration—Concept—Situation activity page can be used as a pre- or post-assessment or as a performance assessment to see whether students can represent the meaning of each operation by using visuals and symbols and explain connections between procedures and concepts. ■

Pearson eText Activity Page: Procedure—Illustration—Concept—Situation

Estimating Division

Understanding division can be greatly supported by using estimation. What does $\frac{1}{6} \div 4$ mean? Will the answer be greater than 1? Greater than $\frac{1}{2}$? Greater than $\frac{1}{6}$? The answer to each of these questions is "no." The answer should be obvious to someone who understands the meaning of this operation (that one-sixth is being shared four ways, resulting in even smaller parts). Conversely, consider what $12 \div \frac{1}{4}$ means. Will the answer be greater than 1? Greater than 12? The answer to these questions is "yes," and again, it should be obvious, because you are actually answering the question, "How many fourths in 12?" (There are 48 fourths in 12 wholes.) Finding compatible numbers is effective for division of fractions. For example, for $5\frac{1}{3} \div \frac{3}{5}$, think, "$5\frac{1}{3}$ is close to five, and $\frac{3}{5}$ is close to $\frac{1}{2}$. How many halves in 5? About 10."

Activity 15.16 encourages estimation with division (and can also be adapted to the other operations).

Activity 15.21 The Size Is Right: Division

CCSS-M: 5.NF.B.7a, b, c; 6.NS.A.1

Place students in partners. Post the dashboard you are going to use and give each group of students a set of expressions to place on the dashboard. Explain that they are to estimate (not solve and then round). After they have sorted, invite pairs to join another pair to see how their dashboards compare. Start with division problems that are alike (e.g., all whole numbers divided by fractions or all fractions divided by whole numbers) but, over time, include mixed problem sets. The following expressions are provided in the Size Is Right activity page, and you can also create your own.

Whole numbers divided by fractions

Dashboard: Less than 1. Greater than 1. Greater than 2. Greater than 4. Greater than 8.

Expressions: $3 \div \frac{1}{3}$ $1 \div \frac{2}{3}$ $2 \div \frac{1}{3}$ $4 \div \frac{7}{8}$ $4 \div \frac{3}{8}$

Fractions divided by whole numbers

Dashboard: Less than $\frac{1}{8}$. Less than $\frac{1}{4}$. Less than $\frac{1}{2}$. Less than 1. Greater than 1. Greater than 2.

Expressions: $\frac{1}{2} \div 3$ $\frac{5}{6} \div 2$ $\frac{7}{9} \div 2$ $\frac{9}{2} \div 3$ $\frac{15}{4} \div 3$

For multilingual learners, provide sentence starters to support the speaking opportunities: "I think the answer is [dashboard choice] because . . . ." For students with special needs in math or students who benefit from visuals, have tools available such as Cuisenaire rods or fraction circles so that they can more readily see the relative size of each fraction.

Pearson eText Activity Page: The Size Is Right
Pearson eText Activity Page: Dashboard

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Activity 15.21 can also be used for fractions divided by fractions, with dashboards such as Quotient < 1, Quotient = 1 and Quotient > 1 (Johanning & Mamer, 2014). Students can write division expressions that fit in each of these categories. The more estimation students do, the more they begin to develop a much-needed number sense for fraction division.

Developing the Algorithms

There are two algorithms for division of fractions. Methods of teaching both algorithms are discussed here.

Common-Denominator Algorithm. Common denominators are an excellent, though less-known, strategy for dividing fractions. Let's revisit $2\frac{1}{2} \div \frac{1}{3}$:

$$2\tfrac{3}{6} \div \tfrac{2}{6} \text{ or } \tfrac{15}{6} \div \tfrac{2}{6}$$

The question becomes, "How many sets of 2 sixths are in a set of 15 sixths?" or, simply, "How many 2s in 15?" $7\frac{1}{2}$. Figure 15.15 illustrates the common denominator approach for $\frac{5}{3} \div \frac{1}{2}$. Notice that once a common denominator is found, the thought process is the same as in the whole-number problem $10 \div 3$. Try using the common denominator approach for $1\frac{2}{3} \div \frac{3}{4}$.

Invert-and-Multiply Algorithm. Although this is a common U.S. method, it is not used in other countries.

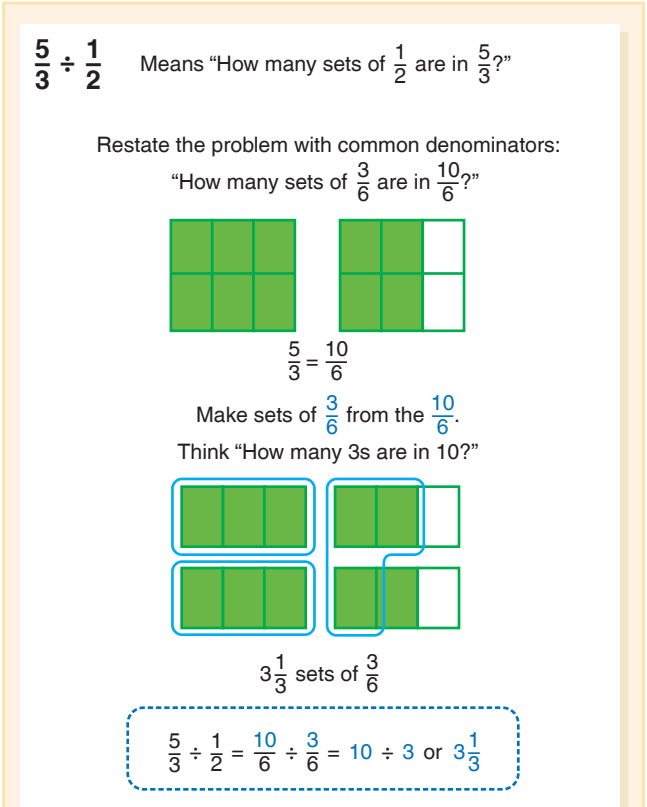
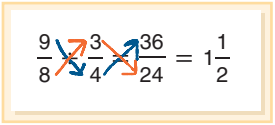

5/3 ÷ 1/2 Means "How many sets of 1/2 are in 5/3?"
Restate the problem with common denominators:
"How many sets of 3/6 are in 10/6?"
5/3 = 10/6
Make sets of 3/6 from the 10/6.
Think "How many 3s are in 10?"
3 1/3 sets of 3/6
5/3 ÷ 1/2 = 10/6 ÷ 3/6 = 10 ÷ 3 or 3 1/3

FIGURE 15.15 Models for the common-denominator method for fraction division

For example, in Mexico, the way division of fractions is notated does not involve using the reciprocal of the divisor (Perkins & Flores, 2002):



In either case, the algorithm must be understood. Use problems with unit fraction divisors to focus on why the procedure involves multiplying by the denominator.

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning.

$3 \div \frac{1}{2} =$ (How many servings of $\frac{1}{2}$ in 3 containers?)

$9 \div \frac{1}{4} =$ (How many servings of $\frac{1}{4}$ in 9 containers?)

$8 \div \frac{1}{5} =$ (How many servings of $\frac{1}{5}$ in 8 containers?)

$3\frac{3}{4} \div \frac{1}{8} =$ (How many servings of $\frac{1}{8}$ in $3\frac{3}{4}$ containers?)

Ask students to look across these problems (and others) for a pattern. They will notice they are multiplying by the denominator of the second fraction. For example, in the third example, a student might say, "You get five for every whole container, so 5×8 equals 40."

Then move to matching problems where the divisor is not a unit fraction:

$9 \div \frac{1}{4}$

$8 \div \frac{2}{5}$

$3\frac{3}{8} \div \frac{3}{8}$

Have students solve these problems and compare their responses with those for the problems in the first set. For example, notice that if there are 40 one-fifths in 8, then when you group the fifths in pairs (two-fifths), you will have half as many—20. Stated in servings, if the serving is twice as big, you will have half the number of servings.

You can also consider these equations as partitive situations, which answers the question "How much in one?" For example,

Mary paid \$9 for a $\frac{1}{4}$ pound of coffee. How much would she pay for one pound?
Mary paid \$9 for $\frac{3}{4}$ pound of coffee. How much would she pay for one pound?

In either the measurement or the partitive division interpretation, the denominator leads you to find out how many fourths, fifths, or eighths you have. So, the process is *multiply by the denominator*. The numerator tells the size of the serving or grouping, so the process is *divide by the numerator*. This procedure can also be explained with the multiplication identity (division is shown as a fraction):

$$\frac{9}{\frac{3}{4}} \times \frac{\frac{4}{3}}{\frac{4}{3}} = 9 \times \frac{4}{3}$$

At some point, this detailed explanation was shortened to "invert and multiply."

Challenges with Dividing Fractions

The biggest misunderstanding with division of fractions is just not knowing what the algorithm means. A critical aspect of understanding division of fractions is understanding that the divisor is the unit (Cramer et al., 2010; Dixon & Tobias, 2013; Philipp & Hawthorne, 2015), and this must be understood to interpret the remainder (Coughlin, 2010/2011; Lamon, 2020; Sharp & Welder, 2014). Table 15.5 provides examples of this challenge, and others, along with ways to help.

TABLE 15.5 COMMON CHALLENGES WITH DIVIDING FRACTIONS

Common Challenges	What It Looks Like	How to Help
1. Think the answer should be smaller	Students solve a problem like $\frac{4}{5} \div \frac{1}{10} = 8$ and do not think the answer can be 8 because it is bigger than the fractions in the problem.	• Devote time to estimating. Estimation can be the goal, as in Activity 15.16, or it can be a beginning to doing a computation to check for reasonableness later. • Use visuals to illustrate. • Add contexts or ask students to create a context to fit the problem.
2. Do not connect the illustration with the correct numeric answer	Student understands that $1\frac{1}{2} \div \frac{1}{4}$ means "How many fourths are in $1\frac{1}{2}$?" They count and get 6, but think the answer is $\frac{6}{4}$ (rather than 6) (Cramer et al., 2010).	• Emphasize the unit. Ask, "How many fourths?" • Use visuals and contexts. • Discuss why it is not $\frac{6}{4}$ (use a worked example to launch a discussion).
3. Do not know what the unit is	A student gets an answer of $\frac{3}{8}$ but cannot answer "$\frac{3}{8}$ of what?"	*Challenges 3 and 4 are inter-related. To support students:* • Emphasize what the unit is (the divisor). • Ask, "What is the unit?" "What is the whole?" • Use contexts for units (e.g., servings of quesadillas, feet, etc.). • For the remainder, ask, "What fraction of the next serving/piece do you have?"
4. Write remainders based on the whole rather than on the share/serving	In the problem $3\frac{3}{8} \div \frac{1}{4}$, a student counts 4 fourths for each whole (12 fourths) and one more for $\frac{2}{8}$ and considers that extra $\frac{1}{8}$ as the remainder: $13\frac{1}{8}$ (rather than realize that $\frac{1}{8}$ is half of the unit $\frac{1}{4}$, so the correct answer is $13\frac{1}{2}$).	

RESOURCES FOR CHAPTER 15

WRITING TO LEARN

1. Why is it important to include estimation with fraction operations?
2. A student adds $\frac{4}{5} + \frac{2}{3}$ and gets the answer $\frac{6}{8}$. How will you help the student understand that this is incorrect? How would you redirect them to do it correctly?
3. For the problem $3\frac{1}{4} - 1\frac{1}{2}$, think of a story problem that would be a take-away situation and one that would be a compare situation.
4. Explain at least one mental method (estimation or mental computation) for each of these:

$$\frac{3}{4} \times \frac{1}{2} \qquad 1\frac{1}{8} \text{ of } 40$$

5. Make up a word problem with a fraction as a divisor. Is your problem a measurement problem or a partitive problem? Make up a second word problem, with fractions of the other type (measurement or partitive).

LITERATURE CONNECTIONS

Alice's Adventures in Wonderland

Carroll and Gray (1865/1992)

In this well-known story, Alice shrinks—an opportunity to explore multiplication by fractions (as scaling). Taber (2007) describes how she did just that. She begins by asking how tall Alice would be if she were originally 54″ tall but was shrunk to $\frac{1}{9}$ of her height. What height will Alice be if she is later restored to only $\frac{5}{6}$ her original height?

Mama Panya's Pancakes: A Village Tale from Kenya

Chamberlin & Chamberlin (2005)

This beautiful picture book is about Mama's son inviting all their friends for pancakes. A wonderful connection to fair sharing (which can be connected to addition and multiplication as described in this chapter).

The Lion's Share

McElligott (2009)

This picture book has animals sharing the Lion's birthday cake, each one eating half of what is left (e.g., $\frac{1}{2}$ *of* $\frac{1}{2}$, $\frac{1}{2}$ *of* $\frac{1}{4}$, etc.). An excellent introduction to multiplying fractions by fractions with an area model.

The Man Who Made Parks

Wishinsky and Zhang (1999)

This nonfiction book explains the remarkable story of Frederick Olmsted, who designed Central Park in New York City. For addition, students can design a park with fractional amounts for what needs to be included in the park—for example, $\frac{2}{5}$ gardens, $\frac{1}{10}$ playgrounds, $\frac{1}{2}$ natural habitat, etc. For multiplication, include guidelines such as that $\frac{3}{4}$ of the park is natural habitat, with $\frac{1}{3}$ of that to be wooded and water features and so on.

RECOMMENDED READINGS

Articles

Cavey, L. O., & Kinzel, M. T. (2014). From whole numbers to invert and multiply. *Teaching Children Mathematics*, *20*(6), 375–383.

An excellent progression of teaching division of whole numbers to division of fractions in a way that makes strong connections between multiplication and division.

Cramer, K., Wyberg, T., & Leavitt, S. (2008). The role of representations in fraction addition and subtraction. *Mathematics Teaching in the Middle School*, *13*(8), 490–496.

Illustrations and student work are used to show how to teach addition and subtraction, using the fraction circle. Essential considerations of effective instruction are emphasized.

Gregg, J., & Gregg, D. U. (2007). Measurement and fair-sharing models for dividing fractions. *Mathematics Teaching in the Middle School*, *12*(9), 490–496.

These authors provide a series of tasks to develop the concept of division of fractions—a must-read for teaching division conceptually.

Imm, K. L., Stylianou, D. A., & Chae, N. (2008). Student representations at the center: Promoting classroom equity. *Mathematics Teaching in the Middle School*, *13*(8), 458–463.

These authors use a park context to explain how to model multiplication of fractions. Equity and a culture for learning are emphasized.

Developing Decimal and Percent Concepts and Decimal Computation

CHAPTER 16

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **16.1** Describe how the place-value system is central to understanding decimals.
- **16.2** Identify concrete and semi-concrete models that connect fractions to decimals.
- **16.3** Demonstrate how to compare and order decimals.
- **16.4** Explain multiple strategies for computing with decimals.
- **16.5** Explain how percents are related to fractions and decimals.

People need to be able to interpret decimals for such varied needs as reading precise metric measures, calculating loans and mortgages, understanding output on a calculator, and interpreting sports statistics such as those at the Olympics, where winners and losers are separated by hundredths of a second. Decimals are critically important in many occupations: nurses, pharmacists, and workers building airplanes, for example; precision affects the safety of the general public. Because students and teachers have been shown to have greater difficulty understanding decimals than fractions (Martinie, 2007; Lortie-Forgues, Tian, & Siegler, 2015; Hurst & Cordes, 2018; Vamvakoussi et al., 2012), conceptual understanding of decimals and their connections to fractions must be carefully developed. Research shows there is a strong relationship between teachers' content knowledge of decimals, including computation with decimals, and their pedagogical knowledge of the ways to teach these ideas to students (Depaepe et al., 2015).

BIG IDEAS

- The base-ten place-value system extends infinitely in two directions: to very small values and to very large values. Between any two place values, the 10-to-1 ratio remains the same.
- Decimals (also called *decimal fractions*) are a way of writing fractions within the base-ten system (denominators of 10, 100, etc.).

- The decimal point is a convention that has been developed to indicate the unit's position. The position to the left of the decimal point marks the location of the units place.
- Addition and subtraction with decimals are based on the fundamental concept of adding and subtracting numbers in like position values—an extension from whole numbers.
- Multiplication and division of two numbers will produce the same digits, regardless of the positions of the decimal point. As a result, multiplicative computations with decimal fractions can be performed as whole numbers, with the decimal placed by way of estimation or by identifying patterns.
- Percents are simply hundredths and as such are a third way of writing fractions and decimals.

Extending the Place-Value System

Learning Outcome 16.1 **Describe how the place-value system is central to understanding decimals.**

Think about the relationship between 3 and 30 or 30 and 300. How does this pattern relate to 0.3 and 3.0? Students need a strong conceptual understanding of decimal place values (Durkin & Rittle-Johnson, 2015), so, let's review how to present the fundamental place value idea of the 10-to-1 multiplicative relationship.

The 10-to-1 Relationship—Now in Two Directions!

As you learned when studying place value, the 10-makes-1 relationship continues indefinitely to larger and larger pieces or positional values. If you use paper base-ten materials, for example, ten of any one piece will make one of the next larger piece (to the immediate left); the strip and square shapes are an infinite progression as they get larger. Likewise, each piece to the right in this continuum is divided by 10 (1 divided by 10 is one-tenth). The critical question becomes, "Is there ever a smallest piece?" In students' prior experience, the smallest piece was the centimeter square or unit piece. But couldn't that piece be divided into 10 small strips? And couldn't those small strips be divided into 10 very small squares, and so on?

The goal of this discussion is to help students recognize that a 10-to-1 relationship extends infinitely in two directions with no smallest piece and no largest piece. The symmetry of the system is around the ones place (tens to the left of the ones place, tenths to the right, and so on)—not the common confusion about symmetry around the decimal point. Figure 16.1 illustrates this idea.

Even at this stage, remind students of the powerful concept of regrouping by having students not only make 1 ten from 10 units (transforming a smaller unit to a larger unit), but flexibly think about equivalent representations such as regrouping 2451 into 24 hundreds, 5 tens, and 1 unit; or 245 tens and 1 unit; or 2451 ones. As you can see, this process is essential in thinking about 0.6 as 6 tenths as well as 60 hundredths and so on.

Units and the Decimal Point

The decimal point marks the location of the ones (or units) place. That is why on a calculator, when there is a whole-number answer, no decimal point appears. Only when the ones place needs to be identified will the decimal point show in the display.

The decimal point is placed between two positions with the convention that the position to the left of the decimal is the units or ones position. Thus, the role of the decimal point is to designate the units position, and it does so by sitting just to the right of that position. The decimal notation of a 0 in the ones place, such as 0.60, is the accepted way to write decimal fractions. This notation is a convention and a way to indicate that the number is less than 1.

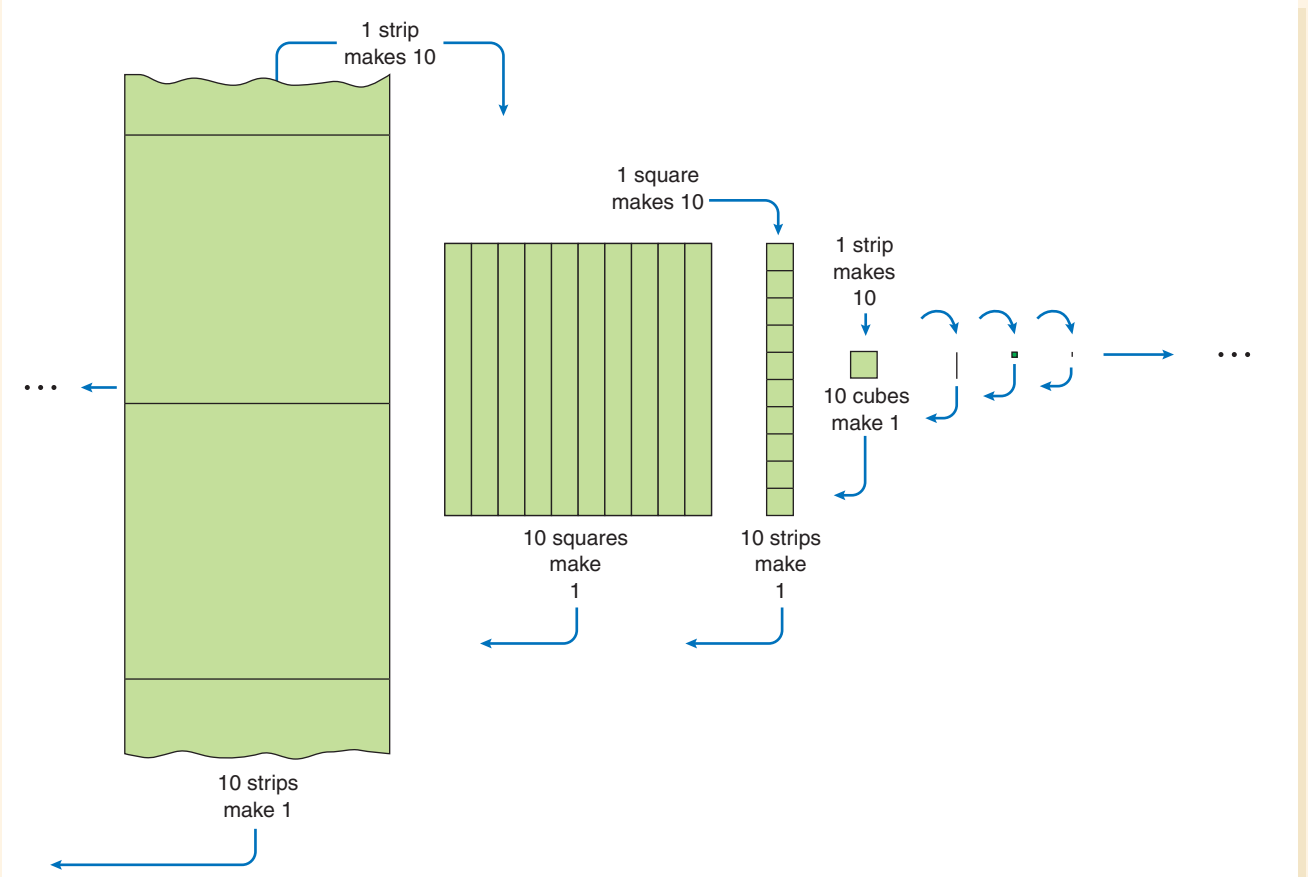


FIGURE 16.1 Theoretically, the strips and squares extend infinitely in both directions

A reminder to help students think about the decimal point is shown in Figure 16.2, with the eyes focusing up toward the name of the units or ones.

Notice that the base-ten pieces in Figure 16.2 have the piece that is often called the strip (it is ten tiny squares long) as the unit. That is a choice made by whoever is reasoning about the problem. Once the strip is identified as the unit, the other pieces have related values: the large square is ten (because it is physically ten units), the tiny square is one-tenth (because it is physically one-tenth of the whole), and so on.

Activity 16.1 illustrates the convention that the decimal indicates the named unit and that the unit can change without changing the quantity.

Activity 16.1 The Decimal Point Names the Unit

CCSS-M: 4.NF.C.6; 5.NBT.A.3a

Give each student a variety of base-10 pieces and a smiling decimal point. Refer to the pieces by their shapes as *squares, strips,* and *tinies,* and reach an agreement on names for the pieces both smaller and larger as the activity requires the values of the materials to shift. For multilingual learners and students with special needs in mathematics, it is important that you write these labels with the corresponding visuals in a prominent place in the classroom for reference. Have students display a collection of base-ten pieces on their desks. For example, put out six squares, two strips, and four tinies. Now ask students to write and say how many squares they have, how many strips, and how many tinies. Next, students position the smiling decimal point and then write and say the amounts for the equivalent representations.

To add more place values to this activity, to the right of tinies can be *tiny strips* and *tiny squares.* To the left of the squares can be *super strips* and *super squares*.

Pearson eText Blackline Master: Smiling Decimal Point

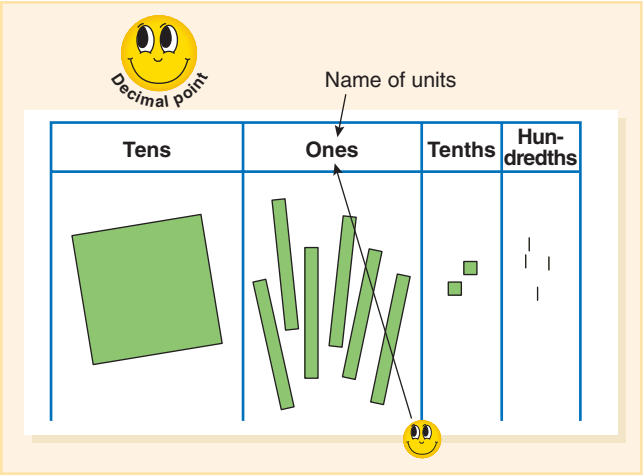


FIGURE 16.2 The decimal point always looks up at the name of the units position—in this case, 16.24

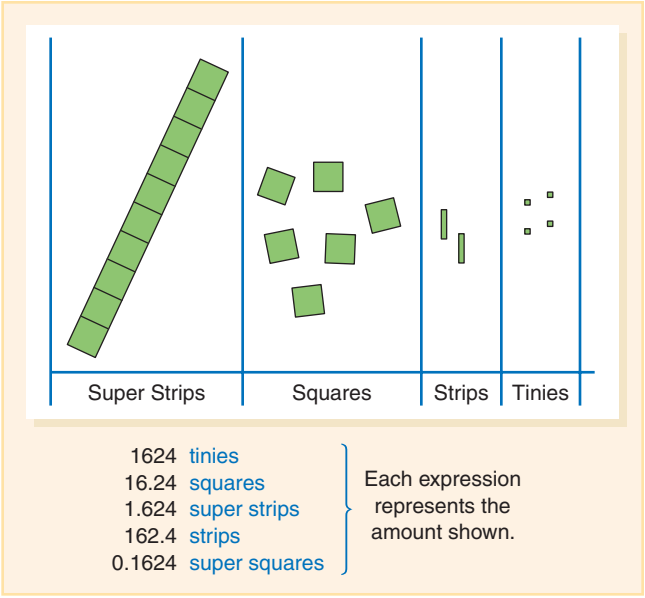


FIGURE 16.3 The placement of the decimal point indicates which position is the unit

Figure 16.3 illustrates how a quantity can be written in different ways as equivalent representations, depending on the choice of the unit or what piece is used to count the entire collection.

FORMATIVE ASSESSMENT Notes. Ask students in an interview or a learning center to write the number that has 3 tenths, 6 hundredths, 7 ones. This task will assess students' grasp of decimal place value parts or whether they become confused and assume the answer uses the left-to-right order in which numbers are commonly written. If they don't look carefully, they may incorrectly write 36.7 or 0.367 instead of 7.36. ■

The calculator can play a significant role in developing decimal concepts, as in the next activity.

Activity 16.2 Calculator Decimal Counting

CCSS-M: 4.NF.C.6; 5.NBT.A.1

Many calculators have a counting function, wherein it counts by the second addend you enter (e.g., [+] 5 will count by 5s every time you press [=]). To begin, demonstrate this counting process with whole numbers (projecting a calculator). Repeat with a count of [+] 0.1 [=]. Ask students what will come next (0.2). Keep going, having students count out loud as you go (a good strategy for all students, in particular students with special needs in math and multilingual learners). Stop at 0.9. Ask students to predict what will happen on the next count- How will we say it? How will we write it? (Students often say ten tenths and write it as 0.10.) Then reveal. Engage students in exploring why $0.9 + 0.1 = 1$ (and not 0.10). Place students in small groups and give students a calculator and base 10 pieces. Ask them to prepare a justification for why the calculator shows 1 and not 0.10. After some time, invite groups to share their explanations. Focus the discussion on trading 10 tenths for one whole. Remind students that a place is full when it has 9 of any unit, and the addition of another unit will push the number to the position that is one unit to the left (like the mileage in a car).

Continue to choral count to 4 or 5 by tenths. Students can follow along on their own calculators. Ask conceptual questions like, "How many presses to get from one whole number to the next?" Once students work well with tenths, try counting by 0.01 or by 0.001, which dramatically illustrates how small one-hundredth and one-thousandth really are! It requires 10 counts by 0.001 to get to 0.01 and 1000 counts to reach 1.

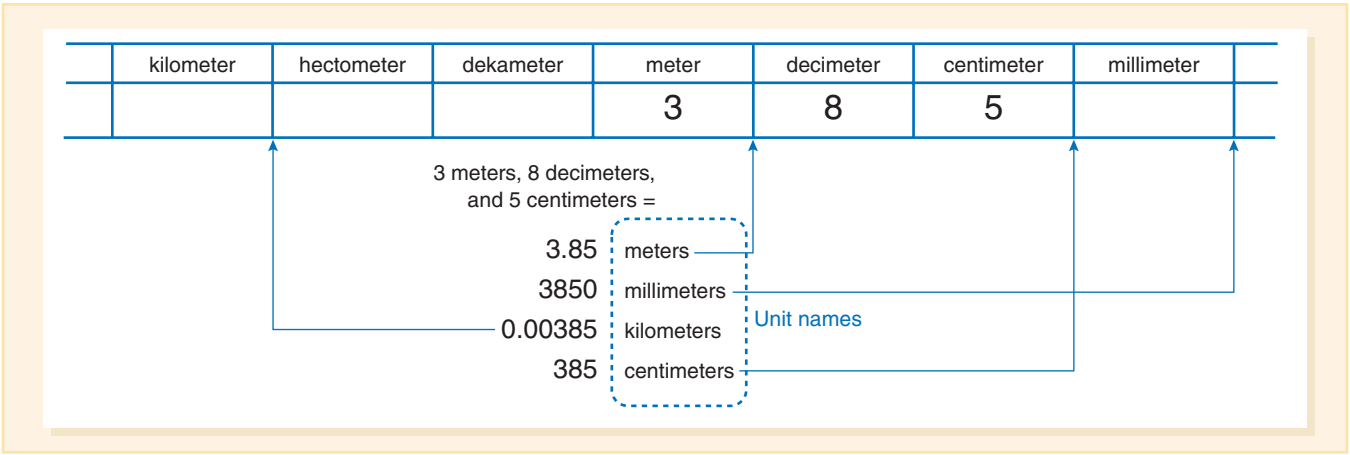


FIGURE 16.4 In the metric system, the decimal point can be placed to designate which length is the unit length

Measurement and Monetary Units

The notion that the decimal point identifies the units place is useful in a variety of contexts, because the unit must be stated in order to understand the number. For example, in the metric system, several place values have names. As shown in Figure 16.4, the decimal point can designate any of these places as the unit without changing the actual measure. In the case of measures such as metric lengths or weights or the U.S. monetary system, the name of the unit is written after the number, and the decimal point is placed to the right of that unit.

Our monetary system is also a decimal system. In $172.95, the decimal point designates the dollars position as the unit. There are 1 hundred (dollars), 7 tens, 2 singles, 9 dimes (one-tenth of a dollar), and 5 pennies (one-hundredth of a dollar) in this amount of money, regardless of how it is written. If pennies were the designated unit, the same amount would be written as 17,295 cents. Equivalent representations could also correctly be 0.17295 thousands of dollars or 1729.5 dimes. In the news, we may hear about Congress spending $7.3 billion. Here the units are billions of dollars, not dollars.

Units (and related use of the decimal point) are used for other subjects beyond measurement and money. For example, a city's population may be 2.4 million (million is the unit) people, which is equal to 2,400,000 people (people is the unit).

Precision and Equivalence

Many state standards include an expectation that students can use an appropriate level of precision based on the context of a problem or situation. Let's think about the two measurements, 0.06 and 0.060. They are equivalent in terms of numerical value, but the second measurement communicates a greater level of precision. In the first case, the measurement was completed to the nearest hundredth, so it might have measured 0.058 or 0.063 if measured more precisely. In the second case, we know it was measured to the nearest thousandth of a unit. It measured precisely 60 thousandths. The context determines the level of precision needed (see Table 16.1). For engineers, it might be ten-thousandths: although 0.04500 may be equivalent to 0.045, the precision communicated by the two measures is not just different; it's important! NASA engineers go out 15 decimal places when using π (NASA/JPL, 2016).

Standards for Mathematical Practice
MP6. Attend to precision

TABLE 16.1 REAL-WORLD DECIMAL CONTEXTS AS MEASURED TO THE FOLLOWING UNITS

Tenths	Hundredths	Thousandths
Human temperature	Distances	Gasoline prices
Sales tax	Times for races	Interest rates
Car odometer	Money	Gymnastic competition scores
Weights	Nutritional information—RDA	Baseball batting averages

Connecting Fractions and Decimals

Learning Outcome 16.2 Identify concrete and semi-concrete models that connect fractions to decimals.

Fractions with denominators of 10, 100, 1,000, and so on—for example, $\frac{7}{10}$ or $\frac{63}{100}$ —can also be written as 0.7 and 0.63. This latter representation is therefore called decimal fractions, a phrase that has been shortened to decimals in school mathematics. The point is that decimals are a special case of fractions (no pun intended!). Because fractions with denominators of tenths or hundredths can often be simplified, the connection isn't always straightforward. For example, 3.75 and $3\frac{3}{4}$ are equivalent. Yet for too many students, fractions and decimals seem like completely separate topics. Even adults tend to think of fractions as sets or areas (e.g., three-fourths of something), but decimals as values or numbers (e.g., weight). When students hear or see that 0.75 is the same as $\frac{3}{4}$, it may seem arbitrary or confusing unless they understand that $\frac{3}{4}$ is equivalent to $\frac{75}{100}$ and that 0.75 is equivalent to $\frac{75}{100}$. Effective teaching of decimals helps students see that both systems represent the same concepts. Here we share ways to ensure that connections between fractions and decimals are understood. Students will later find that not all fractions can be explicitly represented as a terminating decimal.

Say Decimals Correctly

Here are three language considerations that can make a significant difference in helping every student make sense of decimals (and make connections to fractions).

1. **Always say decimals accurately.** Say "five and two-tenths" instead of "five point two." Using the point terminology results in a disconnect from the fractional components that exist in every decimal. This practice of attending to precision in language provides your students with opportunities to hear connections between decimals and fractions repeatedly, so that when they hear "two-tenths," they think of both 0.2 and $\frac{2}{10}$.
2. **Emphasize the *th*.** In your speaking, and in students' speaking, emphasize the *th*s at the end of decimal units as both you and the students talk about decimals. Exaggerate the *th*s in your pronunciations, because initially students are not accustomed to small differences in words such as tens and tenths. Stressing this sound is important for everyone, but especially for multilingual learners and students with special needs in math, who may not initially notice the distinction if it is not clearly emphasized.
3. **Use *and* correctly.** Help students notice that when a number has a whole number part and a decimal part, both parts are stated separately with an "and" connecting them. So 32.16 is read as "thirty-two *and* sixteen hundredths." Decimals, and when reading mixed numbers are times when the word *and* is used in naming numbers. When saying a number such as 116, it is incorrect to say "one-hundred and sixteen."

Representations for Decimals

Not surprisingly, decimal representations also fall in the same types as fractions: area, length, and set. However, the representations must focus on tenths and so on, so many fraction manipulatives do not lend themselves to depicting decimal fractions.

Area Models. Common areas for decimals are base-ten materials, 10×10 grids, and a rational number wheel (circle).

Explicitly link base-ten materials to decimal numbers, the focus of Activity 16.3.

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Activity 16.3 Building Decimals

CCSS-M: 4.NF.C.6; 5.NBT.A.1; 5.NBT.A.2; 5.NBT.A.3a

Put students in pairs and give them a set of building decimal-number cards (see Figure 16.5) and decimal base-ten materials. Provide them with a variety of amounts where they show the value with base-ten materials and place the correct card under each quantity of a given unit, for example, tenths or hundredths. Then the pair should left-justify the cards and show how the decimal number is built. Notice that these materials link to the place-value cards introduced in Chapter 10 for whole numbers.

Pearson eText Activity Page: Building Decimal-Number Cards

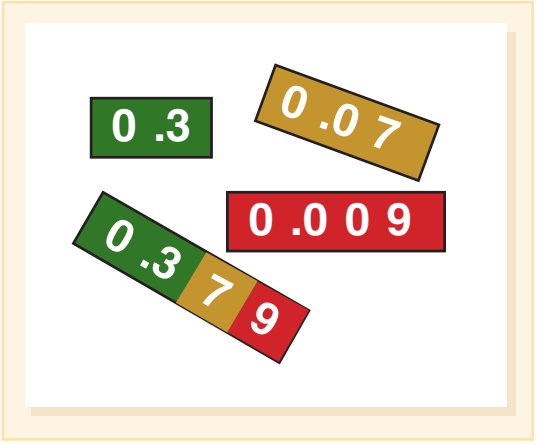


FIGURE 16.5 Decimal-number cards to build the value 0.379

Students benefit from equivalent representation activities in which the unit changes, such as in Activity 16.4.

Activity 16.4 Shifting Units

CCSS-M: 4.NF.C.6; 5.NBT.A.1; 5.NBT.A.2; 5.NBT.A.3a

Give students a collection of base-ten blocks or paper base-ten materials. Ask them to select a particular mix—for example, three paper squares, seven strips, and four tinies. Tell students that you have a unit hidden behind your back; when you show it to them, they are to figure out how much you have and record the value. Hold up one of the units, like the strip. Observe what students record as their value. Ask students to say their quantity aloud. For multilingual learners and students with special needs in mathematics, write labels on the visuals as a reference. Repeat with different collections and units. Include examples in which a piece is not represented, so that students will better understand decimal values like 3.07. Continue playing in partners, with one student selecting the mix of base-ten pieces and the other student deciding on the unit and both writing and saying the number.

Pearson eText Blackline Master: Base-Ten Materials

Importantly, 10×10 grids provide strong visual images for decimals (Cramer et al., 2015; Wyberg et al., 2011). Paper 10-cm squares that were used as the hundreds in the whole-number place value now represent the whole, or 1. Each column, then, is 1 tenth, and each centimeter square is 1 hundredth. Students shade various quantities and can visualize the equivalencies, such as $0.34 = 0.3 + 0.04$, and $\frac{4}{10} = 0.4 = 0.40$ (see Figure 16.6).

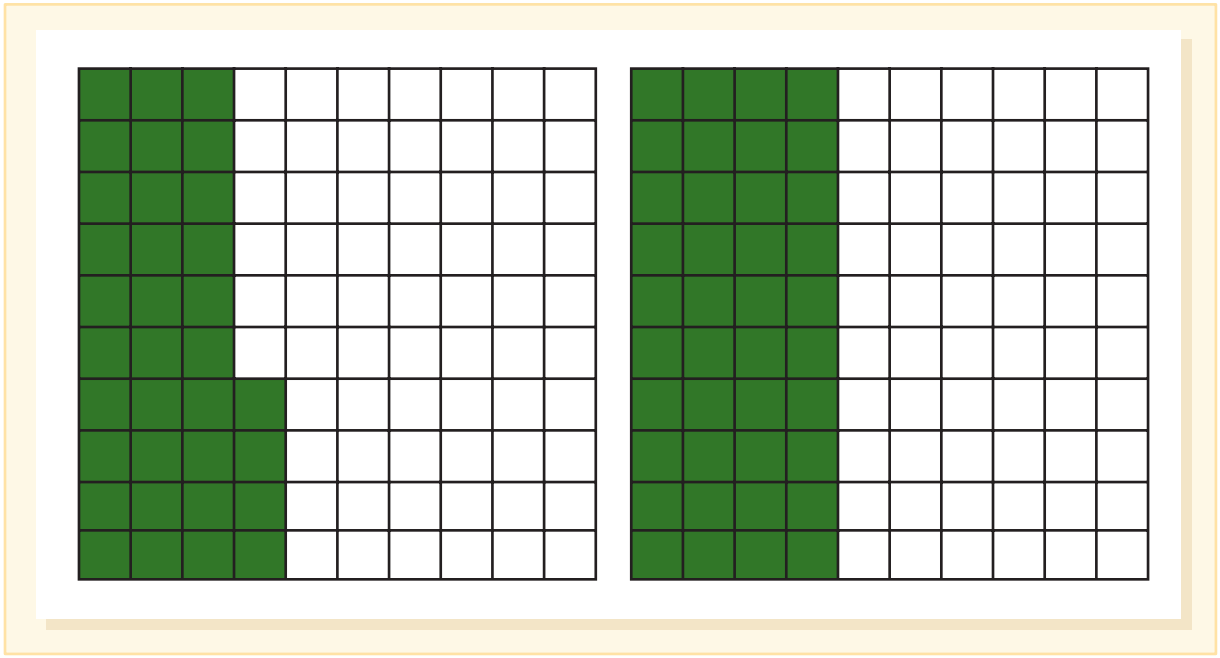


FIGURE 16.6 10×10 grids as one whole to illustrate 34 hundredths and 4 tenths

Students need experiences to visualize decimals beyond hundredths (which are easier due to money and other familiar contexts). Activity 16.4 engages students in partitioning the hundredths in their 10-by-10 grids to visualize thousandths.

Activity 16.5 Decimal Roll and Cover

CCSS-M: 4.NF.C.5; 4.NF.C.6; 5.NBT.A.1; 5.NBT.A.3a

Put students in pairs and give them a 10×10 grid and a die with faces marked with $\frac{1}{10}$, $\frac{1}{100}$, $\frac{5}{100}$, $\frac{1}{1000}$, $\frac{5}{1000}$, $\frac{10}{1000}$, or 0.1, 0.01, 0.05, 0.001, 0.005, and 0.010 (or a combination of fractions and decimals). One student rolls the die and uses a marker to shade the amount rolled on the 10×10 grid. For example, if one tenth is rolled, the student shades a whole column of ten small squares on the grid. Each hundredth rolled results in shading one small square, and each thousandth rolled requires a small square to be first divided into half and then five more divisions made in the other direction to create ten small, equal-sized pieces. Students will eventually note that the $\frac{10}{1000}$ is equivalent to $\frac{1}{100}$. If your students are only working on tenths and hundredths, adjust the die faces to reflect appropriate amounts. The team should try to cover the whole or 1. This activity provides a dramatic visualization with the differences in size of each of these decimals.

Pearson eText Blackline Master: 10×10 Grids

Another way to build an understanding of thousandths and ten-thousandths is to explore the 10,000 grid paper. Students can identify how many squares are needed for 0.1, 0.01, 0.001, and 0.0001, using appropriate names for the values.

Pearson eText Blackline Master: 10,000 Grid Paper

A rational number wheel is two overlapped circles and one circle marked with 100 equal intervals on the circumference and is cut along one radius (see Figure 16.7). Two disks of different colors, slipped together as shown, can model any fraction (including decimal fractions) less than one. Decimals modeled on this rational number wheel can be read by noting the tenths and hundredths marked around the circumference but can also be

stated as fractions, helping students further make the connection between fractions and decimals. For example, the final illustration in Figure 16.7 shows $\frac{25}{100}$ of the blue plate (also $\frac{1}{4}$ or 25% of the circle).

Pearson eText Blackline Master: Rational Number Wheel

Length Models. Because decimals are often used to measure, length models are important representations! You can use base-10 strips and put them end to end to transition to length models, or connect 10 red connecting cubes and 10 blue connecting cubes, and so on to make a physical strip of cubes 100 hundredths long, or use a meter stick.

The meter stick is an important length model because it includes a relevant context. Each decimeter is one-tenth of the meter's length, each centimeter is one hundredth, and each millimeter is one thousandth. Any number-line model broken into 100 subparts is likewise a useful model for hundredths, including a large number line stretched across a wall or on the floor.

For both whole and rational numbers, students must understand the magnitude of a number, meaning they can put numbers in order from least to greatest (Durkin & Rittle-Johnson, 2015; Siegler, Thompson, & Schneider, 2011). Knowledge of decimal magnitude predicts algebra understanding (DeWolf, Bassok, & Holyoak, 2015) and overall mathematics achievement (Resnick et al., 2019). Number lines help students notice the relative size of decimal values (their magnitude) as they relate to whole numbers and other decimals or fractions.

Open number lines (like those used in whole-number computation) help students compare decimals and think about scale and place value (Martinie, 2014). Saying decimals by naming the place-value unit (e.g., four hundredths) is particularly useful in supporting students' understanding of decimal comparisons, especially when the role of a zero is considered (Loehr & Rittle-Johnson, 2017).

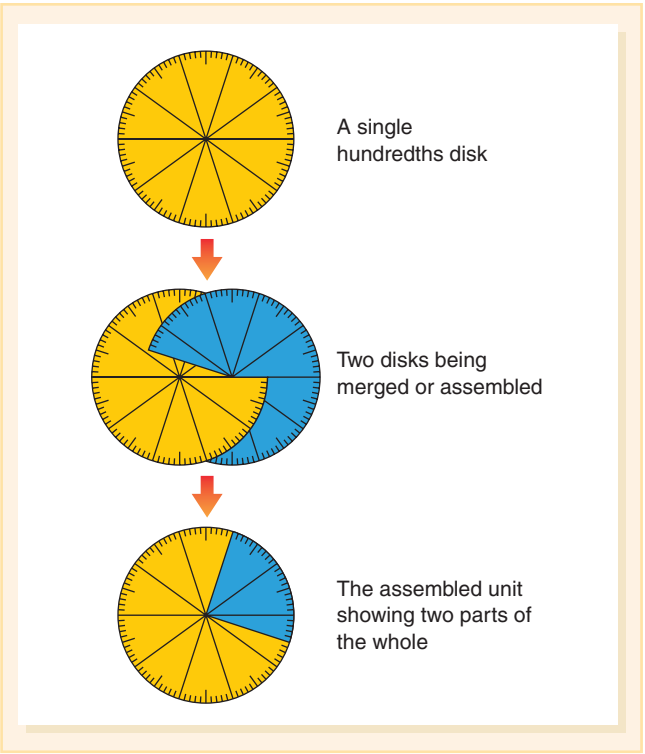


FIGURE 16.7 Rational number wheel

Activity 16.6 Place and Compare

CCSS-M: 4.NF.C.6; 4.NF.C.7

Student pairs should have the Place and Compare activity page. Then they use an open number line to position the values, revealing what they know about the magnitude of these decimals, using zero, one-half, one, other whole numbers, or other decimal values as benchmarks. Example decimals to tenths might include: 0.7, 1.5, and 0.1. To hundredths might include: 0.45, 1.02, 1.25, .08, or 1.89. For a challenge, thousandths can also be placed.

Pearson eText Blackline Master: Number Lines 0–2 and 0–3

Activity 16.7 asks students to find the location of decimals between 0 and 1, in a measurement context.

Activity 16.7 The Amazing Race

CCSS-M: 4.NF.C.6; 4.NF.C.7; 5.NBT.A.3b

Use the Amazing Race activity page or give students cash register tape. The following friends are running, and the decimals next to their names represent what part of a mile they are from the starting line. Who do you think is winning? Can you place these friends on a number line to show where they are between the start and finish?

Aimee—0.34	Hissa—0.12	Leo—0.56
Layla—0.85	Marco—0.45	Gerard—0.73

Differentiate this activity for challenges or supports for students with special needs in math by changing the values of the decimals or including more or fewer friends.

Pearson eText Activity Page: Amazing Race

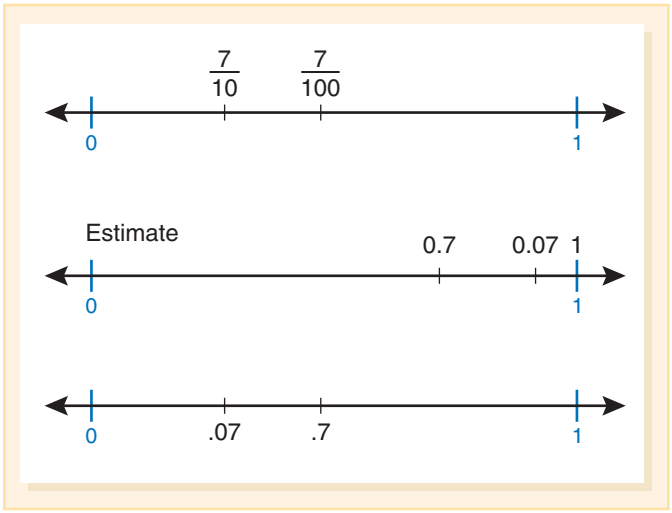


FIGURE 16.8 Three incorrect examples of students locating 0.7 and 0.07 on a number line

FORMATIVE ASSESSMENT Notes. Students may be able to show a decimal value with an area model but not be able to locate it on a number line (see Figure 16.8). Use a multiple representation assessment tool. Give students two related decimal numbers, such as 0.5 and 0.05, and ask them to (1) show both with base-10 pieces, (2) shade both on 10×10 grids, and (3) locate both on the same open number line (Martinie, 2014). If students have significantly more difficulty with one representation over another (typically, the number line), they need more conceptual activities involving length models. ■

Pearson eText Blackline Master: 10 × 10 Grids
Pearson eText Blackline Master: Base-Ten Materials

Set Models. A variety of set models can be used to explore decimals: 100 bead necklaces, lining up 100 cubes, and other sets of 10, 100, or 1000. Also, consider contexts, such as putting crayons or markers in 10-packs or 100-box containers, or loyalty cards—buy 10, get one free: "I am four tenths (0.4) of the way to a free ice cream scoop!"

Money can be a set model for decimals, and it is a real-life context. Let's look at how the connections among decimals, money, and fraction notation can be made through the next activity, adapted from Mathscape (McGraw Hill, 2004).

Activity 16.8 Show Me the Money!

CCSS-M: 4.NF.C.6; 5.NBT.A.3a

Provide student pairs with a baggie of 100 pennies. Ask them to create a rectangular array, using all the pennies, and complete a row of this Show Me the Money table. The following table represents one possible answer:

Rectangle Dimensions	Fraction for the Value of One Row	Decimal for the Value of One Row	Money Value
10×10	$\frac{10}{100} = \frac{1}{10}$	0.10	$.10

Pearson eText Activity Page: Show Me the Money

Keep in mind, though, that money has two limitations when using it as a representation for decimal values: (1) it only goes out to the hundredths place, and (2) it is non-proportional (e.g., one-tenth, a dime, does not physically compare to a dollar). Numbers like 3.2 or 12.1389

do not relate to money and can therefore be confusing to students (Martinie, 2007). So, it is important to use other concrete, proportional models that connect to tenths and thousandths and then integrate money as an important *application* of decimal numeration.

Connecting Visuals to Symbols

As students explore area, length, and set models, help them make sense of the symbols and think flexibly about the quantities in terms of tenths and hundredths (and thousandths, depending on grade level). For example, have students model a decimal fraction, say 0.65, with their choice of visual, and then explore the following ideas:

- What are some different ways to say this number, using tenths and hundredths? ("6 tenths and 5 hundredths," or "65 hundredths".) Again, include thousandths when appropriate.
- What are different ways to write the number (0.65 or $\frac{65}{100}$)?
- What are different ways to write the number as a sum, using tenths and hundredths? ($\frac{6}{10} + \frac{5}{100}$ or $\frac{60}{100} + \frac{5}{100}$ or $0.6 + 0.05$ or $0.60 + 0.05$)

Notice that decimals are usually read as a single value. That is, 0.65 is read "sixty-five hundredths." But to understand the decimal in terms of place value, the same number must be thought of in its decomposed place value as 6 tenths and 5 hundredths (Loehr & Rittle-Johnson, 2017). A mixed number, such as $5\frac{13}{100}$, is usually read the same way as a decimal; 5.13 is "five and thirteen-hundredths." Notice the "and" is read where the decimal point is located. For purposes of considering place-value parts, it should also be understood as $5 + \frac{1}{10} + \frac{3}{100}$.

Highlighting these expanded forms with base-ten materials will be helpful in translating fractions to decimals, which is the focus of Activity 16.9.

Activity 16.9 Build It, Name It

CCSS-M: 4.NF.C.6; 5.NBT.A.1

For this activity, have students use concrete base-ten materials (or create paper versions, using base-ten materials). Assign the large square the value of one (whole). Post a fraction, such as $2\frac{35}{100}$, and instruct students to build it and then name it. Have students cover a decimal amount of the area of the square, using their strips and tinies (call these pieces *tenths* and *hundredths*). Naming it can be written in different forms! The focus of this task is to decide how to write and say this fraction as a decimal in short form and expanded form. This example might be written as 2.35 or $2 + .3 + .05$ (among other options). Continue with examples where the fraction needs to be changed to tenths (e.g., $3\frac{1}{2}$ and $5\frac{2}{5}$) and then hundredths (e.g., $1\frac{1}{4}$). For students with special needs in math, you may want to have amounts shaded rather than have the students try to cover exact amounts; then ask them to name and write the decimal.

Pearson eText Blackline Master: Base-Ten Materials

Pearson eText Activity Page: Build it Name it

In Activity 16.9, a reason $2\frac{35}{100}$ is the same as 2.35 is that there are 2 wholes, 3 tenths, and 5 hundredths. It is important to see this relationship physically. The same materials that are used to represent $2\frac{35}{100}$ of the square can be rearranged or placed on a place-value chart with a paper decimal point used to designate the units position, as shown in Figure 16.9.

***TECHNOLOGY* Note.** Calculators that permit entry of fractions also have a fraction–decimal conversion key, making them valuable tools for connecting fraction and decimal symbolism. Some calculators will convert a decimal such as 0.25 to the fraction $\frac{25}{100}$ and allow for either manual or automatic simplification. Challenge students to explain why 0.25 and $\frac{25}{100}$ are equivalent rather than relying on calculators to do the conversion. ■

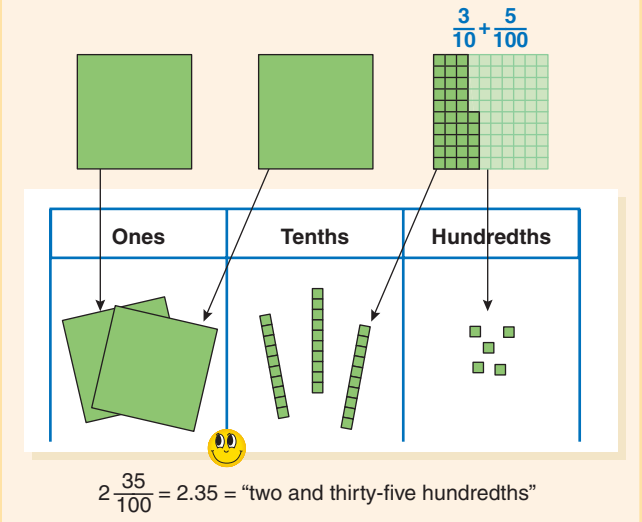


FIGURE 16.9 Translation of a fraction to a decimal, using physical models

Developing Decimal Number Sense

Learning Outcome 16.3 **Demonstrate how to compare and order decimals.**

So far, the discussion has largely focused on connecting decimals to fractions with denominators of 10 and 100. Number sense with decimals implies more—having intuition about, or a flexible understanding of, decimal numbers. To this end, students need ongoing opportunities to explore decimals and fraction equivalencies and see that sometimes one form is more useful than another (in which case, they simply switch to the other form). You do this naturally when you see 1.5, for example, and think, "That is one and a half."

But too many students do not understand this fraction–decimal relationship. Fewer than 30 percent of high school students were able to translate 0.029 to $\frac{29}{1000}$ successfully on a 2004 NAEP item (Kloosterman, 2010). And, in another study, about half of sixth graders could not write $\frac{3}{5}$ as a decimal. Instead, many wrote $\frac{3}{5}$ as 3.5, 0.35, or 0.3 (Shaughnessy, 2009). The researcher also found that more than 25 percent could not write $\frac{3}{10}$ as a decimal. This confusion also appeared in the reverse request when students wrote the decimal 4.5 as $\frac{4}{5}$. Division of the numerator by the denominator may be a means of converting fractions to decimals, but it contributes little to understanding the resulting equivalent relationship.

Familiar Fractions Connected to Decimals

Students should extend their conceptual familiarity with common fractions such as fifths and tenths, halves and fourths, and possibly thirds and sixths. One way to present this knowledge is to have them translate familiar fractions to decimals in a conceptual manner, which is the focus of the next two activities.

Activity 16.10 Familiar Fractions to Decimals

CCSS-M: 4.NF.C.7

Students are given a familiar or commonly used fraction (e.g., $\frac{3}{5}$) to convert to a decimal. Ask students to shade 10×10 grids to illustrate that value (or build it with base-ten materials) and ask them to write the decimal equivalent. A logical sequence is to start with fifths and then halves and fourths. As a challenge, explore thirds and sixths. Students will see they can't exactly shade a 10×10 grid but can estimate the decimal values. (Later, they will learn more about repeating decimals.) For multilingual learners, consider having fractions with words (e.g., $\frac{3}{5}$ is three-fifths or, more generally, $\frac{n}{5}$ is *n* fifths) to help them learn how to say fractions (and decimals). For students with special needs in math, have some 10×10 grids pre-shaded with different fractions and ask students to find, for example, the grid that shows $\frac{1}{2}$. Have students skip-count the squares that are shaded to identify the equivalent of $\frac{50}{100}$. For more details, explore the Expanded Lesson.

Pearson eText Blackline Master: 10×10 Grids
Pearson eText Blackline Master: Base-Ten Materials
Pearson eText Expanded Lesson: Familiar Fractions to Decimals

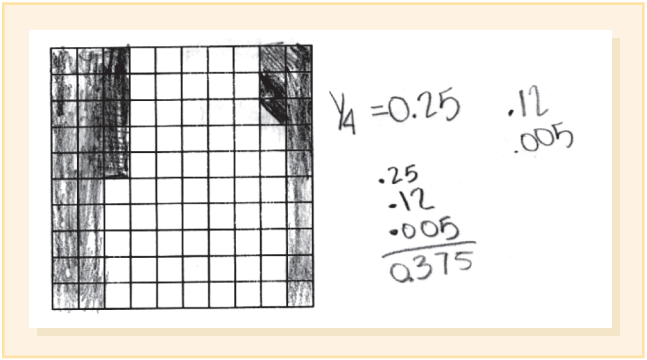


FIGURE 16.10 A student uses a 10×10 grid to convert $\frac{3}{8}$ to a decimal

Figure 16.10 shows how representations in the preceding activity might work with a 10×10 grid. For fourths, students often shade a 5×5 section (half of a half). The question then becomes how to translate this fraction to a decimal. Ask these students how they could think about each half strip as 0.05 and then use models to reason that 2 of the half strips $(0.05 + 0.05)$ equal 0.1. Then you would have $0.1 + 0.1 + 0.05$, or 0.25, and if they counted shaded individual squares, they would find $\frac{25}{100}$. This approach starts where students' prior knowledge is and works from there toward a solution. The fraction $\frac{3}{8}$ represents a wonderful challenge. A hint might be to find $\frac{1}{4}$ first and then notice that $\frac{1}{8}$ is half of a fourth. Remember that the next smaller pieces are tenths of the

tinies (or thousands). Therefore, a half of a tiny is $\frac{5}{1000}$. Note how the student found that $\frac{2}{8} + \frac{1}{8} = \frac{37}{100} + \frac{5}{1000} = 0.375$.

Because the circular model carries such a strong mental link to fractions, it is worth the time to do some fraction-to-decimal conversions, using the rational number wheel shown in Figure 16.7 and used in the next activity.

Activity 16.11 Estimate, Then Verify

CCSS-M: 4.NF.C.7

With the blank side of the rational number wheel facing them, direct students to adjust the wheel to show a given fraction, for example, $\frac{3}{4}$ (see Figure 16.11). Next, ask students to estimate how many hundredths they think are equivalent to that portion. Then students justify how they decided their estimate and the corresponding decimal equivalent. Repeat with other fractions. For students with special needs in math, cut up some rational number wheels into tenths and even hundredths so that these parts can be used as a comparison tool.

Pearson eText Blackline Master: Rational Number Wheel

The number line is a useful model to connect decimals and fraction equivalences, as in the following activity.

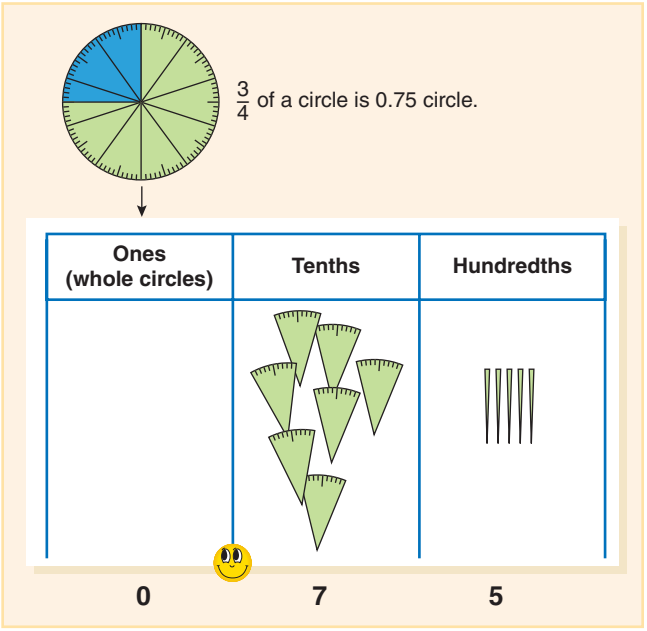
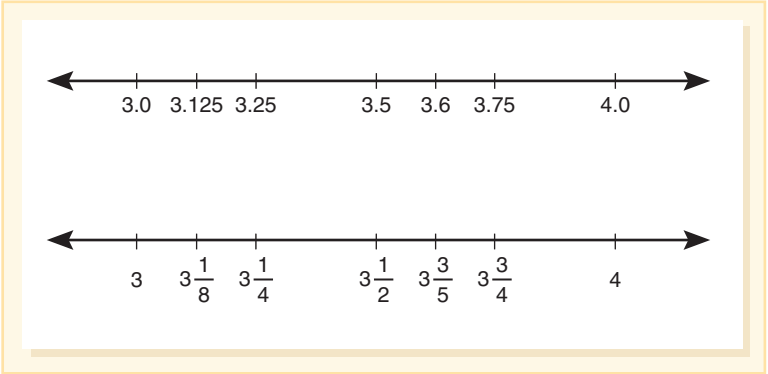


FIGURE 16.11 Using fraction circles to connect fractions and decimals

Activity 16.12 Decimals and Fractions on a Double Number Line

CCSS-M: 4.NF.C.6

Use the Number Lines activity page and give students five decimals that have familiar fraction equivalents. At first, keep numbers between two consecutive whole numbers such as 3.5, 3.125, 3.4, 3.75, and 3.6. Have students mark the first two number lines on the number lines page identically with endpoints, as in this example, starting at 3.0 and ending at 4.0. Then keep the two lines either as an open number line or with tick marks of fourths, thirds, or tenths (no labels). The students' task is to locate each of the five decimals on the first number line and provide the fraction equivalent for each on the parallel number line below it. Use the other number lines to try another set of five decimal amounts with different endpoints on the double number lines.



Pearson eText Activity Page: Number Lines

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

The exploration of modeling $\frac{1}{3}$ as a decimal is a good introduction to the concept of an infinitely repeating decimal, which is commonly a state standard for seventh grade. Try to partition the whole 10×10 grid into three equal parts, using strips and tinies. Each part receives 3 strips with 1 strip left over. To divide the leftover strip, each part gets 3 tinies with 1 left over. To divide the tiny, each part gets 3 tiny strips with 1 left over. (Recall that with base-ten pieces, each smaller piece must be $\frac{1}{10}$ of the preceding piece.) It becomes apparent that this process is never-ending. As a result, $\frac{1}{3}$ is the same as 0.333333… or 0.333 for practical purposes. Similarly, $\frac{2}{3}$ repeats a string of sixes, or about 0.667. Students will eventually discover that many fractions cannot be represented by a finite decimal.

Pearson eText Blackline Master: 10 × 10 Grids

Approximation with a Compatible Fraction. In the real world, decimal fractions are rarely equal to a common fraction, but they are frequently *close to* a common fraction.

Pause & Reflect

What fraction is close to the decimal 0.52? How do you think eighth graders responded to the same question? ●

As you read in the fraction chapters, students tend to struggle more with estimation and approximation than with computation. When this item was on NAEP, 29 percent chose $\frac{1}{50}$ (Kouba et al., 1997). Number lines help students focus on the magnitude of decimal fractions, and you can create number lines that only have benchmark fractions for reference.

The first benchmarks to develop are 0, $\frac{1}{2}$, and 1. Ask students what decimal values are equivalent to $\frac{1}{2}$ (0.5, 0.50, 0.500, etc.). Then ask questions such as, "Is 7.396 closer to 7, $7\frac{1}{2}$, or 8?" and ask them to explain their reasoning. With additional experiences, students will notice that thinking about the tenths can help. In this case, the decimal is *about* 4 tenths, so close to 0.5 or $\frac{1}{2}$. Repeatedly engage students in approximation activities, increasing in precision. Encourage students to consider other common fractions (e.g., thirds, fourths, or fifths) that are close to the decimal. In this example, 7.396 is close to 7.4, which is $7\frac{2}{5}$. These types of activities develop strong decimal sense.

Activity 16.13 Best Match

CCSS-M: 4.NF.C.6

Give students a deck of fraction-decimal cards with familiar fractions on half of the cards and decimals that are close to the fractions (but not exact) on the other half. In this memory-like game, students pair each fraction with the decimal that best matches it. The difficulty is determined by how close various fractions are to one another. Some students might select one pairing and then realize there is a better match. For students with special needs in math, you may need to have them reflect each time on whether what they've turned over is close to 0, close to $\frac{1}{2}$, or close to 1 to help support their matchmaking. Have students share their thinking as their rationales provide strategies that other students will find useful. As a follow-up, have students try the Close to a Familiar Fraction activity page to find out more about the reasoning strategies they are using.

Pearson eText Activity Page: Fraction-Decimal Cards
Pearson eText Activity Page: Close to a Familiar Fraction

Converting Fractions to Decimal Equivalents. Recall that the denominator is a divisor and the numerator is a multiplier. For example, $\frac{2}{5}$ means $2 \div 5$. This originated with sharing tasks (if you share 2 pizzas with 5 friends, each friend gets $\frac{2}{5}$ of pizza). So how

would you express $\frac{2}{5}$ on a simple four-function calculator? Simply enter $2 \div 5$. The display will read 0.4. Too often, students think that dividing the denominator into the numerator is simply an algorithm for converting fractions to decimals, and they have no understanding of why this procedure works. In the pizza context, to have a share as a decimal fraction means the pizza has to be cut in tenths or hundredths. In this case, tenths work, and each friend will get 2 tenths from each pizza, or a total of 4 tenths (0.4). Explore other common fractions, like $\frac{1}{4}$ (this requires going to hundredths). Divide on the calculator and reason about it, using contexts like pizza sharing. This helps students generalize that $\frac{a}{b} = a \div b$, where b is not 0.

The calculator is an important tool when developing familiarity with decimal concepts. Finding decimal equivalents with a calculator can produce interesting patterns. Here are some questions to explore:

- Which fractions have decimal equivalents that terminate? Can you discover a pattern? Is the answer based on the numerator, the denominator, or both?
- For a given fraction, how can you tell the maximum length of the repeating part of the decimal? Try dividing by denominators of 7 and 11 and 13 and then make a conjecture.
- Explore all of the ninths: $\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \ldots \frac{8}{9}$. Remember that $\frac{1}{3}$ is $\frac{3}{9}$ and $\frac{2}{3}$ is $\frac{6}{9}$. Use only the pattern you discover to predict what $\frac{9}{9}$ should be. But doesn't $\frac{9}{9} = 1$?
- How can you find what fraction produces this repeating decimal: 3.454545 . . .?

The last question can be generalized for any repeating decimal, illustrating that every repeating decimal is a rational number.

Comparing and Ordering Decimals

Comparing decimals and putting them in order from least to greatest builds on students' decimal number sense, including their skill at moving between decimals and fractions. Without a strong understanding of the magnitude of decimals, comparing decimals becomes challenging, particularly with decimals with unequal lengths. For example, students will overgeneralize ideas from comparing whole numbers, such as longer is larger.

FORMATIVE ASSESSMENT Notes. A single item interview can give valuable information on student understanding of the magnitude of decimals. Give the student a set of cards with decimal values such as these: 0.36, 0.058, 0.375, 0.97, 0, 2.0, and 0.4. Ask them to order the decimals from least to greatest, thinking aloud as they work. Once placed, ask probing questions like, "How did you decide that _____ was less than _____." Use Table 16.2 to identify whether students are exhibiting any of the common challenges that students often demonstrate when comparing and ordering decimals (Desmet et al., 2010; Durkin & Rittle-Johnson, 2015; Muir & Livy, 2012; Roell et al., 2017; Steinle & Stacey, 2004a, 2004b). ■

Pearson eText Activity Page: Decimal Number Cards

Pearson eText Blackline Master: 10 × 10 Grids

Relative Size of Decimals. Because these common challenges often stem from not understanding the size of such values as tenths, hundredths, and thousandths, it is important to have students find values on the number line or use other models to represent the amounts. The following activity promotes discussion about the relative sizes of decimal numbers.

Pearson eText
Video Example 16.1
In this video consider the common challenge Vanessa exhibits when comparing decimals. What could you do next with Vanessa?

Pearson eText
Video Example 16.2
In this video consider how Megan and Donna think about 4.7 and 4.70. Why was the teacher's choice to make it "You can't add a zero day" a meaningful instructional move? What should be the next instructional steps the students in using the base ten materials to represent decimal values?

Pearson eText
Video Example 16.3
In this video consider how the student is making decisions as to which decimal is larger. What would be your next instructional step with this student?

TABLE 16.2 COMMON CHALLENGES WITH COMPARING AND ORDERING DECIMALS

Common Challenge	What It Looks Like	How to Help
1. Longer is larger	0.375 is greater than 0.97, or 0.44 is less than 0.440.	• Students overgeneralize, using whole-number reasoning, and select the number with more digits as being larger. Use decimal models to show each number and compare. Two 10×10 grids with each number shaded will help students make accurate comparisons.
2. Shorter is larger	0.4 is greater than 0.97 because "a tenth is larger than a hundredth."	• Have students create representations of these two decimals, focusing on the units. For example, ask, "Is any number of tenths automatically larger than any number of hundredths?"
3. Internal zero	0.58 is less than 0.078, thinking that "zero has no impact." Also, suggesting that 34.08 and 34.8 have the same value.	• When students are confused by a zero in the tenths position, for example, have them build the number, using decimal number cards. Match this numerical value to a physical model if needed, or position it on a number line.
4. Less than zero	0.36 is less than 0, because zero is a whole number positioned in the ones column (to the left of the decimal point) and therefore is greater than a decimal (to the right of the decimal point).	• Use contexts; for example, ask students whether they would rather have 0 or 0.50 of a dollar. • Use decimal representations on grid paper to visualize the size of each decimal as compared to zero.
5. Reciprocal thinking	When 0.4 is compared to 0.6, 0.4 is considered larger because students connect 0.4 to $\frac{1}{4}$ and 0.6 to $\frac{1}{6}$ and mistakenly decide 0.4 is greater.	• Use decimal materials such as shading 10×10 grids to visualize the size of each decimal.
6. Equality	Students think that 0.4 is not close to 0.375 and/or that 0.3 is smaller than 0.30.	• Show the connections between these values through shading 10×10 grids or placing these values on the number line to help students see whether the amounts are close in size or, as in the case of 0.3 and 0.30, are equal.

All of these common challenges reflect a need for students to have a conceptual understanding of how decimal numbers are constructed.

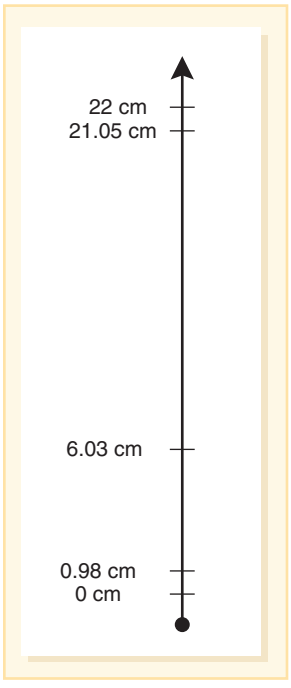


FIGURE 16.12 Plant growth on a vertical number line

Activity 16.14 Line 'Em Up

CCSS-M: 4.NF.C.6; 4.NF.C.7; 5.NBT.A.1; 5.NBT.A.3a; 5.NBT.A.3b

Prepare a list of four or five decimal numbers that students might be challenged to put in order between the same two consecutive whole numbers, or use the Line 'Em Up activity page, which uses the context of the height of plants. First, have students predict the order of the numbers, from least to greatest. Then have them use a model of their choice—for example, an open number line (see vertical number line in Figure 16.12) or 10,000 grid paper—to defend their ordering. As students reason about which value is greater, they develop a deeper understanding of which digits contribute the most to the size of a decimal amount. For students with special needs in math, some systematic instruction might be helpful. Write one of the decimals on the board—6.09, for example. Start with the whole numbers: "Is it closer to 6 or 7?" Then go to the tenths: "Is it closer to 6.0 or 6.1?" At each place-value position, challenge students to defend their choices with the use of a model or other conceptual explanation.

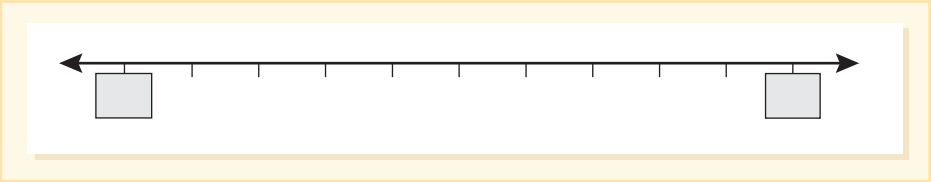
Pearson eText Activity Page: Line 'Em Up
Pearson eText Blackline Master: 10,000 Grid Paper

Density of Rational Numbers. An important concept is that there is always another number between any two numbers (Gorman & Way, 2018). When learners are only familiar with whole-number thinking (Widjaja et al., 2008), or only given examples of decimals rounded to two places, they may think there are no numbers between 2.37 and 2.38 (Steinle & Stacey, 2004b). But if you rewrite these decimals as thousandths (2.370 and 2.380), you can see there are decimal fractions between them (e.g., 2.371), and this continues for infinity. Amazing! Length models help students see that there is always another decimal to be found between any two decimals—a concept emphasized in the following activities.

Activity 16.15 Close Decimals

CCSS-M: 5.NBT.A.3b

This activity can begin with the whole class and then be an engaging exploration by partners. Display a number line, such as this one, or use the Number Lines activity page:



Have a student name a decimal between 0 and 1 (in this case, fill in the end points with 0 and 1). Next, have another student name a decimal that is even closer to 1 than the other decimal. Continue this request for several more decimals, with each decimal being closer to 1 than the previous. Place students with partners and give them an open number line. Pose different targets (e.g., close to but greater than 4, close to 0 or close to 0.5). Students with special needs in math may benefit from also using concrete materials like base-10 blocks to help them with their decision making. Another way to provide a scaffold is to provide two decimals to choose between. After some time, fade the explicit support and transition back to the original tasks.

Pearson eText Activity Page: Number Lines

Activity 16.16 Zoom

CCSS-M: 5.NBT.C3b; 6.NS.C.6

Stretch a number line (e.g., clothesline or cash register tape) across the front of the room or use a virtual number line (https://apps.mathlearningcenter.org/number-line). Ask students to mark where 0.75 and 1 are on the line. Then ask students to zoom in to find and record three more values between those two values. Ask students to share their thinking strategies. Make sure multilingual learners are clear on the meaning of the word *between,* even demonstrating this relationship with students placed in the front of the room. For students with special needs in math, you may need to give them a set of choices and ask them to select three decimals that are between 0.75 and 1.0. See the Expanded Lesson: Zoom: Finding Rational Numbers on the Number Line for procedures, assessment ideas, and questions to pose.

Pearson eText Expanded Lesson: Zoom: Finding Rational Numbers on the Number Line

Computation with Decimals

Learning Outcome 16.4 **Explain multiple strategies for computing with decimals.**

Students can have a solid knowledge of the magnitude of decimals without fully understanding the results of computational operations with decimals (Siegler & Lortie-Forgues, 2015). In the past, decimal computation was dominated by these rules: Line up the decimal points (addition and subtraction), count the decimal places (multiplication), and shift (or use the unfortunate phrase, "get rid of") the decimal point in the divisor and dividend so that the divisor is a whole number (division). When students are only taught strictly procedural approaches, they forget. One study found that in only ten days, students' average daily performance of about 80 percent dropped to about 34 percent (Woodward, Baxter, & Robinson, 1999). That means they remembered less than half of what they were told to remember. These rule-based approaches are still common, but if computation is built on a firm understanding of place value and a connection between decimals and fractions, students will become flexible and efficient in their computation. State standards expect students to understand and be able to explain why procedures make sense. This section focuses on conceptual and developmentally appropriate ways to teach decimal operations.

There are three main types of problems, delineated by their quantities used in the operation (Tian et al., 2021) and adapted here as a progression of problem types.

Pearson eText
Video Example 16.4
In this video author Jennifer Bay-Williams discusses the role of estimation in computation with decimals.

PROGRESSION OF DECIMAL COMPUTATION PROBLEM TYPES

Type	Example
Whole-decimal problem	$4 \times 5.7 =$
Decimal–decimal problem with equal digits	$3.15 + 4.62 =$
Decimal–decimal problem with un-equal digits	$2.8 - 0.87 =$

Addition and Subtraction

There is much more to adding and subtracting decimals than knowing to line up the decimal points. Most state standards expect fifth graders to add and subtract decimals to hundredths (some may say thousandths), using models, place-value knowledge, and the properties of the operations. Importantly, they should be able to make reasonable estimates of the answers, which helps them judge whether the decimal point is placed properly.

Contextual Examples and Visuals. As always with computation, start with problems in a context and provide meaning and relevance with situations that can be easily imagined and visualized with a variety of representations. But there is an order in which you should consider presenting the decimals used in your problems. First, start with decimals that have an equal number of digits to the right of the decimal (i.e., $2.5 + 1.4$ or $0.45 - 0.33$) and then move to ragged decimals, where the number of digits differs (i.e., $1.836 + 0.7$). Subtraction with regrouping such as $0.5 - 0.321$ may require more sophisticated thinking and therefore come later. Having learners keep the ideas in their minds that are needed for decimal computation, which differ in some ways from what they've learned with whole numbers, is a challenge. As you might expect, students sometimes overgeneralize whole-number thinking, such as when they add $0.26 + 0.3$ and incorrectly get 0.29 as the answer.

Consider this problem:

Pearson eText
Video Example 16.5
In this video author Karen Karp discusses a variety of ways to think about addition and subtraction of decimals.

Jessica and Sumiko each timed their own quarter-mile run with a stopwatch. Jessica says she ran the quarter mile in 74.5 seconds. Sumiko was more precise in her timing, reporting she ran the quarter mile in 81.34 seconds. Who ran it the fastest and how much faster was she?

Students who understand decimal numeration should be able to estimate the approximate difference—close to 7 seconds. Then they should be challenged to figure out the exact answer by using a variety of strategies. The advantage of the initial estimate helps them avoid the common error of lining up the 5 under the 4. Instead, students might note that 74.5 and 7 equals 81.5, then figure out how much extra that is (0.16) and subtract that to get the difference of 6.34. Other students may count on from 74.5 by adding 0.5, which equals 75, and then add 6 more seconds to reach 81, and finally add on the remaining 0.34 seconds. This strategy can be effectively represented on an open number line, which aligns with the problem's context. Another strategy is for students to change 74.5 to 74.50 and subtract, using their prior knowledge regarding whole-number regrouping. Similar story problems for addition and subtraction, some involving different numbers of decimal places, will help develop students' understanding.

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

After students have had several opportunities to solve addition and subtraction story problems, it is important to see whether they can think flexibly about a problem through multiple representations, as in the next activity.

Estimating Decimal Sums and Differences. Estimation is important, because often an estimate is all that is needed. Students should become skillful at estimating decimal computations before they learn to compute with the standard algorithm. This thinking is aligned with state standards, where fifth graders focus on decimal computation but not until sixth grade is the standard algorithm required. Until students have a sound understanding of place value, equivalence, and relative size of decimals, they are not ready to develop an understanding of the operations (Cramer & Whitney, 2010). An emphasis on estimation is particularly important for students who have learned the procedures for decimal computation yet cannot decide whether

their answers are reasonable. An initial minimum goal for a reasonable estimate is that it contains the correct number of digits to the left of the decimal—the whole-number part.

Pause & Reflect

Before continuing, determine a whole-number estimate for each of the following expressions. ●

1. $4.907 + 123.01 + 56.123$
2. $459.8 - 12.345$
3. $0.607 + 0.18$
4. $89.1 - 0.998$

Your estimates might be:

1. Between 175 and 200
2. A little less than 450
3. Close to 0.8
4. About 88

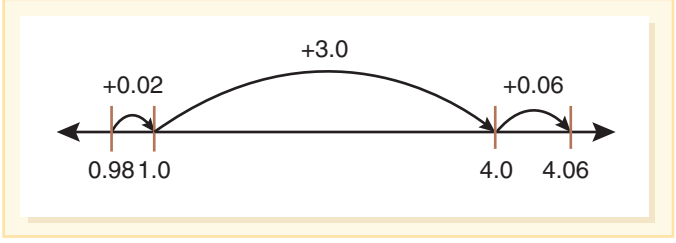
In these examples, an understanding of decimal numeration and basic whole-number estimation skills (e.g., front-end, rounding, and compatible numbers) can produce reasonable estimates. When encouraging students to estimate, focus on the size of the numbers, the meaning of the operations, and the use of a variety of strategies. Also, when initially teaching addition and subtraction with decimals, present problems to students horizontally to emphasize the need to align the numbers according to place value.

Developing Addition and Subtraction Strategies and Algorithms. Reasoning strategies receive significant attention when developing whole-number computation skills, and that same focus should continue with decimal computation. Reasoning strategies are grounded in place value, are efficient, and are often more conceptual for students than standard algorithms. Research shows that intermingling lessons on decimal place value and decimal addition and subtraction results in stronger student performance (Rittle-Johnson & Koedinger, 2009). So even after the standard algorithm is learned and understood, students should be encouraged to use their collection of reasoning strategies so they can pick the best method, given the situation. This decision making is what mathematically proficient students do. In Table 16.3 are some reasoning strategies students can consider. Notice that you've seen these before with whole numbers and fractions!

Standards for Mathematical Practice
MP1. Look for and make sense of structure

TABLE 16.3 REASONING STRATEGIES FOR ADDITION AND SUBTRACTION OF DECIMALS

Reasoning Strategy	Example
Break Apart	Given $4.5 + 3.7$, students group the units into $(4.0 + 3.0) + (0.5 + 0.7) = 7 + 1.2 = 8.28$
Make a Whole	Given $17.9 + 5.8$, students move one tenth over to 17.9 and rethink the problem as $18 + 5.7$.
Think Addition	Given $4.06 - 0.98$, students use an open number line to add .02 (to reach 1), add 3.0, and then add 0.6 to find the difference of 3.8.



Activity 16.17 Estimate Illustrate Situate

CCSS-M: 5.NBT.B.7

Give students the Estimate Illustrate and Situate activity page, which is a 2-by-2 grid in which students record the following:

Equation	Estimate
Illustrate	Situate

Put a problem such as $17.9 + 5.8 =$ in the upper left-hand quadrant. Then estimate and calculate the answer. In the third quadrant, they can illustrate the operation using, for example, an open number line or base-ten pieces. Finally, they write a word problem that situates the problem in a context. The same approach can be used with subtraction.

Pearson eText Activity Page: Estimate Illustrate Situate

FORMATIVE ASSESSMENT Notes. As students complete Activity 16.17, use an observation look-for tool to record whether students are able to (a) estimate the sum and (b) use efficient strategies to add. Note whether students get a correct sum by using an algorithm they learned but are challenged to estimate a reasonable answer, or if they are unable to write a word problem that matches the computation or create a corresponding concrete or semi-concrete model. If there are difficulties in several areas of the task, shift attention back to providing additional experiences in foundational decimal concepts. ■

Pearson eText Teacher Resource: Observation Look-For Tool

As students become more proficient in adding and subtracting with the standard algorithm, continue to ask them to estimate, illustrate using models, use reasoning strategies, and explain a context to fit the situation. For example, SplashLearn has over three dozen decimal operations games here: https://www.splashlearn.com/math/decimal-operations-games. Experiences such as these will support students' development of procedural proficiency for decimal addition and subtraction.

Multiplication

Multiplication of decimals tends to be poorly understood for many reasons. For example, students are initially confused when a multiplication problem with two decimal factors of two digits to the right of the decimal point can result in an answer with four digits to the right of the decimal point (Lortie-Forgues, Tian, & Siegler, 2015). This answer is also confusing when students overgeneralize whole-number thinking of "multiplication makes bigger" because the answer is smaller than either factor. They need to grasp that multiplying a number by 1 gives the answer of the same number, so multiplying two numbers between 0 and 1 will generate an answer smaller than either factor (Lortie-Forgues & Siegler, 2017). When students are just told to count how many decimal places they have in the problem to place the decimal in the answer, this idea of scaling does not develop. But being mathematically proficient means having a much deeper understanding of multiplication of decimals. Students need to be able to use concrete or semi-concrete models, reasoning strategies based on place value and properties of operations, and explain the reasoning used.

Contexts and Visuals. Explore multiplication of decimals by using problems in a context and by returning to concrete and semi-concrete models that were useful in thinking about whole-number multiplication. The area model is particularly useful in illustrating decimal multiplication (Rathouz, 2011b) and can link to the base-ten models used previously with multi-digit

whole numbers as well as to the algorithm. Here is an area model for 2.7×2.4, using base-ten materials (see Figure 16.13). Partial products reveal that the answer is $4 + 1.4 + 0.8 + 0.28 = 6.48$.

Consider using a semi-concrete model in a scenario such as this one that aligns with the rectangular array:

A gardener has 1.5 m^2 of his garden where he can plant flowers. He decides to plant bluebells on an area that is 0.6 of the garden. On how many total square meters can he plant bluebells?

See a student's solution (Figure 16.14), using a grid diagram to model the problem 0.6×1.5. Each large square represents 1 m^2, with each row of 10 small squares as 0.1 m^2 and each small square as 0.01 m^2. The shaded section shows $0.6\,m^2 + 0.3\,m^2 = 0.9\,m^2$. Notice that this model is proportional, allowing students to visualize the factors' values.

Also use problems that can be illustrated with an open number line, such as:

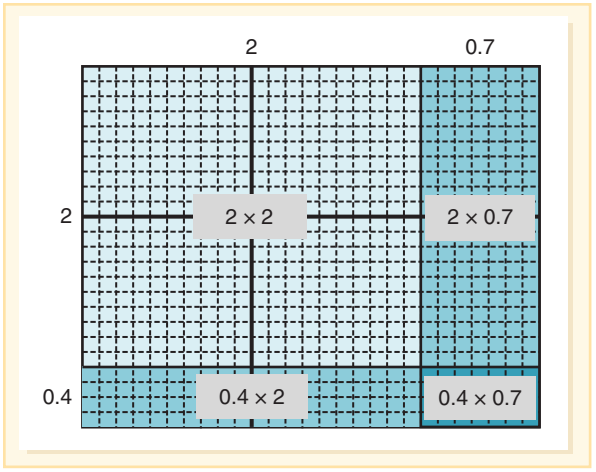


FIGURE 16.13 Area model for 2.7×2.4 that aligns with whole-number concrete models used previously

A frog always hops 4.2 inches. How far away is she from her starting point after 5 hops?

***TECHNOLOGY* Note.** Explore PhET's "Area Model Decimals" interactive simulation tool (https://phet.colorado.edu/en/simulation/area-model-decimals). There you can build a variety of rectangular arrays that help students make connections between whole-number multiplication and decimal multiplication. ■

Figure 16.15 shows an open number line to illustrate the frog hops. This illustration of decimal multiplication should remind students of strategies they used for whole-number multiplication; this connection is useful in developing meaning for the standard algorithm for decimal multiplication.

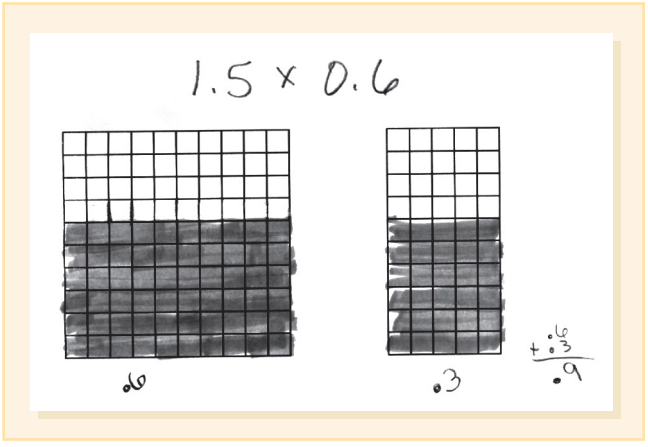


FIGURE 16.14 A student's use of 10×10 grids to reason about 1.5×0.6

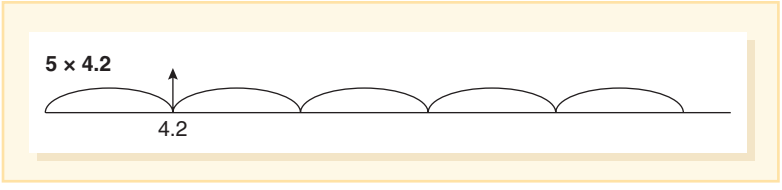


FIGURE 16.15 A number line is used to illustrate multiplication of decimals

Estimating Products. It might be argued that much of the estimation in the real world involves fractions, decimals, and percent. A key consideration in estimating is using whole numbers to estimate rational numbers.

Decide what numbers you would use in each case as you estimate the following problems. Which ones were easy to estimate? Difficult?

1. 5.91×6.1
2. 145.33×0.109
3. 0.54×9.871

Pearson eText
Video Example 16.6
In this video author Karen Karp discusses decimal multiplication using estimation and place value knowledge.

A student's reasoning might be similar to the following:

1. This problem is about 6 times 6, so the answer is about 36.
2. This is like 145 dimes, so divide by 10 and it is about 14.50. *Or*, this is about one tenth of 145, so 14.5.
3. The first value is about one-half, so half of about 10 is about 5.

When problems involve two very small decimals, estimation is difficult, but it is still possible to look at the answer to see whether it is relatively smaller than the initial factor (taking a small part of a small part results in an even smaller part).

Activity 16.18 Hit the Target: Continuous Input

CCSS-M: 5.NBT.B.7; 6.NS.B.3

Select a target range. Next, enter the starting number in the calculator and hand it to the first player. For multiplication or division, only one operation is used through the whole game. After the first or second turn, decimal factors are usually required. This variation provides excellent understanding of multiplication or division by decimals. A sequence for a target of 262 to 265 might be like this:

Start with 63.

Player 1 [×] 5 [=] 315 (too high)
Player 2 [×] 0.7 [=] 220.5 (too low)
Player 1 [×] 1.3 [=] 286.65 (too high)
Player 2 [×] 0.9 [=] 257.985 (too low)
Player 1 [×] 1.03 [=] 265.72455 (very close!)
(What would you press next?)

This game can be played using division. Adapt the game for addition and subtraction; the first player then presses either [+] or [−], followed by a number and then [=].

Developing Strategies and Algorithms. Estimation should play a significant role in developing a multiplication of decimals algorithm. Consider this problem:

The farmer fills each jug with 3.7 liters of cider. If you buy 4 jugs, how many liters of cider is that?

Ask students, "Is the answer more than 12 liters? What is the most it could be?" Once an estimate of the result is decided, students use their own methods for determining a precise answer (based on place value and properties). One strategy might be to double 3.7 (which equals 7.4), double it again, and total. Another is to multiply 3×4 and then add on 0.7 four times. Eventually, students will agree on 14.8 liters. Connect these strategies to the number line, showing how jumps on the decimal number line match the reasoning strategies.

Ask students to compare a decimal product with one involving the same digits but no decimal. For example, how are 23.4×6.5 and 234×65 alike? Interestingly, both products have exactly the same digits: 15210. (The zero may be missing from the decimal product.) Have students use a calculator to explore other products that are alike except for the location of the decimal points involved. Let them notice the pattern that the digits in the answer are always alike. After students confirm this regularity with other problems, do the following activity.

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

Activity 16.19 Where Does the Decimal Go? Multiplication

CCSS-M: 5.NBT.B.7; 6.NS.B.3

Have students compute the following product: 24×63. Using only the result of this computation (1512) and estimation, have them give the exact answer to each of the following problems:

0.24×6.3 24×0.63 2.4×63 0.24×0.63

For each computation, they should describe their thinking for how they placed the decimal point. For example, on the first problem, a student might explain that 0.24 is close to one-fourth and one-fourth of 6 is less than two, so the answer must be 1.512. They can check their results with a calculator. Multilingual learners may apply a different mental strategy that is common in their country of origin. Even if they have trouble articulating their reasoning, it is important to consider alternative ways to reason through the problem.

Pause & Reflect

What is the value in having students explain how they placed the decimal point rather than just having students count over the number of places? ●

Another way to support full understanding of the algorithm is to rewrite the decimals in their fraction equivalents. So, if you are multiplying 3.4×1.7, that is the same as $\frac{34}{10} \times \frac{17}{10}$. When multiplied, you would get $\frac{578}{100}$, which, rewritten as a decimal, is 5.78, which corresponds to moving the decimal two places to the left (Rathouz, 2011b).

The method of placing the decimal point in a product by way of estimation is more difficult as the product gets smaller. For example, knowing that 37×83 is 3071 does not make it easy to place the decimal in the product 0.037×0.83. But the standard algorithm can be developed from this problem, all the while helping students understand the properties of multiplication.

Here is the process:

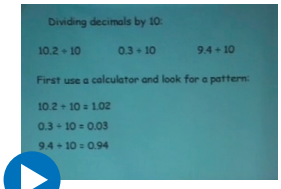
$$0.037 \times 0.83 = (37 \times \tfrac{1}{1000}) \times (83 \times \tfrac{1}{100})$$
$$(37 \times \tfrac{1}{1000}) \times (83 \times \tfrac{1}{100}) = 37 \times 83 \times \tfrac{1}{1000} \times \tfrac{1}{100}$$
$$37 \times 83 \times \tfrac{1}{1000} \times \tfrac{1}{100} = 37 \times 83 \times (\tfrac{1}{1000} \times \tfrac{1}{100})$$
$$(37 \times 83) \times (\tfrac{1}{1000} \times \tfrac{1}{100}) = 3071 \times (\tfrac{1}{100{,}000}) = 0.03071$$

These calculations may look complicated, but if you carefully follow what is happening with the decimal fractions, you can grasp why you count the number of values to the right of each factor and then place the decimal in the product so that it has the same number of decimal places. The standard algorithm for decimal multiplication is: Do the computation as if all numbers were whole numbers. When finished, place the decimal by reasoning or estimation if possible. If not possible to estimate, count the decimal places, as illustrated. Even if students have already learned the standard algorithm, they need to continue to consider when they want to use it, and when a reasoning strategy makes more sense. Sometimes the standard algorithm is much more tedious, as in this example: 4×9.25. A fluent student would notice that a break-apart strategy can be done mentally: $(4 \times 9) + (4 \times 0.25) = 37$.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Questions such as the following focus on number sense and provide useful information about your students' understanding.

1. Consider these two computations: $3\frac{1}{2} \times 2\frac{1}{4}$ and 2.27×3.18. Without doing the calculations, which product do you think is larger? Provide a reason for your answer.
2. How much larger is 0.26×8 than 0.25×8? How can you tell without doing the computation?



Pearson eText
Video Example 16.7
In this video the late author John A. Van de Walle discusses teaching division of decimals by ten using patterns and a problem-based approach.

Division

Like multiplication of decimals, division of decimals is often carried out just as a series of rules, and not well understood. Returning to whole-number understanding of the meaning of the operation of division can help students make sense of decimal division.

Estimating Quotients. Estimation and the use of concrete materials are both needed to build a strong understanding of division of decimals. In fact, the best approach to an estimate for a division problem generally comes from thinking about multiplication rather than division.

Consider the following problem:

Pearson eText
Video Example 16.8
In this video author Karen Karp discusses the role of estimation in division of decimals.

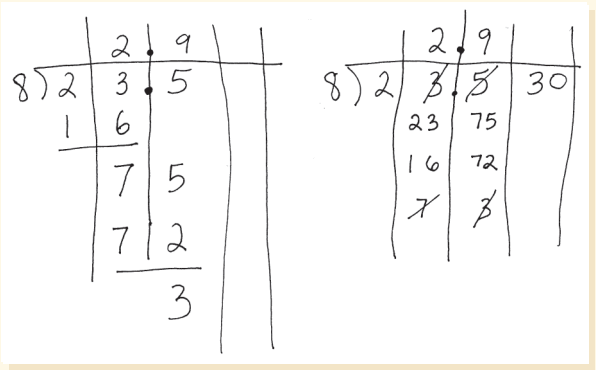
The trip to Washington was 282 miles. It took 4.5 hours to drive. What was the average miles per hour?

To make an estimate of this quotient, think about what times 4 or 5 is close to 280. You might think $60 \times 4.5 = 240 + 30 = 270$, so maybe about 61 or 62 miles per hour.

Here is a second example without a context.

Make an estimate of $45.7 \div 1.83$. Think only of what times $1\frac{8}{10}$ is close to 46.

23.5 ÷ 8



Trade 2 tens for 20 ones, making 23 ones.
Put 2 ones in each group, or 16 in all.
That leaves 7 ones.

Trade 7 ones for 70 tenths, making 75 tenths.
Put 9 tenths in each group, or 72 tenths in all.
That leaves 3 tenths.

Trade the 3 tenths for 30 hundredths.

(Continue trading for smaller pieces as long as you wish.)

FIGURE 16.16 Extension of the division algorithm

Pause & Reflect

Will the answer be more or less than 46? Why? Will it be more or less than 20? Now think about 1.8 being close to 2. What times 2 is close to 46? Use this approach to produce an estimate. ●

Because 1.83 is close to 2, the estimate is near 23. And because 1.83 is less than 2, the answer must be greater than 23—say, 25 or 26. (The actual answer is 24.972677.)

Developing the Algorithm. Although estimation can produce a reasonable result, students should learn a standard algorithm in the same way it was needed for multiplication. Figure 16.16 shows division by a whole number and how that can be carried out to as many places as you wish. (The explicit-trade method described in Chapter 12 is shown on the right in the figure.) Through reasoning, you are placing the decimal point—that is, trade 2 tens for 20 ones and then put 2 ones in each group—so you know the 2 in the quotient is in the ones place.

An algorithm for decimal division is parallel to that for multiplication: Ignore the decimal points and do the computation as if all numbers were whole numbers. When finished, place the decimal by using estimation. This approach is reasonable for divisors greater than 1 or close to a familiar value (e.g., 0.1, 0.5, 0.01). If students have a method for dividing by 45, they can divide by 0.45 and 4.5, thinking they will get ten times as many (because 4.5 is one-tenth of 45) and therefore multiply the answer by ten.

Activity 16.20 Where Does the Decimal Go? Division

CCSS-M: 5.NBT.B.7; 6.NS.B.3

Provide a quotient such as $146 \div 7 = 20857$ that is correct to five digits but without the decimal point. The task is to use only this information and estimation to give a fairly precise answer to each of the following:

$146 \div 0.7$ $1.46 \div 7$ $14.6 \div 0.7$ $1460 \div 70$

For each computation, students should write a rationale for their answers and then check their results with a calculator. As noted in multiplication, multilingual learners may apply a different mental strategy, and it is important to value alternative approaches. Again, engage students in explicit discussions of common challenges and how to think past them.

Introducing Percents

Learning Outcome 16.5 **Explain how percents are related to fractions and decimals.**

The term *percent* is simply another name for *hundredths* and, as such, is a standardized ratio with a denominator of 100. If students can express fractions and decimals as hundredths, the term *percent* can be substituted for the term *hundredth*. Consider the fraction $\frac{3}{4}$. As a fraction expressed in hundredths, it is $\frac{75}{100}$. When $\frac{3}{4}$ is written in decimal form, it is 0.75. Both 0.75 and $\frac{75}{100}$ are read in exactly the same way, "seventy-five hundredths." When used as operators, $\frac{3}{4}$ of something is the same as 0.75 or 75 percent of that same thing. Thus, percentages are merely a new notation and terminology, not a new concept.

Models and Terminology

Physical models provide the main link among fractions, decimals, and percents, as shown in Figure 16.17, the rational number wheel, bar diagrams, and 10×10 grid. Base-ten models are suitable for fractions, decimals, and percents, because they all represent the same idea. The same is true of a 10×10 grid, where each little square inside is 1 percent of the grid and each row or strip of 10 squares is not only a tenth but also 10 percent of the grid. The bar diagram is aligned with common images of a download bar or battery life indicator bar on a computer or smart phone. The rational number wheel with 100 markings around the circumference is a model for percents as well as a fraction model for hundredths.

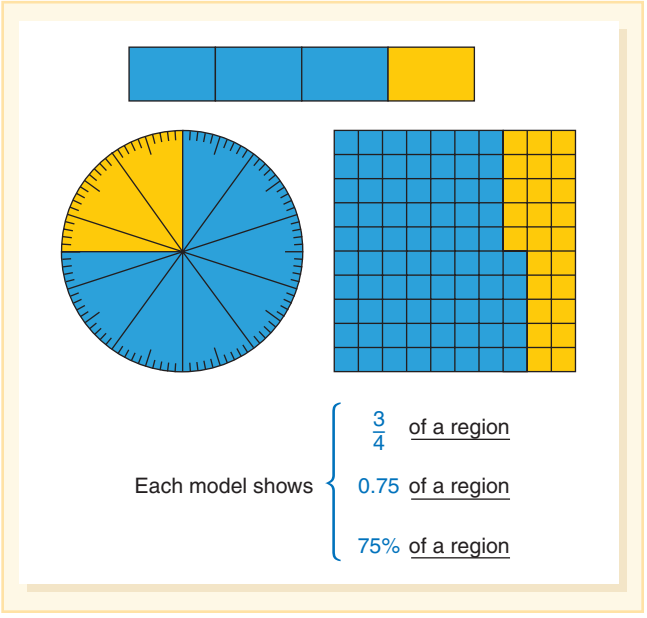


FIGURE 16.17 Models connect three kinds of notations

Pearson eText Blackline Master: 10 × 10 Grids
Pearson eText Blackline Master: Blank Bar Diagrams
Pearson eText Blackline Master: Rational Number Wheel

Fractions to Percents. Zambo (2008) recommends linking fractions to percent, using a 10×10 grid. Like with decimals, shading can be translated in either direction. For example, you can mark out one out of every four squares on the chart or shade a 5×5 region in the corner of the grid to show $\frac{25}{100}$ or 25 percent or $\frac{1}{4}$.

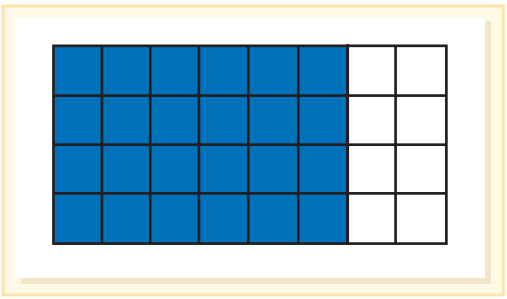
Students benefit from being able to move between common fractions (halves, fourths, fifths, and tenths) and the related decimals and percents automatically. This should be the main focus of translating among representations. For example, $\frac{3}{5}$ is 60 percent as well as 0.6.

After significant experiences with a 10×10 grid, students benefit from exploring other sized grids. Let's look at such an activity (based on Smith, Silver, & Stein, 2005).

Activity 16.21 Name the Shaded Region

CCSS-M: 6.RP.A.3c

Use the Name the Shaded Regions activity page to explore students' reasoning about fractions, decimals, and percents without the ease of the 10×10 grid.



What percent of this region is shaded? What is the decimal that shows how much of the area is shaded? Explain your thinking.

Students might note that in this example, two columns is one-fourth or 25% or the area of the rectangle, so 6 columns is 75%. The activity page provides other regions to consider.

Pearson eText Activity Page: Name the Shaded Regions

The bar diagram representation is a useful connection to real-world visualizations of percents, because students are likely familiar with the rectangle that fills as we check our technology power or download a program app. Try this next activity.

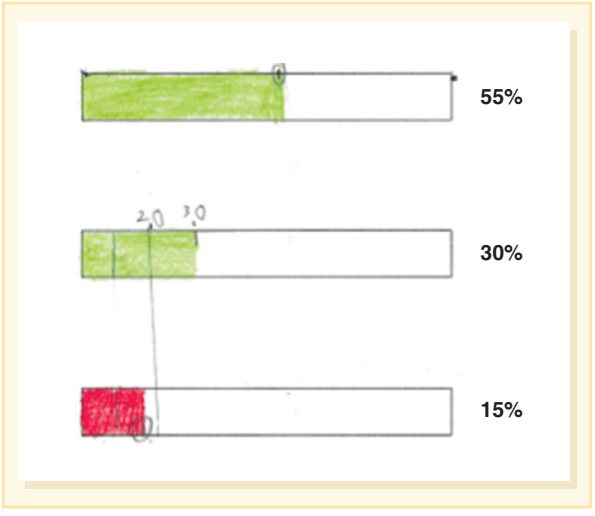


FIGURE 16.18 Kieran estimates percentages on bar diagrams, where he decides to use green and red to show when the device needs to be recharged!

Activity 16.22 Charged Up?

CCSS-M: 6.RP.A.3c

Put students in groups of two and give them each a copy of the Charged Up? activity page. Explain that each of the bar diagrams represents a download bar or battery charge bar from a phone or tablet. Have them look at the percent charged and represent it on the bar diagram by shading in a matching area. Tell them that they don't have to go in order of the problems on the sheet. After a short period, have them compare one of their shadings to their partner's. Ask them to talk about the similarities and differences. Also, notice if students are starting with 90% or if they use another benchmark such as 50 and begin by shading 55%. Note whether they mark interim units as references along the way. Look at this student's work as they decide how much of the bar diagram is shaded for each percentage (Figure 16.18). We've shared a blank bar diagram page so you can tailor the percentages to different students for a ramp-up or a challenge. Students can create their own percentages to share with a partner.

Pearson eText Activity Page: Charged Up?

Pearson eText Blackline Master: Blank Bar Diagrams

When you have data representations with circle graphs, the rational number wheel is a tool that can be used to estimate (or represent) percentages, as is a hundred-bead percent necklace as described in the next activity.

Activity 16.23 Percent Necklaces

CCSS-M: 6.RP.A.3c

Using fishing cord or sturdy string, thread 100 same-sized beads and tightly knot them together in a circular necklace for a useful percent estimation tool. Given any circle graph, even a human circle graph as shown in Figure 20.10 on page 570, place the necklace so that its center coincides with the center of the circle graph (don't try to align the necklace with the outside edge of the circle graph). If the necklace makes a wider concentric circle, students use a straight edge to extend the lines that distinguish different categories on the graph out to meet the necklace. If the circle graph is larger than the necklace, as it would be in Figure 20.10, use the lines that denote the categories. Have students count the number of beads between any two lines that represent a category. For example, they might count 24 beads in the section of the circle graph that shows how many students selected the Grand Canyon as where they want to go on a vacation. That count is an estimate suggesting that approximately 24 percent of the students favor the Grand Canyon trip. Counting the beads in a given category provides an informal estimation of the percentage while providing a meaningful model for thinking about the per-one-hundred concept.

***TECHNOLOGY* Note.** "Fraction Models" on the NCTM Illuminations website (http://illuminations.nctm.org) explores equivalence of fractions, mixed numbers, decimals, and percents and can be used on Chromebook or a tablet. Select the fraction and pick the type of model (length, area [rectangle or circle], or set), and it shows the corresponding visual and all the equivalences. ■

Here's another activity that focuses on representations of percent:

Activity 16.24 Percent Memory Match

CCSS-M: 6.RP.A.3c

Use the percent cards that include circle graphs with a percentage shaded in (card with $\frac{1}{2}$ shaded) and matching percents (card with 50% written on it). Students pair each circle graph with the percent that best matches it in a memory-like game in which they flip over cards to find a matching pair. To support students with special needs in math, provide rational number wheels (Figure 16.7) as a movable representation. For a virtual experience, consider the "Percentage Fraction Decimal Grid" demo from Visnos (https://www.visnos.com/demos). Here you use a square or circular figure to partition into pieces. These parts can be colored and represented as fractions, decimals, or percents.

Pearson eText Activity Page: Percent Cards

Pearson eText Blackline Master: Rational Number Wheel

Another helpful approach to developing the terminology involved with percentages as compared to decimals is having students consider the role of the decimal point. Recall that the decimal point identifies the units position. When the unit is ones, a number such as 0.659 means more than 6 tenths of 1. The word *ones* is understood (6 tenths of 1 unit of one or one whole). But 0.659 is also 6.59 tenths and 65.9 hundredths and 659 thousandths. The name of the unit must be explicitly identified. Because *percent* is another name for hundredths, when the decimal point identifies the hundredths position as the units, the word *percent* can be specified as a synonym for hundredths. Thus, 0.659 (of some whole or 1) is 65.9 hundredths or 65.9 percent of that same whole. As illustrated in Figure 16.19, the notion of placing the decimal point to identify the percent position

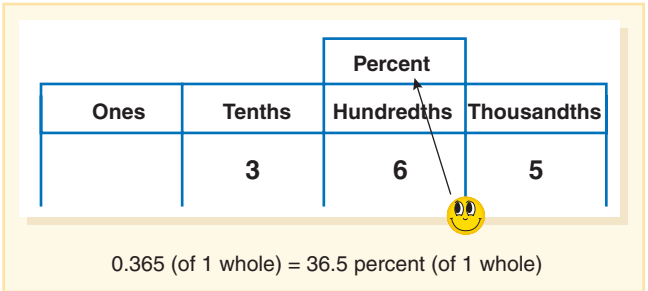


FIGURE 16.19 Hundredths are also known as percents.

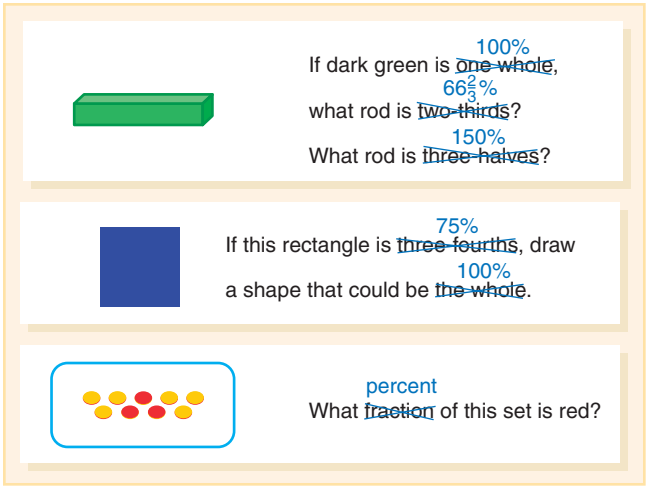


FIGURE 16.20 Part-whole fraction exercises can be translated into percent exercises

(e.g., hundredths) is conceptually more meaningful than the rule: "To change a decimal to a percent, move the decimal two places to the right." A more conceptually focused idea is to equate hundredths with percentages both orally and in notation.

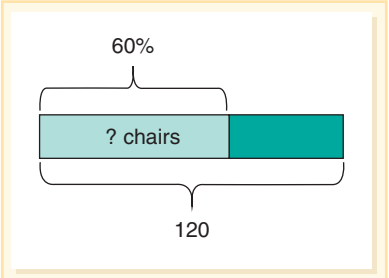
Percent Problems in Context

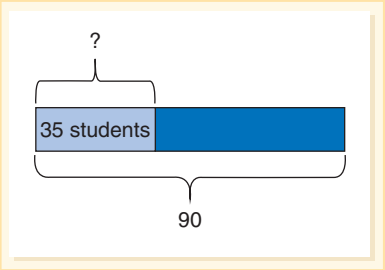
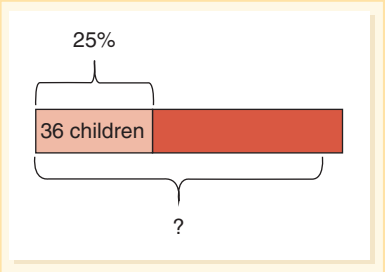
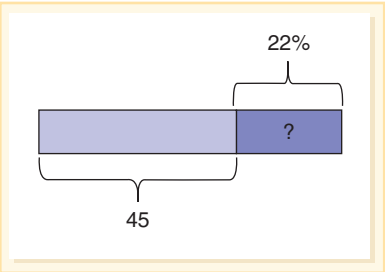
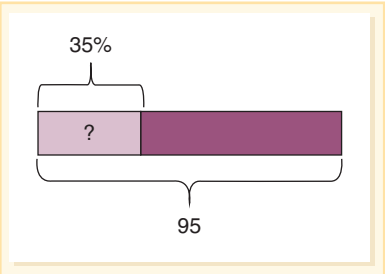
Students should have some experience with situations without contexts (see Figure 16.20) and realistic contexts (Shahbari & Peled, 2016). Find or create percent problems, and present them as they appear in the news, on television, and in other real-world settings. Follow these guidelines:

- Initially limit the percents to familiar fractions (halves, thirds, fourths, fifths, and eighths) or simple percents ($\frac{1}{10}$, $\frac{1}{100}$) and use numbers compatible with these fractions. The focus of these exercises is the relationships involved, not complex computations.
- Require students to use models, drawings, and contexts to explain their solutions. It is wiser to assign three problems requiring a drawing and an explanation of their thinking than to give 15 problems requiring only computation and answers. Remember that the purpose is the exploration of relationships, not just computational skill.
- Do not rush to sharing rules or developing algorithms for different types of problems—encourage students to notice patterns.
- Use the terms *part*, *whole*, and *percent* (or *fraction*). Help students see these percent exercises as the same types of exercises they did with fractions.
- Encourage mental computation and estimation.

Percents must be considered in light of the context (Bu & Marjanovich, 2017). Yet some of you may have been taught a classic and formulaic focus on "three-percent problems" because the sentence "_____ is _____ percent of _____" (20 is 25 percent of 80) has three spaces for numbers. Traditionally, there was an attempt to reduce the process to thinking about these three cases, but students were often connecting understanding with neither the problem nor the procedure. Furthermore, commonly encountered percent situations, such as sales figures, taxes, food composition (% of fat), sports reporting, and economic trends are rarely in that three-part format. So, let's focus on visual representations that help students think about percent situations and logically progress (see Table 16.4, adapted from Mula & Hodnik, 2020; Petit et al., 2020; Pöhler et al., 2017).

TABLE 16.4 PROGRESSION OF PERCENT PROBLEM TYPES

Problem Type	Example	Bar Diagram Representation	Equation
Find the percent of the whole amount (unknown part)	60% of the 120 chairs at the concert were empty. How many chairs were empty?		$0.60 \times 120 = ?$

Problem Type	Example	Bar Diagram Representation	Equation
Find the percentage, given the whole and the proportional amount (unknown percent)	If 35 of the 90 students were absent, what percentage of the students were absent?		$? \times 90 = 35$
Find the whole amount, given the percent (unknown whole)	25% of the people at the zoo exhibit were children. If there were 36 children, how many people were there?		$0.25 \times ? = 36$
Find the percent increase	Last year, there were 45 members in the club. This year, there was a 22% increase. Now how many club members are there?		$45 + (0.22 \times 45) = ?$
Find the percent decrease (i.e., discount)	The bicycle was on sale for 35% off. What is the cost of a bike that was originally \$95?		$\$95 - (0.35 \times \$95) = ?$

Try working each of the six following problems, first identifying which problem type. Then draw bar diagrams to explain or describe your thought process. Examples of student reasoning about two of the problems are illustrated in Figure 16.21.

1. The PTA reported that 75 percent of the total number of families was represented at the meeting. If students from 320 families go to the school, how many were represented at the meeting?
2. The baseball team won 80 percent of the 25 games it played this year. How many games were lost?
3. In Mrs. Carter's class, 20 students, or $66\frac{2}{3}$ percent, were on the honor roll. How many students are in her class?
4. Zane bought his new computer at a $12\frac{1}{2}$ percent discount. He paid \$700. How many dollars did he save by buying it at a discount?
5. If Nicolas has read 60 of the 180 pages in his library book, what percent of the book has he read so far?
6. The hardware store bought widgets at 80 cents each and sold them for \$1 each. By what percentage did the store mark up the price of each widget?

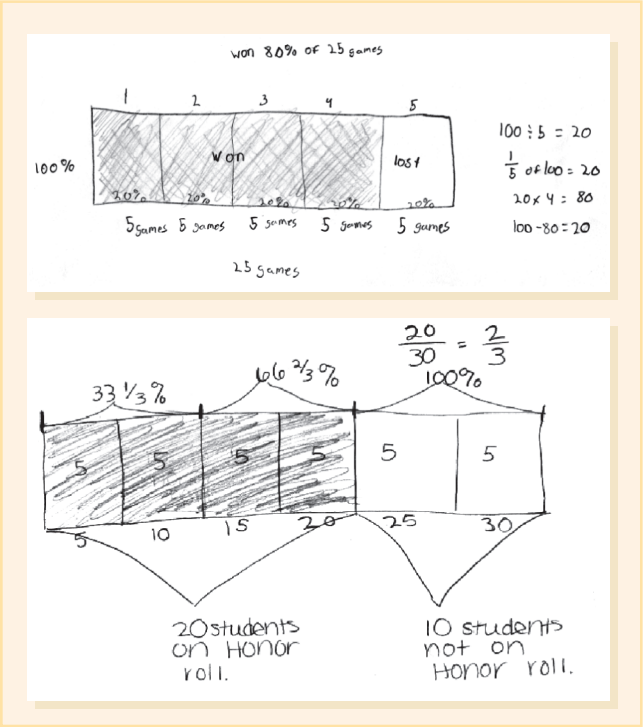


FIGURE 16.21 Students use bar diagrams to solve percent problems

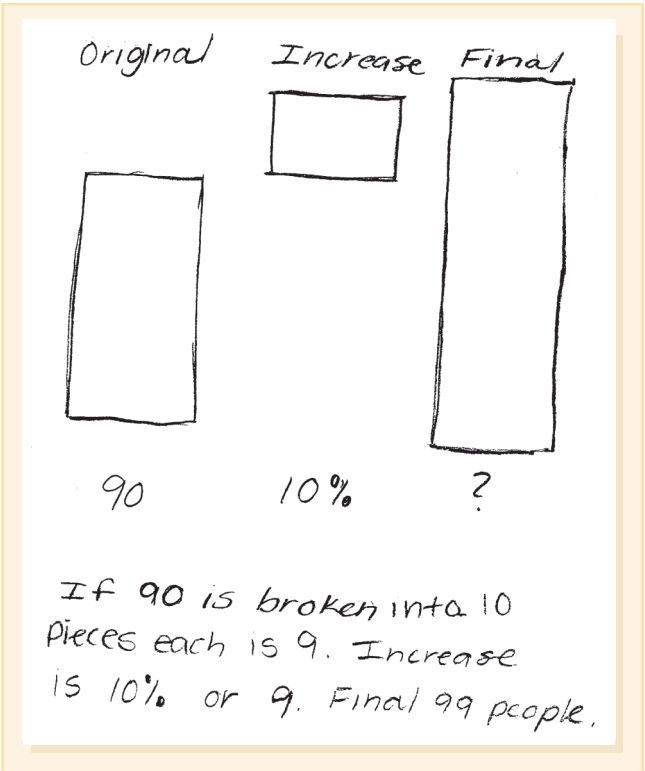


FIGURE 16.22 A student uses a proportional model to reason about a percent increase problem

Here is a powerful visual that links to proportional thinking by using a three-part model to represent the original amount, the percentage of decrease/increase, and the final amount (Lo & Ko, 2013; Parker, 2004). Using three rectangles as representations that can be positioned and divided, students can imagine and analyze aspects of a situation and then consider the relationships between each component of the model. This model is particularly useful for percent increase problems where the task is to find an amount greater than the original. For example, here is a 2005 NAEP item: How many employees were there at a company whose workforce increased by 10 percent over the previous level of 90 workers? Using Parker's (2004) approach, Figure 16.22 shows how a student used this three-part model to arrive at a correct solution.

Estimation

Many percent problems do not have simple (familiar) numbers. Frequently in real life, an approximation or estimate in percent situations is enough to help one think through the situation. An estimate based on an understanding of the relationship confirms that a correct operation was performed or that the decimal point was positioned correctly.

To help students with estimation in percent situations, two ideas that have already been discussed can be applied. First, when the percent is not a simple one, substitute a close percent that is easier to work with. Second, select numbers that are compatible with the percent involved to make the calculation easy to do mentally. In essence, convert the complex percent problem into one that is more familiar. Here are some examples.

1. The 83,000-seat stadium was 73 percent full. How many people were at the game?
2. The treasurer reported that 68.3 percent of the dues had been collected, for a total of $385. How much more money could the club expect to collect if all dues are paid?
3. Max McStrike had 217 hits in 842 at bats. What was his batting average?

Pause & Reflect

Use familiar percents, fractions, and compatible numbers to estimate solutions to each of these last three problems. ●

Here are some possible estimates:

1. Use $\frac{3}{4}$ and 80,000, so about 60,000 people
2. Use $\frac{2}{3}$ and \$380; the club will collect $\frac{1}{3}$ more, so about \$190 more
3. The number of hits (217) is about $\frac{1}{4}$ the number of at-bats (842); $\frac{1}{4}$ is 25 percent, so Max is getting a hit a little more than 25% of the time.

There are several common uses for estimating percentages in real-world situations. As students gain full conceptual understanding and flexibility, there are a variety of useful ways to think about percents:

Tips. For 15%, you can find 10 percent of the amount and then half of that again or, for 20%, double 10%.

Taxes. Depending on the sales tax rate, you can find 10 percent, take half of that, and then find 1 percent and add or subtract that amount as needed. Students should also realize that finding percents is a process of multiplication; therefore, finding 8 percent of \$50 will generate the same result as finding 50 percent (half) of 8, or \$4.

Discounts. A 30 percent decrease is the same as 70 percent of the original amount, and depending on the original amount, using one of those percents may be easier for mental calculations than the other. If a \$48 coat is 30 percent off, for example, you are paying 70 percent. Round \$48 to \$50 and you can calculate 0.70×50 (think 7×5), so your coat is less than \$35.

These are not rules to memorize, but reasoning strategies to try and discuss. Such discussions develop a full understanding of percent concepts and, in this case, the commutative property.

Although we've mentioned some common challenges about comparing decimals earlier, we include others here in Table 16.5. Because incomplete understandings often provide powerful learning opportunities, we share ways to help students add to their ever-expanding knowledge of these ideas.

Common Challenges in Decimals and Percents

TABLE 16.5 COMMON CHALLENGES IN DECIMALS AND PERCENTS AND HOW TO HELP

Common Challenge	What It Looks Like	How to Help
1. Deciding when zero is important in interpreting the decimal quantity and when it can be omitted from the decimal	Students think that 0.80 is ten times larger than 0.8 (Irwin, 2016) Students are not sure that 0.6 and 0.60 are equal	● This confusion links to the inaccurate rule that to multiply any number by 10, just put a zero at the end of the number. ● Students should continue to model what these values look like, using shading of 10×10 grids and then compare.
2. Recording decimal value from hearing it	Students write one hundredth as 0.100 (Irwin, 2016) Students write 43 tenths as 0.43 instead of 4.3.	● Present two "anonymous" students' answers and have students talk about why the reasoning for one is correct and why the other is not.
3. Writing equivalent decimals and fractions	Students write: $\frac{4}{5}$ as 4.5 $\frac{1}{4}$ as 1.4 or 0.4	● Discuss what $\frac{4}{5}$ of a pizza and 4.5 pizzas look like, using models. ● Shade in 10×10 grids to compare $\frac{1}{4}$, 1.4, and 0.4.
4. Treating two components of the decimal (whole number and fractional part) as two separate numbers	$4.6 + 5.8 = 9.14$	● Use base-ten materials and a place value matt with the decimal number cards shown in Figure 16.5.

Common Challenge	What It Looks Like	How to Help
5. Aligning addition or subtraction problems as right justified rather than by lining up the decimal points to align place value parts	$5.17 + 0.2 = 5.19$ $0.25 - 0.1 = 0.24$ $5.56 + 2 = 5.57$ $.3 + 0.46 = 0.49$	• Notice that students are using whole-number thinking, and they need more experiences using models and explicitly translating actions and representations to symbols. Work problems with the base-ten materials and show how to record with partial sums (or differences). • Use a context, because these numbers are best interpreted within a situation.
6. Thinking multiplication makes bigger and division makes smaller	$0.5 \times 0.2 = 0.7$ or Students would incorrectly choose the answer "a" for: $8 \div 0.4 =$ a. less than 10 b. greater than 10 (Fagan & Tobey, 2015)	• Use easily understood numbers and, with models, estimation, or reasoning strategies, show how, in the case of the first example given, half of two tenths is one tenth (not 7 tenths). • 10×10 grids can be used to represent the multiplication problems (see Figure 16.14). • Students use whole-number reasoning and overgeneralize. Use base-ten materials to represent and act out division problems.
7. Understanding a fractional percent	$\frac{1}{10}$ or 0.1 are both equal to $\frac{1}{10}\%$	• Students should use 10×10 grids to shade in each amount and compare to see if they are equivalent quantities.
8. Percents cannot be greater than 100	1.45 is written as 0.145%	• Use base-ten models to investigate both values. • Set the problem in a context such as a plant growing more than 100% in a single year.
9. There is confusion about thinking single-digit percents are tenths	Students write 9% as 0.9	• Use the rational number wheel, base-ten materials, or 10×10 grids to explore these values.
10. Distinguishing between percent changes (increases and decreases)	Given these problems: "There are 50 people in the restaurant, and then the group increases by 10%. Now how many people are in the restaurant?" Students would incorrectly answer 60. or "What is the price of a magic set that costs $25.00 when the store offers a 10% discount?" Students would incorrectly answer $2.50 instead of $22.50. or "Mary's salary of $55,000 decreased by 20% last year due to budget cuts. This year she got a 20% raise, what is her salary now?" Students would incorrectly answer $55,000.	• Use models and discussion. Help students understand the nature of a percent increase. Use Figure 16.22 to show the model. • Model the problem using an array with 5 columns and 5 rows in the second example. The full array would represent $25.00. Color in 2.5 to show 10%. Discuss differences between the discount and the final price. • Students do not recognize the whole the percent refers to in the third example, and that the second percent change refers to a different whole than the first. Have students act this situation out, using a proportional model as shown in Figure 16.22. Talk about the intermediate referent and label each whole.

RESOURCES FOR CHAPTER 16

WRITING TO LEARN

1. List three models that help support students' exploration of decimals.
2. How can we help students think about very small place values such as thousandths and millionths in the same way we get students to think about very large place values such as millions and billions?
3. Use an example involving base-ten pieces to explain the role of the decimal point in identifying the units position. Relate this idea to changing units of metric measurement.
4. Explain how the line-up-the-decimals rule for adding and subtracting can be developed through practice with estimation.
5. Give an example of how multiplication and division with decimals can be replaced with estimation and whole-number methods.

LITERATURE CONNECTIONS

There are many authentic, real-world connections for decimal and percent situations. Money-related increases and decreases are interesting to project over several years. For example, if the consumer price index rises approximately 3 percent a year, how much will a $100 shopping cart of groceries cost by the time you are 21 years old? Here are other contexts from children's literature:

The Phantom Tollbooth

Juster (1961)

Mathematical ideas abound in this story about Milo's adventures in Digitopolis, where everything is number-oriented. He meets a half of a boy, cut from top to bottom. The boy is actually 0.58 because he is a member of the average family: mother, father, and 2.58 children. One advantage is that he can drive 0.3 of a car, as the average family owns 1.3 cars. Explore decimal averages that are interesting to students (average number of siblings, etc.) and illustrate the results. Illustrating an average number of pets can be very humorous!

Piece = Part = Portion: Fraction = Decimal = Percent

Gifford & Thaler (2008)

Illustrated with vivid photos, this book relates fractions to decimals and percents through common representations, such as one sneaker representing $\frac{1}{2}$ or 0.50 or 50 percent of a pair of shoes. Real-world links such as one-seventh of a week and one-eleventh of a soccer team will connect with students.

RECOMMENDED READINGS

Articles

Cramer, K., Monson, D., Wyberg, T., Leavitt, S., & Whitney, S. B. (2009). Models for initial decimal ideas. *Teaching Children Mathematics*, *16*(2), 106–116.

This article describes ways of using 10×10 grids and decimal addition and subtraction boards to enhance students' understanding. Several diagnostic interviews are included.

Martinie, S. (2014). Decimal fractions: An important point. *Mathematics Teaching in the Middle School*, *19*(7), 420–429.

By offering assessment tasks that include using multiple representations, the author taps into common challenges students face. A summary of samples of student work help pinpoint common problems and next instructional steps.

CHAPTER

17 Ratios and Proportional Reasoning

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **17.1** Describe the types of ratios and the relationship among ratios, rates, unit rates, and fractions.
- **17.2** Contrast proportional and non-proportional situations and describe ways that students reason proportionally as they compare ratios.
- **17.3** Illustrate different ways to represent and reason about proportional problems across a developmental learning progression.
- **17.4** Connect proportional reasoning to concepts in geometry and algebra.

Proportional reasoning goes well beyond the notion of setting up a proportion to solve a problem—it is a way of reasoning about multiplicative situations, like how long it will take to arrive at your destination based on how fast you are traveling, or how much a certain number of avocados will cost, based on their unit price. Proportional reasoning is used in many important concepts, including "similarity, relative growth and size, dilations, scaling, pi, constant rate of change, slope, speed, rates, percent, trigonometric ratios, probability, relative frequency, density, and direct and inverse variations" (Heinz & Sterba-Boatwright, 2008, p. 528). Wow! Yet, sadly, over half the population of adults are not proportional thinkers (Lamon, 2020). We must move beyond teaching an algorithm to solve missing value proportions to teaching students to reason proportionally—that is the focus of this chapter.

BIG IDEAS

- A ratio is a multiplicative comparison between two or more quantities or measures.
- Rate is a fixed ratio and, as such, represents an infinite number of equivalent ratios.
- Ratios and proportions involve multiplicative rather than additive comparisons. This means that equal ratios result from multiplication or division, not from addition or subtraction.

- Mathematical models, such as ratio tables, double number lines, graphs, and equations, support student reasoning about proportional situations.
- There are several ways to solve missing proportion problems; they should each be understood and taught in a developmentally appropriate sequence. Additionally, students need ample opportunity to learn to choose when they want to use each method.

Ratios

Learning Outcome 17.1 Describe the types of ratios and the relationship among ratios, rates, unit rates, and fractions.

Ratio reasoning grows out of students' prior understanding of multiplicative compare situations (discussed in Chapter 8). A *ratio* is a multiplicative comparison of two or more quantities—for example, comparing numbers of apples to number of bananas as 5 to 2 (or 5:2 or $\frac{5}{2}$). Reasoning with ratios involves paying attention to how the quantities covary.

Types of Ratios

Part-to-Part Ratios. A ratio can relate one part to another part. For example, the ratio of spring flowers as 9 tulips to 7 daffodils can be represented as $\frac{9}{7}$, meaning "a ratio of nine to seven," but not nine-sevenths (the fraction). A part-to-part ratio is *not* a fraction, although it can be written with the fraction bar; the context is what tells you it is a part-to-part ratio.

Part-to-part ratios occur across the curriculum. In geometry, corresponding parts of similar geometric figures are part-to-part ratios. Pi (π) is the ratio of circumference to diameter of a circle. In algebra, the slope of a line is a ratio of *rise* to *run*. In probability, *odds* are the chance something will occur to the chance they will not occur (a part-to-part ratio).

Part-to-Whole Ratios. Ratios can express comparisons of a part to a whole—for example, a household might have 4 cats and 2 dogs. Part-to-whole describes the ratio of cats to pets (4:6) or dogs to pets (2:6). Part-to-whole ratios can also be stated as fractions (two-thirds of their pets were cats). Percentages and probabilities are also examples of part-whole ratios.

Ratios as Quotients. Ratios are multiplicative comparisons and, as such, describe two quantities that are related by division. For example, if you can buy 4 kiwis for $1.00, the ratio of money to kiwis is $1.00 to 4 kiwis, or $\frac{1}{4}$. Dividing the values results in $0.25 for one kiwi, the unit rate.

Ratios as Rates. A rate is a fixed ratio that compares two quantities measured in different units, such as miles compared to gallons, number of roses to number of bouquets made. Relationships between two units of measure are fixed ratios, so they are also rates—for example, inches per foot, milliliters per liter, and centimeters per inch.

A rate represents an infinite set of equivalent ratios (Lobato, Ellis, Charles, & Zbiek, 2010). For example, a person's rate of walking might be described as "walking 9 miles every 3 days," which means this person walks 18 miles in 6 days, 27 miles in 9 days, and so on. A *unit rate* is a rate in which the second quantity in the comparison is one unit, for example, 3 miles in one day or 3 miles per day.

Ratios involve a variety of comparison phrases. All of these statements represent ratios (Petit et al., 2020):

The ratio of bananas **to** apples is 5:3	12 books **in** 5 weeks
5 apples **for every** 3 bananas	25 miles **per** hour
0.5 cups of flour **per** dozen cookies	4 **out of** 5 doctors
3 parts lemonade **to** 2 parts tea	Read 20 minutes **each** day

It is worthwhile to begin exploration of ratios and rates by helping students understand the language and meanings of ratios, rates, and unit rates (Jensen, 2018). Activity 17.1 engages students in such an experience.

Activity 17.1 Which Is It: Ratio, Rate, or Unit Rate?

CCSS-M: 6.RP.A.2

Prepare a collection of examples that fits each category. Examples are provided in the activity page. A sampling includes:

8 cars, 3 motorcycles	$1.50 for one avocado	$8.40 for a dozen doughnuts
4:5:9	$\frac{1}{5}$	Miles per hour
1 ½ cups flour to ½ cup butter	inches: feet	$\frac{3\,feet}{1\,yard}$

Provide students with definitions for each term and have them compare the definitions. They may notice the nesting of these definitions. Place students in small groups and ask them to find the most specific label for each of the values on their recording table. If the value is not a ratio (or rate), place it in the None category. As students work in groups, look for values that appear in different columns for different groups. After adequate time in small groups, begin a whole-class discussion about where the values go. Begin with examples on which the groups agreed, and then move to one that was placed differently and have students justify their decisions. For more examples and teaching ideas, see Jensen, 2018.

Pearson eText Activity Page: Ratio, Rate, Unit Rate Cards
Pearson eText Activity Page: Ratio, Rate, Unit Rate Recording Table

How Do Ratios Relate to Fractions?

Ratios are related to fractions, but ratios and fractions should be thought of as overlapping concepts with important distinctions (Lobato et al., 2010; Petit et al., 2020). As you read in Chapter 14, a fraction is a number expressing a part-whole relationship (one of two parts), a measurement (one-half mile), or an operator (e.g., one-half of 24). Because they are represented symbolically with a fraction bar, it is important to help students see that fractions and ratios are related, but not the same thing. In the following three examples, ask, "Is this a fraction, a ratio, or both?"

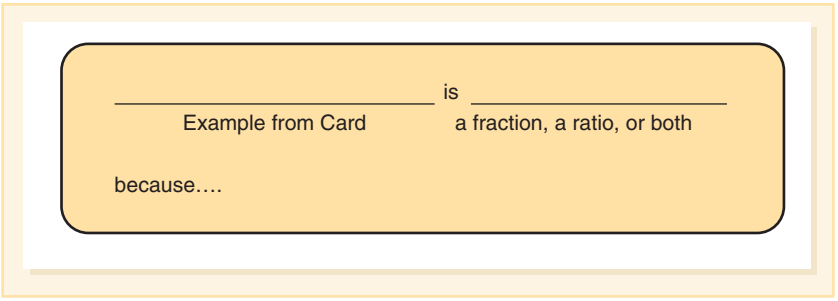
1. The ratio of cats to dogs at the pet store is $\frac{3}{5}$.
2. The ratio of cats to pets at the pet store is $\frac{3}{8}$.
3. Mario walked three-eighths of a mile ($\frac{3}{8}$ miles).

In the first example, the answer is ratio. It is not a fraction because fractions are not part-to-part comparisons. Number 2 is both. The ratio is part-to-whole, and it can be adapted to say three-eighths of the pets are cats. The third example is a fraction. There is no multiplicative comparison, only a measurement. Engaging students in a task that compares ratios to fractions helps students connect their prior knowledge of fractions to new learning on ratios. Activity 17.2 uses a Venn diagram format to compare ratios and fractions.

Activity 17.2 Fraction, Ratio, or Both?

CCSS-M: 6.RP.A.1

Prepare a collection of examples like the ones just shared and that are provided in the Fraction, Ratio, or Both activity page. Cut them into cards and place them in an envelope. Place students in partners or triads and have them sort the cards into three categories (using the Fraction, Ratio, or Both sorting mat). After the cards are sorted, ask students to select one from each category and explain why it belongs in that category. A sentence frame, such as the one provided here, can support multilingual students and students with special needs. To challenge students, use the sentence frame and have students create their own example of each.



Pearson eText Activity Page: Fraction, Ratio, or Both
Pearson eText Activity Page: Fraction, Ratio, or Both Sorting Mat

Two Ways to Think about Ratio

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

As Lobato and colleagues (2010) state, "forming a ratio is a *cognitive task*—not a *writing task*" (p. 22, emphasis added). What they mean is that a ratio communicates a relationship, and that relationship can be thought of in different ways, regardless of whether it is notated as $\frac{2}{5}$, or 2:5, or $2 \div 5$. Ratios must move beyond ways to notate them to focus on multiplicative reasoning. This includes two constructs, multiplicative comparisons and composed units.

Multiplicative Comparison. Recall that one of the multiplication situations is a multiplication comparison (see Chapter 8). An example of a multiplicative comparison is, "Timothy had three times as many books as Jeremiah." This can be restated as the ratio 3:1. It could mean that Timothy has 3 books and Jeremiah has 1 book, or Timothy has 6 books and Jeremiah has 2 books, and so on. It is also true that Jeremiah has one-third as many books as Timothy. Thus, questions related to multiplicative comparisons can be worded in two ways (Lobato et al., 2010, p. 18):

- "How many times greater is one thing than another?" or
- "What fractional part is one thing of another?"

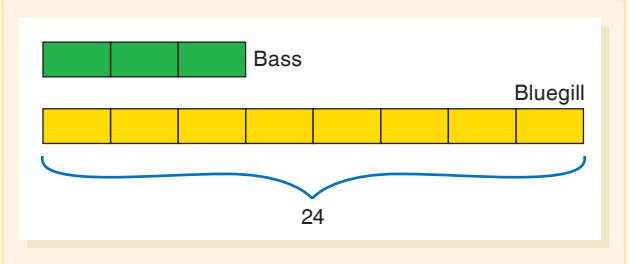
Students need to solve real-world problems involving multiplicative comparisons, and those problems should increase in difficulty with experience (Cohen, 2013).

Activity 17.3 Stocking the Pond

CCSS-M: 6.RP.A.1; 6.RP.A.2; 6.RP.A.3

Use the Stocking the Pond activity page or simply pose each of the three problems, one at a time, for students to solve. Ask students to use tape diagrams (also referred to as *bar model* or *strip diagrams*) as a tool to represent each problem. The first task is included here:

For every 3 bass that are put in a pond, Environmental Edwin puts in 8 bluegill. If Edwin puts 24 bluegill in the pond, how many bass does he put in the pond?



After students have solved this problem and the others, ask questions such as, "What fractional part is the number of bass in the pond as compared to the number of bluegill?" (three-eighths) and "What fractional part is the bluegill compared to the bass?" (eight-thirds). For multilingual students, it is helpful to use visuals and keep the same context (fish in the pond) to reduce the linguistic load. Also, be sure that the word *if* is understood in the situation to mean "assume."

Pearson eText Activity Page: Stocking the Pond

Composed Unit. The term *composed unit* refers to thinking of the ratio as one unit—in other words, joining the two quantities in a way that maintains the multiplicative relationship. Imagine a store is selling a sack of 4 kiwis for $1.00. The ratio 4 to 1 is a composed unit. You can find equivalent ratios by finding multiples of that composed unit, such as 8 kiwis for $2.00, 16 kiwis for $4.00, and so on. This process is called *building up*. Or you can partition the unit to other equivalencies such as 2 kiwis for $0.50 and 1 kiwi for $0.25. This process is called *breaking down*. Any number of kiwis can be priced by using the composed unit.

Activity 17.4 provides a context for engaging students in thinking of a ratio as a composed unit and then as a multiplicative comparison, relating three quantities (batches, green drops, and blue drops).

Activity 17.4 Making Icing

CCSS-M: 6.RP.A.1; 6.RP.A.2; 6.RP.A.3

Describe a context for icing (e.g., making cupcakes for a bake sale). Give students a color ratio for the icing, such as for 1 batch of cupcake icing with aqua-colored icing, you will need 2 drops of green food coloring and 5 drops of blue food coloring. Ask students to figure out how many drops of each color food coloring will be needed for 1 batch of cupcakes, 2 batches, 5 batches, and so on (composed unit thinking). Use the Making Icing activity page.

After exploring the ratios, ask students to create lists of comparison statements about the situation, such as, "There is 2.5 times as much blue food coloring as green food coloring." Students—particularly students with special needs in math—may benefit from visualizing this comparison, which can be done by lining up green and blue tiles or counters (or use drawings).

Pearson eText Activity Page: Making Icing

***TECHNOLOGY* Note.** Desmos (desmos.com) has several activities that focus on ratios and proportions. Check out *Balloon Float*, an activity in which students figure out how many balloons it takes to float ducks and, eventually, people safely. *Turtle Time Trials* focuses on proportional and non-proportional situations; *Click Battle* uses unit rates to engage in battles with two bots, and *Lego Prices* uses a slider tool to explore the relationship between price and number of pieces. This is just a sampling (and new ideas continue to be added!) ■

Proportional Reasoning

Learning Outcome 17.2 Contrast proportional and non-proportional situations and describe ways that students reason proportionally as they compare ratios.

Proportional reasoning is a unifying theme in mathematics and plays a critical role in developing algebraic thinking (Lanius & Williams, 2003; Blanton et al., 2015). It begins in early childhood with one-to-one correspondence and continues through the grades with place value, fraction concepts, and multiplicative reasoning (Seeley & Schielack, 2007). In middle school, it is perhaps the most significant topic, providing the foundation for algebra and geometry and relating to important real-life contexts such as interest, taxes, and tips.

Proportional reasoning is difficult to define in a simple sentence or two. According to Lamon (2020), students who are proportional thinkers:

- Understand a *ratio as a distinct entity* representing a relationship different from the quantities it compares. (See the earlier discussion about composed units.)
- Recognize *proportional relationships as distinct from non-proportional relationships* in real-world contexts.

- Have a sense of *covariation*. They understand relationships in which two quantities vary together and are able to see how the variation in one quantity coincides with the variation in another quantity.
- Develop *a wide variety of strategies* for solving proportions or comparing ratios, most of which are based on reasoning strategies rather than on prescribed algorithms.

In other words, this list describes what fluency looks like related to ratios and proportions. These last three areas are addressed in the sections that follow.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

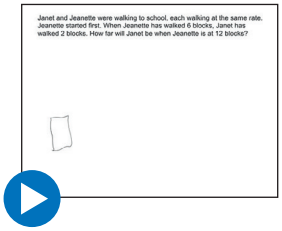
Types of Comparing Situations

Students need opportunities to compare situations and discuss whether the comparison is due to an *additive or constant situation (non-proportional) or a multiplicative* (proportional) relationship (Van Dooren, De Bock, Vleugels, & Verschaffel, 2010).

Pause & Reflect

Match each of the following situations to one relationship: *additive, constant*, or *multiplicative*. What wording in these situations helped you decide?

1. Janet and Jeanette were walking to school, each walking at the same rate. Jeanette started first. When Jeanette walked 6 blocks, Janet walked 3 blocks. How far will Janet be when Jeanette is at 12 blocks?
2. Lisa and Linda are planting corn with the same kind of seed on the same farm. Linda plants 4 rows and Lisa plants 6 rows. If Linda's corn is ready to pick in 8 weeks, how many weeks will it take for Lisa's corn to be ready?
3. Kendra and Kevin are baking cookies, using the same recipe. Kendra makes 6 dozen cookies and Kevin makes 3 dozen. If Kevin is using 8 ounces of chocolate chips, how many ounces of chips will Kendra need? ●


Janet and Jeanette were walking to school, each walking at the same rate. Jeanette started first. When Jeanette has walked 6 blocks, Janet has walked 2 blocks. How far will Janet be when Jeanette is at 12 blocks?

Pearson eText
Video Example 17.1
In this video a student solves the Janet and Jeanette problem. What is the student doing to support their reasoning and what makes this problem challenging?

Standards for Mathematical Practice
MP7. Look for and make use of structure

The first situation is *additive*. The situation describes the distance (difference) between the two walks. Janet will still be 3 blocks behind Jeanette because they are walking at the same rate, so the answer is 9 blocks. The second situation is *constant*. It will take 8 weeks for the corn to grow, regardless of how many rows were planted. The final situation is *multiplicative*. A ratio of dozens to ounces is given, and so the answer is 16 ounces.

Situations That Are Additive *and* Multiplicative. An effective way to start helping students distinguish between these types of reasoning is to provide opportunities for them to make both additive and multiplicative observations about a situation and compare the responses. This first task, similar to ones commonly used in classrooms in China, engages students in both kinds of reasoning (Cai & Sun, 2002).

Task 1: Mr. Miller's 25 students are asked if they are soccer fans (yes or no). Of these students, 20 say "yes" and 5 say "no." Describe as many relationships as you can about those students who are soccer fans and those who are not.

Students might generate these relationships (among others). As you read each, ask yourself which is additive and which is multiplicative:

1. There are 15 more fans than nonfans.
2. There are 15 fewer nonfans than fans.
3. There are 4 times as many fans as nonfans.
4. There are $\frac{1}{4}$ as many nonfans as fans.
5. For every 4 students who like soccer, there is 1 who does not.

The first two focus on the difference between the two numbers—the additive relationship. The other three describe a multiplicative relationship, each expressing the 4-to-1 ratio of fans to nonfans in a slightly different way. A student may also incorrectly say "is four times more than," perhaps because they are confused about whether the situation is additive or multiplicative (Rathouz, Cengiz, Krebs, & Rubenstein, 2014). Explicitly ask students to compare these statements and identify which are additive and which are multiplicative (and why).

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

Task 2, adapted from *Adding It Up* (NRC, 2001), also engages students in considering both additive and multiplicative interpretations.

Task 2: Two weeks ago, two flowers were measured at 8 inches and 12 inches, respectively. Today, they are 11 inches and 15 inches tall. Did the 8-inch or 12-inch flower grow more?

Additive reasoning leads to the response that they both grew the same amount—3 inches. Multiplicative reasoning leads to the first flower growing more: The first flower grew $\frac{3}{8}$ of its height whereas the second grew $\frac{3}{12}$ of its height. Both additive and multiplicative reasoning produce valid answers for this context.

Activity 17.5 can also be examined additively or multiplicatively.

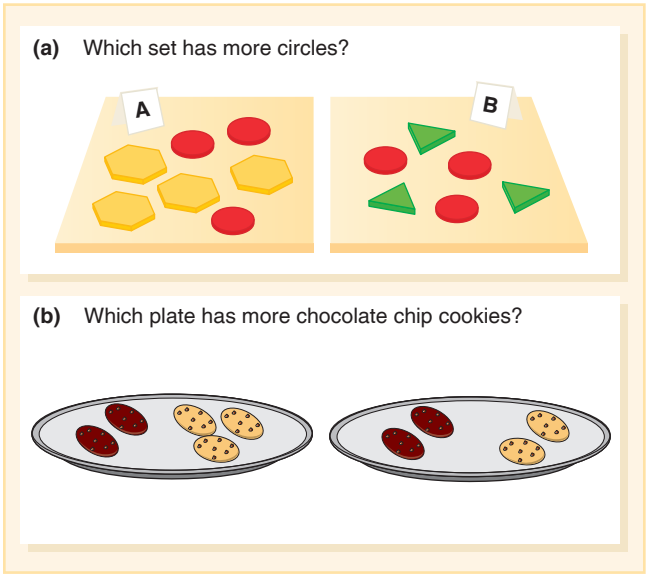


FIGURE 17.1 Pictorial situations that can be interpreted with additive or multiplicative comparisons

Activity 17.5 Which Has More?

CCSS-M: 6.RP.A.1

Provide students with situations similar to those in Figure 17.1. Ask students, "Which plate has more chocolate chip cookies?" and explain that they need to justify their thinking. Reasoning can be modeled through illustrations or explanations. As students explain, make explicit connections between finding the difference and additive reasoning and between using the ratio and multiplicative reasoning. If no one suggests one of the options, introduce it. For example, say, "Why might someone say…?" For multilingual students, support language by connecting *additive* to the word *add*, and *multiplicative* to *multiple* and *multiply*.

Students benefit from critiquing authentic issues, in particular as they relate to *social justice*, such as population growth, crime rates, racial representation in the U.S. Congress, or immigration rates. These explorations provide valuable information about the world we live in (Simic-Muller, 2015). Activity 17.6 (based on Simic-Muller) engages students in making distinctions between additive reasoning and multiplicative reasoning while they consider social issues.

Activity 17.6 Applying Reasoning to Explore Social Issues

CCSS-M: 6.RP.A.1

Provide students with real issues to explore. Such investigations can be organized in a variety of ways. First, you can either supply the data or you can task students to find the data. Second, you can have students (1) explore all problems; (2) explore one that you select; (3) explore one that they select; or (4) join expert groups, wherein each group takes one problem. Four such situations are offered on the Applying Reasoning to Explore Social Issues activity page. Two are shared here for reference:

- Share the population and number of crimes in two cities/counties in your area and ask, "Which community is safer and how did you decide?"
- Compare the racial and ethnic composition of the students in your school or district and compare to the racial and ethnic composition of teachers and/or district employees.

After data have been explored, discuss the findings in terms of (1) additive versus multiplicative reasoning and (2) social implications.

Pearson eText Activity Page: Applying Reasoning to Explore Social Issues

For each problem, circle the correct answer.

1. Mrs.Allen took a 3-inch by 5-inch photo of the Cape Hatteras Lighthouse and made an enlargement on a photocopier, using the 200% option. Which is more square, the original photo or the enlargement?

a. The original photo is more square.
b. The enlargement is more square.
c. The photo and the enlargement are equally square.
d. There is not enough information to determine which is more square.

2. The Science Club has four rectangular plots for experiments with growing plants:

1 foot by 4 feet	7 feet by 10 feet
17 feet by 20 feet	27 feet by 30 feet

Which rectangular plot is most square?

a. 1 foot by 4 feet
b. 7 feet by 10 feet
c. 17 feet by 20 feet
d. 27 feet by 30 feet

3. Sue and Julie were running equally fast around a track. Sue started first. When Sue had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?

a. 45 laps
b. 24 laps
c. 21 laps
d. 6 laps

4. At the midway point of the basketball season, you must recommend the best free-throw shooter for the all-star game. Here are the statistics for four players:

Novak: 8 of 11 shots	Peterson: 22 of 29 shots
Williams: 15 of 19 shots	Reynolds: 33 of 41 shots

Which player is the best free-throw shooter?

a. Novak
b. Peterson
c. Williams
d. Reynolds

5. Write your answer to this problem.
A farmer has three fields. One is 185 feet by 245 feet, one is 75 feet by 114 feet, and one is 455 feet by 508 feet. If you were flying over these fields, which one would seem most square? Which one would seem least square? Explain your answers.

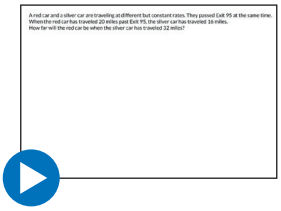
FIGURE 17.2 Five items to assess proportional reasoning
Source: Reprinted with permission from G. W. Bright, J. J. Joyner, & C. Wallis. (2003). "Assessing Proportional Thinking." *Mathematics Teaching in the Middle School,* 9(3), p. 167. Copyright © 2003 by the National Council of Teachers of Mathematics. All rights reserved.

Tasks such as the ones in Activity 17.6 can activate students' own ideas about real-life situations to explore mathematically (Beigie, 2016; Simic-Muller, 2015; Turner & Strawhun, 2007).

Situations That Require Noticing Comparison Types. Whereas the preceding tasks are effective in helping students understand the differences between additive and multiplicative relationships, most situations cannot be solved both ways—students have to recognize which relationship is described in the story and solve it accordingly (like the three problems in the earlier Pause and Reflect). One of the most common errors students make is to use additive reasoning when multiplicative reasoning is needed (Canada, Gilbert, & Adolphson, 2008; Dougherty, Bryant, Bryant, & Shin, 2016; Kaput & West, 1994; Ölmez, 2016). Conversely, students can overgeneralize proportional situations and incorrectly use multiplicative reasoning to solve additive situations (Heinz & Sterba-Boatwright, 2008; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010).

Pause & Reflect

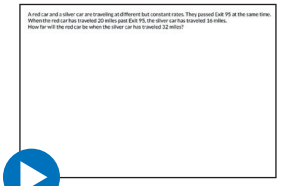
Solve the problems in Figure 17.2. Which ones are multiplicative situations? Additive? ●



Pearson eText
Video Example 17.2
In this video a student solves the Task B car problem. How is this student reasoning about this problem? How does their thinking compare to how the student solves the same problem in Video Example 17.3.

The items involving rectangular representations (1, 2, and 5) are multiplicative, looking at the ratios of the sides of the rectangles (part-to-part). The walking comparison (3) is additive, not multiplicative. The basketball task (4) can be reasoned either way. Thus, the answers are: 1. **c**; 2. **d**; 3. **c**; 4. **a** (missed fewest) or **d** (highest percentage made); 5. most square: 455 by 508; least square: 75 by 114. See one way to solve #5 in Figure 17.3. Jacob noticed that each length was divisible by 5; therefore, he simplified each ratio to have a side of 5 and then compared the widths.

FORMATIVE ASSESSMENT Notes As can be seen by Jacob's response, any or a combination of the tasks in Figure 17.2 can be effective performance assessments or used in diagnostic interviews. Other tasks in this section that focus on constant, additive, and multiplicative reasoning will also work as more in-depth assessments. ■



Pearson eText
Video Example 17.3
In this video a student solves the Task B car problem. How is this student reasoning about this problem? How does their thinking compare to how the student solves the same problem in Video Example 17.2.

There are (at least) five things you can do to help students determine whether a situation is additive or multiplicative:

1. Use problems like Activity 17.4 and Activity 17.5 that involve and compare both ways of reasoning.
2. Pair problems that are very similar, where one is additive and the other is multiplicative, and then have students contrast the two (either before or after solving them):

Task A: A red car and a silver car are traveling at the same constant rate. When the red car has traveled 20 miles, the silver car has traveled 12 miles. How far will the red car have traveled when the silver car has traveled 32 miles?

Task B: A red and a silver car are traveling at different but constant rates. They pass Exit 95 at the same time. When the red car has traveled 20 miles past exit 95, the silver car has traveled 16 miles. How far will the red car have traveled when the silver car has traveled 32 miles?

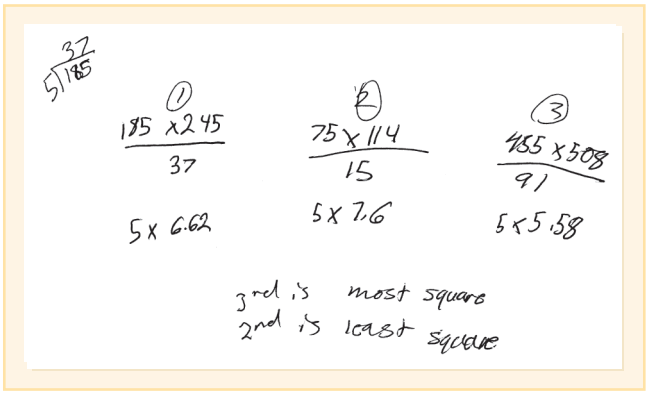


FIGURE 17.3 Jacob's work on the farmer's field problem

3. Have students write the relationships, using variables to illuminate the relationships (Lim, 2009). In the preceding tasks, the relationship in Task A is expressed as red = silver + 8 because the red car is 8 miles in front of the silver car. In Task B, the relationship is red = $\frac{5}{4}$ silver because for every five miles the red car travels, the silver car travels four.
4. Focus students on solving missing value proportion problems while continuing to include constant and additive situations.
5. Debrief from exploring tasks as a whole class, comparing additive to multiplicative reasoning and when each is appropriate.

Activity 17.7 (based on Che, 2009) provides an engaging context that helps students see the difference between additive and multiplicative reasoning (and see why the situation calls for multiplicative reasoning).

Activity 17.7 Pencil-to-Pencil

CCSS-M: 6.RP.A.3a; 7.RP.2b

Begin with a book or poem about giants (see Table 17.1). Hold up a cutout of a very large pencil (e.g., 30 inches in length). Explain to students that this is the exact size of a pencil used by a giant. Ask, "If this is her pencil, how tall is the giant?" "How long would her hand be?" Place students in partners and have them determine at least one measurement for the giant, based on knowing her pencil length (and their own pencil length). Have students post one of their giant measurements on a poster, using an illustration of their choice along with an explanation of how they found the measures. Students from other countries are likely more familiar with centimeters, so allow students to choose which unit to use (inches or centimeters). In debriefing this activity, discuss their thought processes and why the situation is a multiplicative comparison and not an additive comparison.

TABLE 17.1 CHILDREN'S AND ADOLESCENTS' LITERATURE FOR RATIOS AND PROPORTIONAL REASONING

Literature with Proportional Situations	
Many books include multiplicative comparisons and therefore can be used for proportional reasoning activities. See the e-book, *Exploring Math through Literature* (Thiessen, 2017) for lessons and activity pages for many of these books (select the Ratios and Proportions tab).	
If You Hopped Like a Frog Schwartz, 1999	Schwartz compares features of various creatures with those of humans. For example, in the title comparison, he deduces that if a person had a frog's jumping ability, they could jump from home plate to first base on a baseball diamond in one hop. Twelve other animal–human comparisons are shared, along with factual data on which the proportions are based. Students can figure out how strong or tall they would be if they were one of the featured animals.
Animal Farm Orwell, *1946*	This classic novel can be an interdisciplinary opportunity with a language arts class. The animals chase off the humans to run the farm themselves. Over time, the pigs take on a dictatorial role, for example telling the animals their foodstuff had increased by 200%, 300%, or 500%. This provides an opportunity to explore proportions (and percent increase) (Martinie & Bay-Williams, 2003).
Holes Sachar, 2000	A popular book and movie, this novel is about boys in a camp who dig holes every day. This setting provides an opportunity to look at daily rates of dirt removal—only one boy's hole is smaller by a fraction of an inch. How does this difference compare over time? See Pugalee, Arbaugh, Bay-Williams, Farrell, Mathews, and Royster (2008) for a full activity.
Wilma Unlimited: How Wilma Became the World's Fastest Woman. Krull, 2000	Wilma Rudolph survived polio, paralyzing her left leg. Determined to overcome her physical challenges, she goes from hopping to walking to running, eventually winning three gold medals in the 1960 summer Olympics. This phenomenal true story can launch a lesson in which students measure their hopping, walking, and running rates and use proportional reasoning to determine how quickly they would travel various distances (Martinie & Bay-Williams, 2003).
Literature with Large and/or Small People	
There is a plethora of literature involving very little or very big people (or animals). With any of these books, lengths of arms, legs, or noses can be compared as a way to explore within and between ratios.	
Alice's Adventures in Wonderland Carroll, 1865/1982	In this classic, Alice becomes very small and very tall, opening doors to a variety of ratio and proportion investigations. See Taber (2005) for ideas on using this book to teach proportions.
The Borrowers Norton, 1953	A classic tale of little folk living in the walls of a house. Furnishings in the story are created from odds and ends of the full-size human world.
Gulliver's Travels Swift, 1726; amended 1735/1999	Yet another classic story. In this case, Gulliver first visits Lilliput, where he is 12 times the size of the inhabitants, and then goes to Brobdingnag, where he is one-tenth the size of the inhabitants. A perfect opportunity for a variety of comparisons.
Harry Potter and the Sorcerer's Stone Rowling, 1997	Hagrid is described as twice as tall and nearly five times as wide as the usual man. Students can measure their own shoulder width and height and create a scatter plot. Placing Hagrid's measurements on the graph illustrates that he is not made in proportion to the usual person.
James and the Giant Peach Dahl, *1961*	This classic story has many recent reprints and a movie. James accidentally drops magic crystals by a peach tree and a peach grows as big as a house. Inside the peach are oversized creatures—a grasshopper, a ladybug, and more.

Literature with Large and/or Small People	
Jim and the Beanstalk Briggs, 1970	What happened to the giant after Jack? Jim comes along. Jim wants to help the poor, pessimistic giant. This heartwarming story is great for multiplicative or proportional reasoning.
Kate and the Beanstalk Osborne & Potter, 2000	This version of the traditional *Jack and the Beanstalk* tale includes a giantess. The giantess falls to earth, and Kate finds out that the castle belongs to her family.
The Lord of the Rings Tolkien, 1965	Hobbits are described as approximately 3 feet tall (which can be estimated at 100 cm). This height can be used to set up a ratio with the height of the typical sixth grader and the ratio then used to determine the size of various objects. To connect to the movie, ask students to be in the role of producer and figure out the size of objects and props used in the movie. (Beckmann, Thompson, & Austin, 2004).
"One Inch Tall" in *Where the Sidewalk Ends* Silverstein, 1973	Shel Silverstein is a hit with all ages. This poem asks what it would be like if you were one inch tall. This height provides an opportunity to use proportional relationships to figure out the size of other objects, from pencils to shoes.
Swamp Angel Isaacs & Zelinsky, 1994	A swamp angel named Angelica grows into a giant. Students can compare birth height with current height or compare Angelica's measurements with their own.

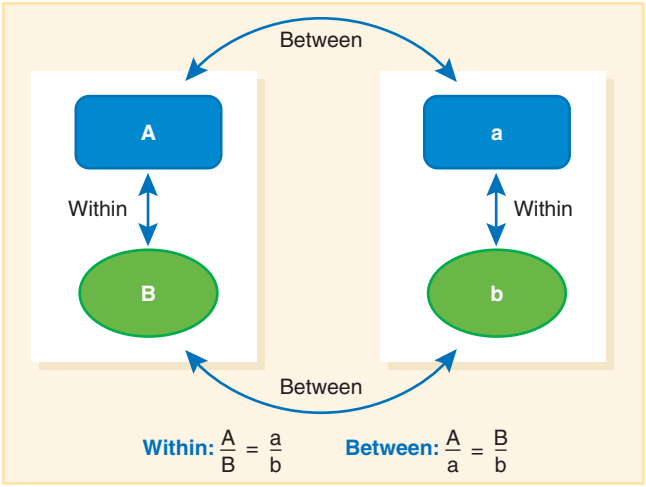


FIGURE 17.4 In a proportional situation, the between and within ratios are equivalent

Covariation

Covariation may sound like a high school concept, but it simply means that two quantities vary together. For example, five mangos cost $2.00 (two quantities in a multiplicative relationship); as the number of mangos varies (for example, to 10 mangos), so does the cost. And as the cost changes, so does the number of mangos you will get. The number of mangoes and the cost covary.

Within and Between Ratios. A ratio of two measures in the same situation is a *within* ratio. For example, in the case of the mangos, the ratio of 5 mangos to $2.00 is a within ratio; that is, it is within the context of that example.

A *between* ratio is a ratio of two corresponding measures in different situations. For example, the ratio of the original number of mangos (5) to the number of mangos in a second situation (10) is a between ratio; that is, it is between the two situations.

Figure 17.4 illustrates these two ratios. In a proportional situation, the between and within ratios are equivalent. A drawing similar to this will be very helpful to students in setting up proportions, especially for students who struggle with abstract representations.

Standards for Mathematical Practice
MP4. Model with mathematics

Activity 17.8 asks students whether the ratios represented in pictures are, in fact, equivalent.

Activity 17.8 Different Objects, Same Ratios

CCSS-M: 6.RP.A.1; 6.RP.A.2

Use the illustrations on the Ratio Cards activity page (see Figure 17.5) (or create your own, using clipart of objects that are relevant to your students). Ask students to match cards that show equivalent ratios. After they have their matches, engage students in justifying how they decided the cards represented equivalent ratios. To challenge students, invite them to create their own sets that are and are not equivalent ratios (using their own clipart).

Pearson eText Activity Page: Ratio Cards

FORMATIVE ASSESSMENT Notes. An observation tool can capture whether students are using within ratios, between ratios, or both. Use a class list and check off which they are using. This same process can be used for the various ways to solve proportional situations (described later in this chapter). ■

Strategies for Comparing Ratios

Comparing ratios to determine whether they are equal prepares students for later solving missing-value proportion problems. Students use a variety of strategies for comparing ratios (Martinez & I, 2018), which are summarized in Table 2 for the question:

Is a paint mixture of 3 parts yellow to 2 parts red the same shade as a paint mixture of 5 parts yellow to 3 parts red?

Each of these methods needs to be taught and understood.

Creating paint shades, the focus of Activity 17.9, is an engaging way to have students compare ratios, where they can really *see* if they are equivalent (Beswick, 2011).

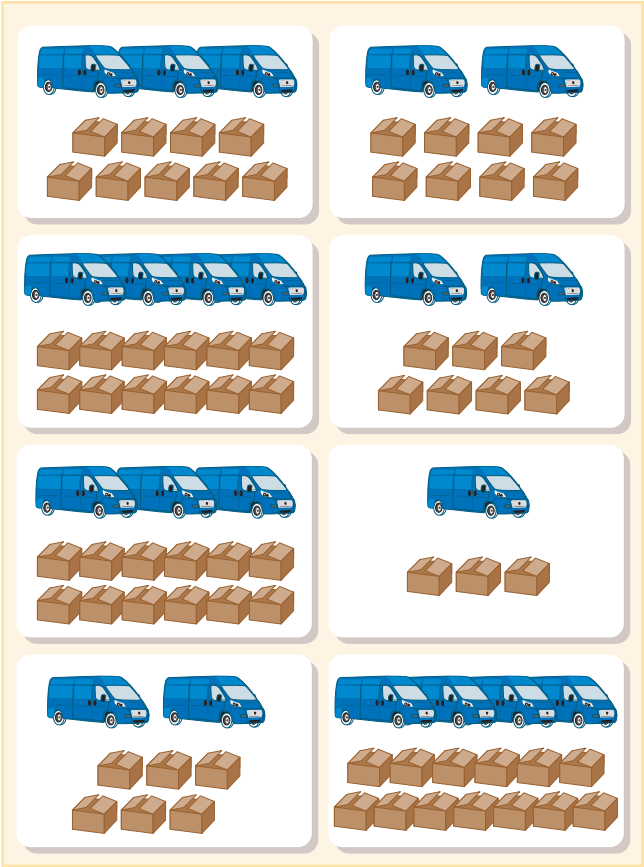
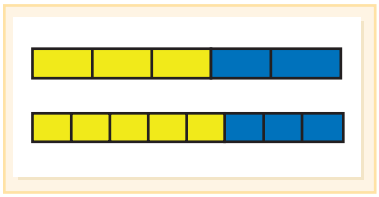


FIGURE 17.5 Ratio cards for exploring ratios and rates

TABLE 17.2 STRATEGIES STUDENTS USE TO COMPARE RATIOS (USING THE EXAMPLE, IS 3:2 EQUIVALENT TO 5:3?)

Strategy	Description	Example
Comparing the value of a ratio	Find the quotient of each ratio to see if they are the same.	$3 \div 2 = 1.5$ $5 \div 3 = 1.67$
Creating visual representations	Use bar diagrams or circles, shade and compare. Organized lists or comparison tables may also be used.	
Multiplying by one whole	Find a common denominator (or numerator) to compare.	$\frac{3}{2} = \frac{9}{6}$ and $\frac{5}{3} = \frac{10}{6}$
Using cross-multiplication	Compare the cross products.	$\frac{3}{2} = \frac{5}{3}$ and $9 \neq 10$

Selecting problems that can be represented and solved in many ways is important. Activity 17.10 is one such task, appearing for years in many textbooks and articles (e.g., Brakoniecki, Amador, & Glassmeyer, 2021).

Activity 17.9 Creating Paint Swatches

CCSS-M: 6.RP.A.1; 6.RP.A.3a, b; 7.RP.1; 7.RP.A.2a

Ask students to create five shades, using two primary colors (e.g., use red and yellow and make recipes for shades of orange). Ask students to order the shades (e.g., from the reddest orange to the yellowest orange), and try to include some that are equivalent ratios. Use real paint drops with a dropper and have students test their ratios and create their own paint chart. After each group has finished their collection of shades, they can look for matching shades on other posters. Additionally, in a debrief conversation, you can post two combinations and ask whether these two options make the same shade. Ask, "Which strategies did you use to compare?"

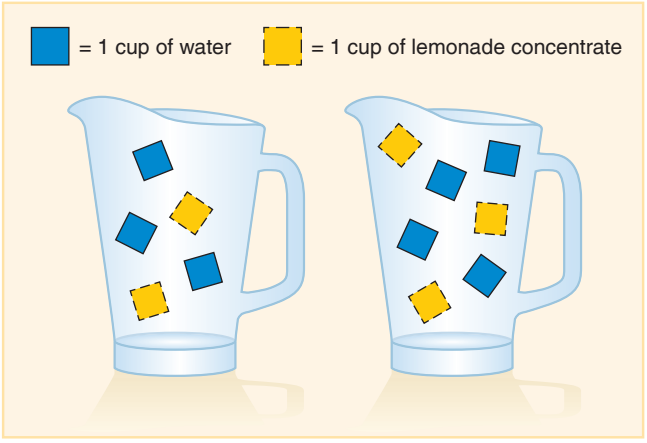


FIGURE 17.6 A comparing ratios problem: Which pitcher of lemonade will have the stronger lemon flavor, or will they be the same?

Activity 17.10 Comparing Lemonade Recipes

CCSS-M: 6.RP.A.1; 6.RP.A.3a, b; 7.RP.1; 7.RP.A.2a

Show students a picture of two lemonade pitchers, as in Figure 17.6. The little squares indicate the recipes used in each pitcher. In this case, the recipes are:

3 cups water	4 cups water
2 cups lemon concentrate	3 cups lemon concentrate

A yellow square represents a cup of lemonade concentrate and a blue square represents a cup of water. Ask whether the lemonade in one of the pitchers will have a stronger lemon flavor than other pitchers or taste the same.

To differentiate or scaffold this activity, alter the recipes. For example, recipes of 3 to 6 and 4 to 8 are easier to compare (and equal). Another option is 2 to 5 and 4 to 9 recipes, where it is easier to double the first and compare it to the second.

Pause & Reflect

Solve the lemonade problem and write down your reasoning. Is there more than one way to justify the answer? ●

Here are a few ways students can compare ratios in the Lemonade problem, connected to the strategies in Table 17.2:

1. Figure out how much water goes with each cup of concentrate, using division: $1\frac{1}{2}$ versus $1\frac{1}{3}$ [unit rate thinking].
2. Use part-part fractions (concentrate to water) and compare the fractions ($\frac{2}{3}$ versus $\frac{3}{4}$ or the reverse), using visual representations or equivalent fractions to determine which has more concentrate.
3. Use part-whole fractions (concentrate to whole mixture) ($\frac{2}{5}$ versus $\frac{3}{7}$) and use visual representations or equivalent fractions to determine which has more concentrate.
4. Use multiples of one or both of the pitchers until either the amounts of water or the amounts of lemonade concentrate are equal (common numerator or common denominator) [building-up thinking].

Recipes can use fractional values. For example, ask students to reason about which of these recipes has a stronger lemon flavor:

Recipe A:
$\frac{3}{4}$ cups water
$\frac{1}{8}$ cups of concentrate

Recipe B:
$1\frac{1}{2}$ cups water
$\frac{1}{4}$ cups of concentrate

Look at the between and within ratios for these recipes. What do you notice? The between ratio for both water and concentrate is $\times 2$ or $\div$ 2. In other words, recipe B is doubling recipe A (they will taste the same).

The next activity is not a mixture problem but instead compares food portions, and it also can be approached in a variety of ways.

Activity 17.11 Which Camp Gets More Pizza?

CCSS-M: 6.RP.A.1; 6.RP.A.3a, b; 7.RP.1; 7.RP.A.2a

Pose the following story (or adapt it to be about your own students). Before solving the problem, ask students to identify the camp they *think* gets more pizza. Record the information for later. Ask students to prepare a convincing argument for which camp gets more pizza.

Two camps are having pizza parties. The leader of the Bear Camp ordered enough so that every 3 campers will have 2 pizzas. The leader of the Raccoon Camp ordered enough so that there will be 3 pizzas for every 5 campers. Did the Bear campers or the Raccoon campers have more pizza?

After students have each found their own way to decide, pair them with another student to compare strategies and see if they agree on which campers get more pizza. Finally, ask students to reflect back on their original prediction and how it relates to what they figured out.

Figure 17.7 shows two reasoning strategies. Figure 17.7(a) shows an approach using division (pizzas per camper). But notice that this problem does not say that the camps have only 3 and 5 campers, respectively (this is simply the ratio). Any multiples of 2 to 3 and 3 to 5 can be used to make the appropriate comparisons, as illustrated in Figure 17.7(b). This is like finding common numerators. Because there are more campers in the Raccoon Camp ratio (larger denominator), there is less pizza for each camper.

This section has focused on comparing ratios when there are no missing values, with a focus on covariation. In the next section, we expand on these ideas as students begin to explore problems with missing values.

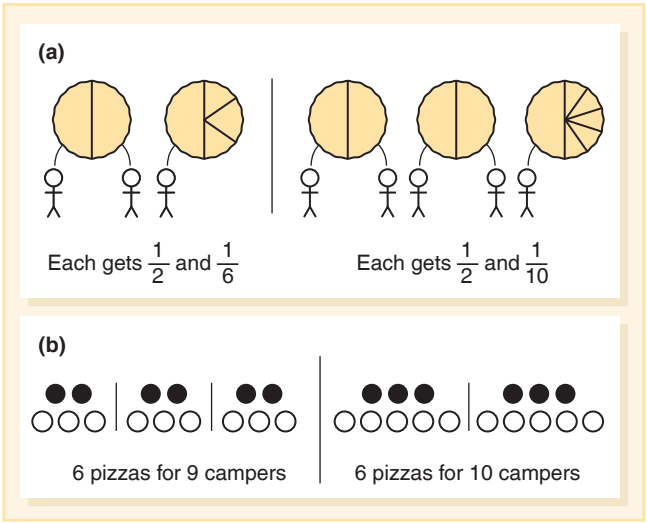


FIGURE 17.7 Methods for comparing pizzas per camper

***TECHNOLOGY* Note.** PhET Interactive Simulations (phet.colorado.edu) has an engaging Ratio and Proportion simulation that can be used on devices with or without touchscreens. Students predict different hand positions by plucking the strings on the screen where they will get success for each challenge ratio, and then test each position to see whether they are correct. Students come up with a rule for finding new successful hand positions. A second PhET simulation, called Proportion Playground, provides a rich environment for exploring ratios, unit rates, and proportions. ■

Proportional Reasoning Strategies and Representations

Learning Outcome 17.3 **Illustrate different ways to represent and reason about proportional problems across a developmental learning progression.**

This section addresses representations and methods for solving problems involving proportions, in particular missing-value proportion problems. Recall that procedural fluency includes the components of *efficiency*, *flexibility*, and *accuracy*. Fluency is the goal as we teach students to solve proportional situations: sometimes the numbers in the situation lend themselves to considering unit rates; at other times, a build-up method is efficient, and then sometimes we need to implement the cross-product algorithm. Representations support reasoning, linking concrete (situation) to semi-concrete (visual) to abstract (proportions). In brief, a proportional reasoning learning progression has these phases (Petit et al., 2020):

Early ratio strategies: Iterates (counts) composed units, using an additive strategy throughout the problem. May use visuals (e.g., tables) and build up or down, using addition.

Transitional strategies: Uses multiplicative relationships to reason. May use visuals (e.g., ratio tables and tape diagrams), combined with scaling up or down, using multiplication.

Proportional strategies: Uses a multiplicative relationship within the problem context and then elects and uses an efficient method to solve. Possibilities include using unit rates, comparing fractions, using a between or within ratio (scale factor), employing the cross-product method, using the equation $y = mx$ or related graph to solve.

The strategies mentioned here are illustrated, using the following tasks (and others).

1. **Keira sees an advertisement: Bowling, $35 for 4 people. What is the cost for 12 people?**
2. **Tamara sees an advertisement: 3 T-shirts for $24. At that rate, what will 10 T-shirts cost?**

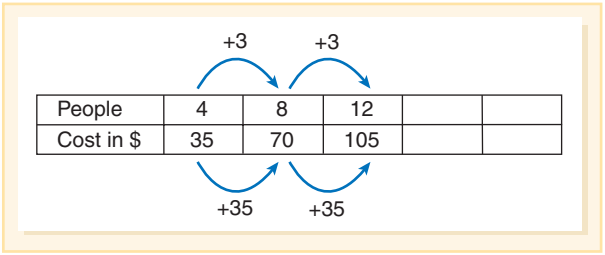
Pause & Reflect

Think of (at least) three ways students might figure out the costs in each situation. ●

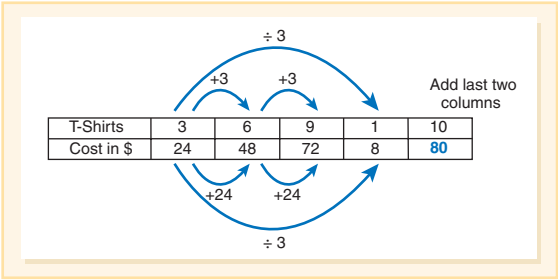
Early and Transitional Reasoning with Ratio Tables

Ratio tables provide an organized manner in which students can generate equivalent ratios, using build-up or break-down methods, until they find the ratio they need in order to answer the missing-value proportion problem.

Early Ratio Strategy. To solve the bowling situation, a student might reason: I can build up: $35 for 4 people, so $70 for 8 people, so $105 for 12 people. Build-up reasoning is supported by using a ratio table:

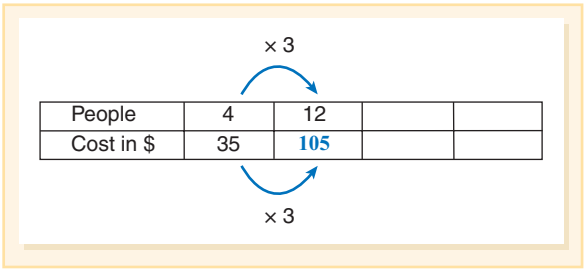


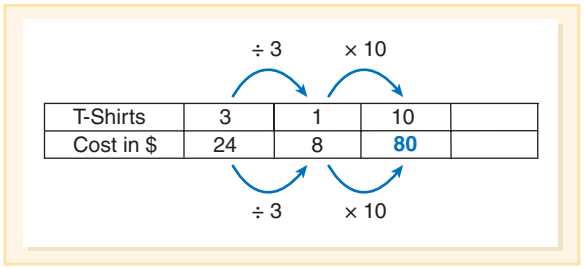
The T-shirt problem involves build-up and break-down reasoning, which may be done mentally or in a ratio table.



Ratio tables show how two quantities covary. In this case, the student uses the table to build up and break down, using addition primarily.

Transitional Strategy. At the transitional phase, the representations may be the same, but the reasoning shifts to using multiplicative relationships. For example, here is how the two problems might be solved.





A ratio table is an important representation in helping students reason proportionally, especially as they are transitioning from using addition to using multiplication to generate equivalent ratios. Activity 17.12 provides a set of proportional reasoning tasks for which a ratio table can be used as a tool to support student reasoning.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

Activity 17.12 Solving Problems Using Ratio Tables

CCSS-M: 6.RP.A.3a, b; 7.RP.A.1; 7.RP.A.2b, c; 7.RP.A.3

Prepare a set of cards, using the ones provided on the Proportional Problems Task Cards activity page. One example task:

> **A person who weighs 160 pounds on Earth will weigh 416 pounds on the planet Jupiter. How much will a person weigh on Jupiter who weighs 120 pounds on Earth?**

You can use these cards in a variety of ways. One option is to have students pick 3 to solve. Another option is to give each group one to solve and then present their work on a poster. Students can then analyze the reasoning of another group's finished problem to provide peer feedback. You can explain that they need to show their thinking by using a ratio table. Figure 17.8 shows three ways to use ratio tables to solve the Jupiter and Earth weight task.

Pearson eText Activity Page: **Proportional Problems Task Cards**

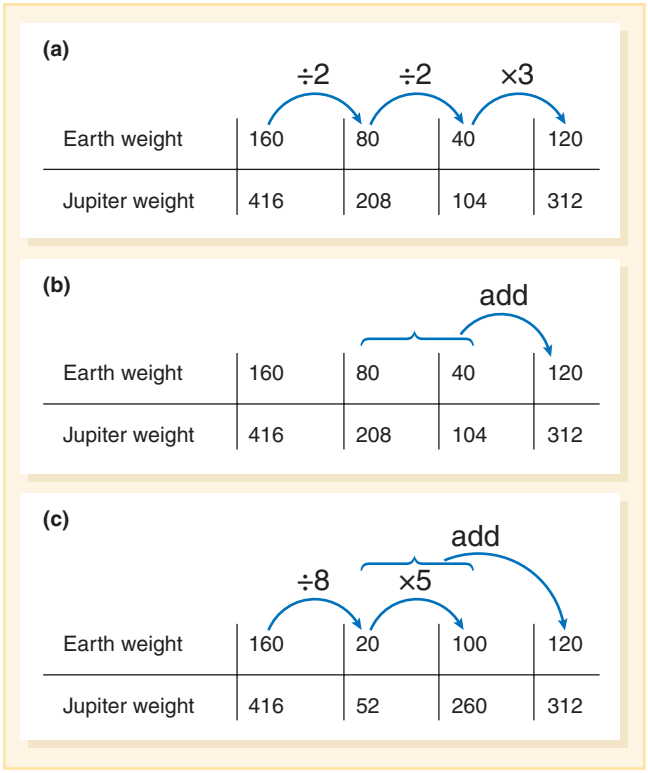


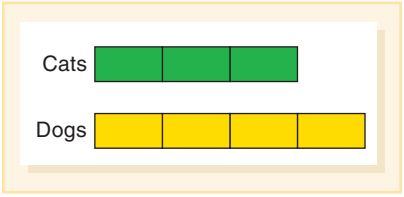
FIGURE 17.8 Three solutions to the Jupiter and Earth weight task, using ratio tables

Early and Transitional Reasoning with Tape Diagrams

Tape diagrams have numerous other names: bar model, strip diagrams, or length model. The term *tape* is used in many middle school standards documents. A tape diagram is a drawing that looks like segments of tape, used to illustrate number relationships (NGA Center & CCSSO, 2010). A tape diagram is an excellent model to support reasoning with part-part ratios. For example, consider this task:

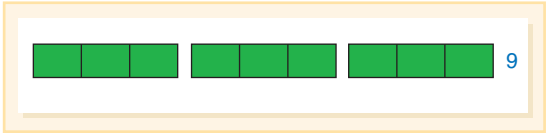
At the Pet Adoption Center, the ratio of available cats to dogs is 3 to 4. There are 24 dogs. How many cats does the Adoption Center have?

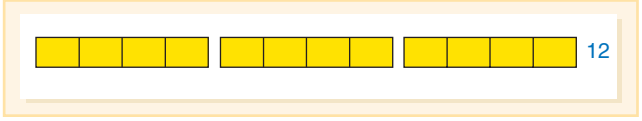
Early Ratio Strategy. Recall that early ratio strategies involve addition. For this problem, a student might build the tape, using same-sized Cuisenaire rods, linking cubes, or color tiles to create concrete strips. Or they may draw the tapes, in which case, offer grid paper or dot paper to help them create accurate representations.
To begin, students build the ratios of the two parts:



Pearson eText Blackline Masters: 1-Centimeter Grid Paper
Pearson eText Blackline Masters: 0.5-Centimeter Grid Paper
Pearson eText Blackline Masters: Dot Paper

Recall that early ratio reasoning uses addition. Thus, the tapes for both cats and dogs can be added on until there are 12 dogs:

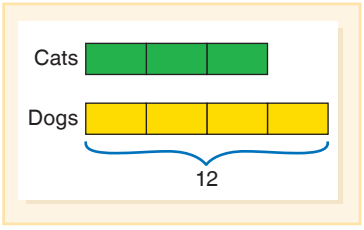




Like with ratio tables, perhaps even more concretely, you can see the covariation; as each new group of 4 dogs is added, a group of 3 cats is added.

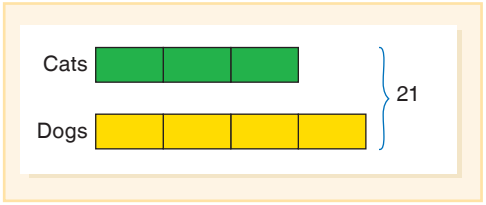
Transitional Strategy. **Tape** diagrams are particularly useful once students begin using multiplicative reasoning. In particular, they are versatile and can help students attend to parts and wholes as well as to what is missing in the situation. All problems begin representing the part-to-part ratio. Students use this sketch to fill in known information and solve for the unknown. Let's look at some examples with different unknown quantities.

1. If there are 12 dogs, how many cats? (One part is given, the other part requested)



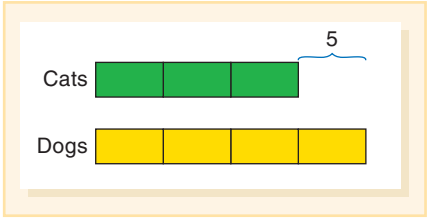
In observing the dogs tape, students see the relationship of 12 shared 4 ways, placing 3 in each section. Therefore, there will be 9 cats.

2. If there are 21 pets (cats and dogs), how many are cats? (The whole is given, a part is requested)



Story problems that involve parts and wholes, called proportion division problems, are very challenging for students (I, Martinez, & Dougherty, 2018; Cohen, 2013). In particular, it can be difficult for students to set up proportions to represent these situations, so tape diagrams are a good option.

3. There are 5 more dogs than cats. How many dogs are there? (The difference between the parts is given, a part is unknown)



We can see that the five represents one-fourth of the dogs, so there are 20 dogs. Notice how the diagrams make each of these situations easier to compare parts to parts and parts to the whole.

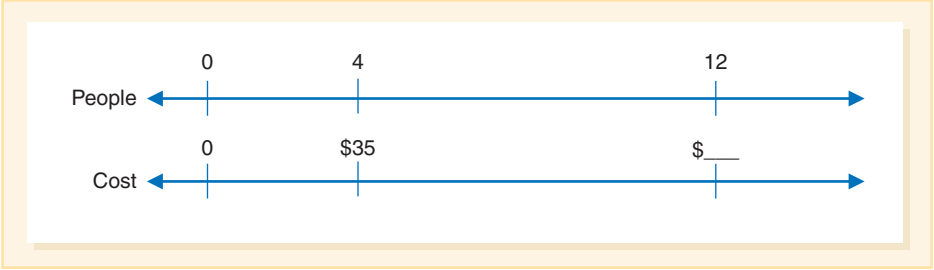
Standards for Mathematical Practice
MP4. Model with mathematics

***TECHNOLOGY* Note.** Math Playground Thinking Blocks Ratios allows you to select which style of story situations to explore and uses bar models (tape diagrams) to illustrate. ■

Transitional Reasoning with Double Number Lines

Double number lines are similar to tape diagrams in that there is one length for each quantity (part), but double number lines are designed to be the same length and use different units on each length. This allows students to see matching ratios. Labeling the two lines becomes critical to being able to compare parts to parts. Let's revisit the bowling problem:

Keira sees an advertisement: Bowling, $35 for 4 people. What is the cost for 12 people?



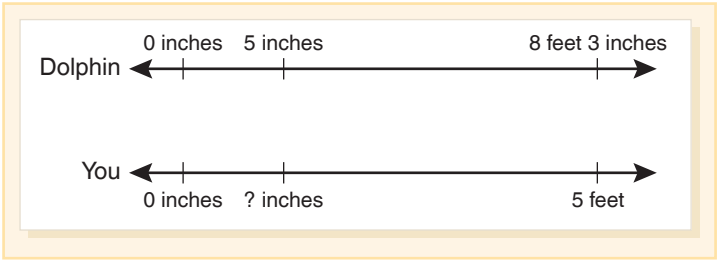
Using transitional reasoning, a student might solve this problem by thinking, "12 is three times 4, so I multiply 35 by three to determine the cost." Or, more partitions can be inserted on the number line (e.g., 8 people and $70), just as combinations can be inserted in a ratio table. Activity 17.13 involves proportional reasoning, using double number lines (though the same task can be applied to the other representations, and/or students can select the representation).

Activity 17.13 You and the Zoo

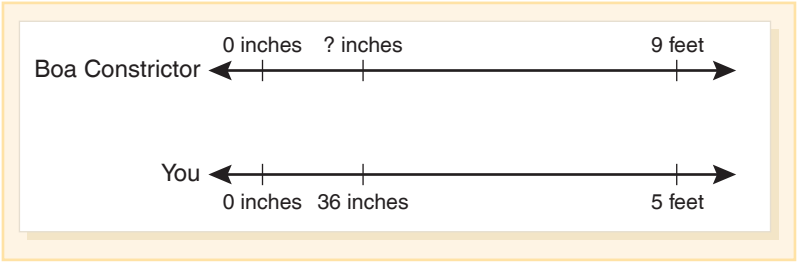
CCSS-M: 6.RP.A.3a, d; 7.RP.A.2b; 7.RP.A.3

Bring in plastic animals that are several inches long (approximate) and are in scale with each other. This activity works well with play dough but can also be done with construction paper cutouts. Explain to students that they will use double number lines to prepare a zoo scene—starting with preparing a replica of themselves to use as a scale!

Each pair or group of students is given a plastic animal and asked to make a representation of themselves that is in proportion to the animals. The first task is for them to determine the actual height of the animal (they may look it up online, or you may prepare this data in advance and post it) as well as their own actual height. Here is an example:



Once the You number line is known, students can identify other animals they would like to create for their zoo. Challenge students to select two favorite animals, or assign animals that vary in size and might not be too hard to sketch or mold. Examples include: bald eagles, tree frogs, boa constrictors, pandas, tigers, and lemurs.



Note: Consider which measurements your students need to practice; this activity can be done with metric or U.S. units.

The remainder of this section focuses on the strategies students use to reason proportionally in a developmental sequence, based on the learning progression described earlier. The goal for fluency is that students know each of these methods and are given opportunities to choose one that makes sense to them. In other words, it is important to teach several methods and avoid requiring students to use a particular method.

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Proportional Strategies

The remainder of this section is focused on the ways a student might reason to solve missing-value proportion problems. For any of these reasoning methods, they might be mental, and they might employ one of the representations (ratio tables, tape diagrams, or double number lines).

Using Unit Rates. The unit-rate, or "How many for one?" is the most widely used by students who have not yet learned cross products (Post, Behr, & Lesh, 1988). Back to these two tasks:

1. **Keira sees an advertisement: Bowling, $35 for 4 people. What is the cost for 12 people?**
2. **Tamara sees an advertisement: 3 T-shirts for $24. At that rate, what will 10 T-shirts cost?**

In both cases, the answer can be found by using division to find the cost for 1 and then multiplying to determine the cost for 10 or 12, respectively. This is certainly an efficient option for the T-shirt problem, because finding the unit rate can be done mentally, as can multiplying 8 times 10.

Scale Factor. This reasoning, also called factor-of-change thinking, involves noticing a *within ratio* or a *between ratio* to solve the problem (see Figure 17.4). Let's look at the ratios for the bowling and T-shirt problems:

Bowling Problem	T-shirt Problem
$\frac{35}{4} = \frac{n}{12}$	$\frac{24}{3} = \frac{n}{10}$

In the bowling problem, the between ratio (4 to 12) is a known multiplicative relationship of "times 3." That is, the factor of change and 35 must also be multiplied by 3 to have an equivalent ratio. In the T-shirt problem, this between ratio is not helpful, but the within ratio (24 to 3) has a known relationship of "times 8" and, thus, the 10 must also be multiplied by 8.

Notice that using a scale factor involves looking at the numbers in the problem to determine whether the within or between ratio is helpful, or whether a unit rate approach might be more efficient. This is procedural fluency in action! To support the development of such fluency, provide students with tasks that lend to these different methods, like the examples in Activity 17.14.

Activity 17.14 It's a Matter of Rates

CCSS-M: 6.RP.A.3a, b; 7.RP.1; 7.RP.A.2b

Pose a set of problems to students, such as those on the activity page. Two are shared here:

At the office superstore, you can buy 4 pencils for $0.59, or you can buy the same pencils in a large box of 5 dozen for $7.79. How much will you save per pencil if you buy the large box?

The price of a box of 2 dozen gumballs is $4.80. Bridget wants to buy 5 gumballs. What will she have to pay?

Ask students to analyze the values in the problem and look for multiplicative relationships that will help them determine the answer. See the Expanded Lesson: It's a Matter of Rates and the It's a Matter of Rates activity page for a full lesson.

To challenge students, pose tasks with problems that have more difficult numbers, asking students to apply the same strategies to reason to an answer. For example:

Brian can run 5 km in 17.4 minutes. If he keeps running at the same speed, how far can he run in 23 minutes?

Pearson eText Expanded Lesson: It's a Matter of Rates

Pearson eText Activity Page: It's a Matter of Rates

Using $y = k\,x$. An equation is yet another representation, as is its related graph, but it is also a way to reason about missing value situations. For the T-shirt problem, for example, the student may have used the unit rate and recognizes that the cost of the T-shirts is $y = 8x$. Then substitute the 10 in the equation (or any other value). This representation has the advantage of being able to find not just one equivalent ratio, but many of them. Graphs of these equations go through the origin and show the relationship between the two quantities. Activity 17.15 offers a task that uses tables, graphs, and equations to reason.

Activity 17.15 Factory Ratios

CCSS-M: 7.RP.A.2a, b, c, d

Use the Factory Ratios activity page, which begins with a ratio table displaying times and number of doohickeys produced. Because different ratios are represented in the ratio table, those ratios can be compared or a missing value found. Then students graph the ratios and write equations. After students have completed the representations, ask them to connect the context to each representation, asking questions such as, "How can you use the graph to find . . .?" and "In this context, what does the k in the equation $y = kx$ mean?"

Pearson eText Activity Page: Factory Ratios

The next activity is an engaging project in which students use tables, graphs, and equations to explore reason proportionally about how far a toy will travel over time.

Activity 17.16 Rates of Wind-Up Toys

CCSS-M: 6.RP.A.3a, 7.RP.A.1, 7.RP.A.2b, d

This activity requires bringing in (or having families donate) small wind-up toys (that travel at a constant rate). Distribute one such toy and yardstick/meter stick to each group of three students. Explain that students are going to record time and related distance (e.g., inches or centimeters) on a table, noting how far the toy travels in 3 seconds, 6 seconds, 9 seconds, and so on. Have students create a related coordinate graph (x = time, y = distance). Once students have gathered data and created a graph, ask, "What is your toy's unit rate? How can you use the rate to determine how far it will travel in 25 seconds, 50 seconds, x seconds (if it kept going at the same rate)?" Students can prepare a poster with a picture of their toy, their data in a table, their graph, the unit rate, and the equation. As students explore each group's findings, ask students what connections they see across representations on each poster. For example, they might note that the unit rate is the k in $y = kx$. See Swanson (2015) for more ideas on this activity

Pearson eText Blackline Master: Coordinate Grid—Quadrant 1

Certainly, this reasoning process is an overlap with algebraic reasoning. In fact, proportional situations such as the ones in this section are a good beginning to exploring linear equations, because they are all linear situations that form lines going through the origin.

Cross Products. Setting up a proportion (an equation with one fraction equal to another fraction) is another way to solve for a missing value in a proportional situation. This strategy is described last in this chapter—developmentally, it should be the last one taught, because it is the least intuitive. Students are often introduced to this strategy soon after the other strategies or, worse, they only learn this method. When using cross products, students should understand why the algorithm works (Martinez & I, 2018).

To build understanding of cross products, connect to students' understanding of equivalent ratios. For example, students can illustrate the problem to show how it covaries. Figure 17.9(a) shows a simple sketch, or they can use a double number line or other representation of the situation as in Figure 17.9(b). Notice that in both examples in the figure, the proportions are set up as within ratios and between ratios.

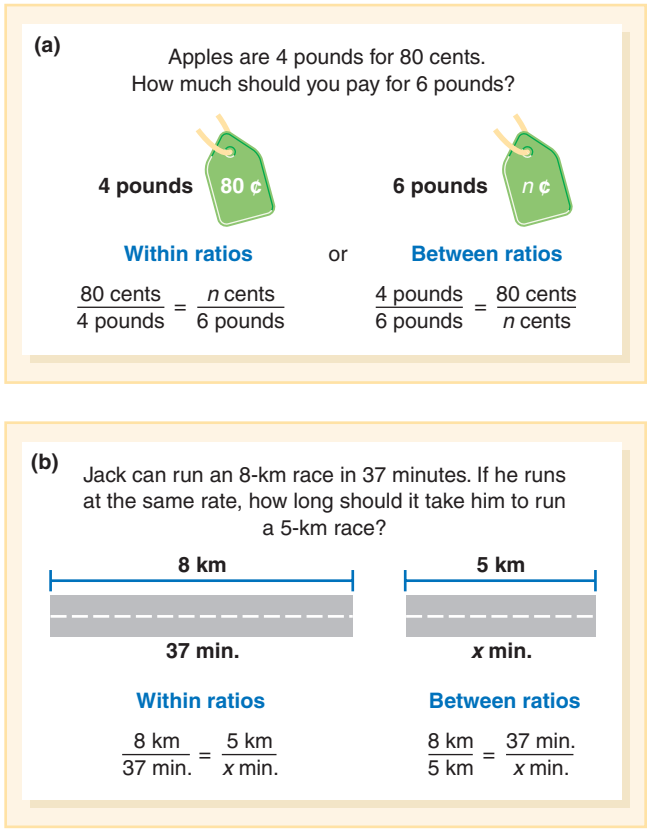

(a)
Apples are 4 pounds for 80 cents.
How much should you pay for 6 pounds?
4 pounds
80 ¢
6 pounds
n ¢
Within ratios
or
Between ratios
80 cents / 4 pounds = n cents / 6 pounds
4 pounds / 6 pounds = 80 cents / n cents
(b)
Jack can run an 8-km race in 37 minutes. If he runs at the same rate, how long should it take him to run a 5-km race?
8 km
5 km
37 min.
x min.
Within ratios
Between ratios
8 km / 37 min. = 5 km / x min.
8 km / 5 km = 37 min. / x min.

FIGURE 17.9 Drawings can help in setting up proportion equations

Once the proportions are created, students still have options for how to determine the missing value—they don't have to use a cross product. The problem in Figure 17.9(a) can be solved with *unit rate*, or you can use within ratio and notice the "times 20" *scale factor*. But such relationships are not present in the problem in 17.9(b). Unit rates are a good option, or we can use equivalent fractions or cross products. Here is how equivalent fractions become cross products:

$$\frac{8}{5} = \frac{37}{x}$$

The common denominator is the product of the two denominators, $5x$. Multiplication is used to create equivalent fractions that now have a common denominator:

$$\frac{8x}{5x} = \frac{37 \times 5}{5x}$$

With the denominators equal, we know the numerators are also equal: $8x = 185$. Therefore $x = 23.125$. As students explore this process across examples, and you ask them to compare these before and after the equations, as illustrated here, they notice the cross-product phenomenon, which leads to the shortcut notation of not recording the denominator.

FORMATIVE ASSESSMENT Notes. Because there are many ways to reason about proportional situations, it is important to capture *how* students are reasoning. Writing is an effective way to do this. You can simply ask students to explain how they solved a problem, or you can provide more structure by using specific writing prompts or sentence starters or asking students to describe two ways they could arrive at the solution. ■

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

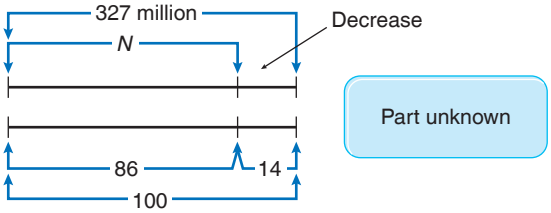
Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

Pearson eText
Video Example 17.4
In this video consider how the class discusses proportions and finding cross products.

Percent Problems

Percent problems can be solved using various representations (e.g., ratio table, double number line) or strategies (e.g., use scale factors). The double number line is particularly useful in helping students figure out which part is unknown. The units of one number line correspond to the numbers or measures in the problem. On the second number line, the units indicate the corresponding values in terms of percents (with a whole of 100) (see Figure 17.10).

In 1960, U.S. railroads carried 327 million passengers. Over the next 20 years, there was a 14 percent decrease in passengers. How many passengers rode the railroads in 1980?

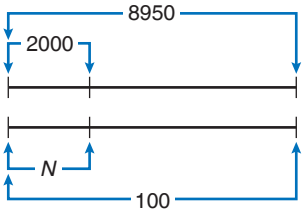


Part unknown

$$\frac{N}{327 \text{ million}} = \frac{86}{100} = 0.86$$

$N = 0.86 \times 327$ million → about 281 million

Sylvia's new car cost $8950. She made a down payment of $2000. What percent of the sales price was Sylvia's down payment?



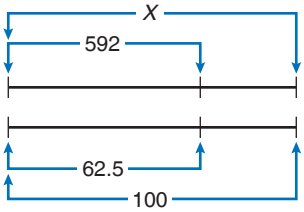
Percent (fraction) unknown

Part/Whole

$$\frac{\$2000}{\$8950} = \frac{N}{100}$$

$8950N = 200{,}000$ → $N = 22.35$, or about 22%

The seventh- and eighth-grade classes at Robious Middle School had a contest to see which class would sell more raffle tickets at the school festival. The eighth grade sold 592 tickets. However, this turned out to be only 62.5 percent of the number of tickets sold by the seventh grade. How many tickets did the seventh grade sell?



Whole unknown

Comparison

$$\frac{592 \text{ tickets}}{X \text{ tickets}} = \frac{62.5}{100}$$

$62.5X = 59{,}200$ → $X = 947.2$, or 947 tickets

FIGURE 17.10 Percentage problems represented as double number lines and equations

Activity 17.17 Making Sense of Percents

CCSS-M: 6.RP.3c; 7.RP.A.2c; 7.RP.A.3

Use the Making Sense of Percents activity page stories, as illustrated in Figure 17.10. You can give all three problems to every student or have expert groups working on each task and then have students trade tasks and critique each other's solutions. Afterward, discuss connections among the story, the number lines, and the solutions.

Pearson eText Activity Page: Making Sense of Percents

The double number line is useful across different types of percent problems. It allows modeling of not only part-to-whole scenarios but also percent increase–decrease situations. The number lines also work effectively for percents greater than 100 (Parker, 2004).

Proportional Reasoning across the Curriculum

Learning Outcome 17.4 Connect proportional reasoning to concepts in geometry and algebra.

Beyond finding percents and other concepts in the number strand, proportional reasoning is used in geometry and algebra. Measurement conversions, similarity, and linear situations all involve covariation. Each idea is briefly discussed here.

Measurement and Geometry

Within and between ratios apply to *measurement conversions*. How might you use within and between ratios to determine the number of feet in 60 inches?

$$\frac{12 \text{ inches}}{1 \text{ foot}} = \frac{60 \text{ inches}}{? \text{ feet}}$$

Setting up a proportion and analyzing the between and within relationships can help students see the options they have for converting between measurement units.

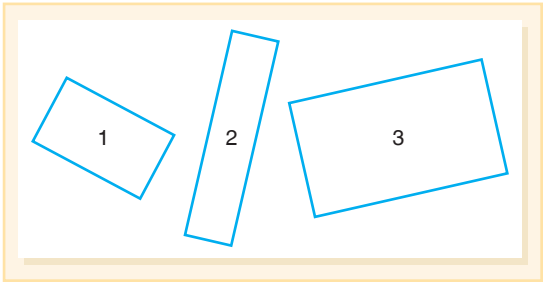
Similarity is another topic that involves covariation. Students often struggle to determine which features to compare to which features. A within- and between-ratio equation, like the one in Figure 17.4, can be used with a visual exploration, such as in Activity 17.18, to help students make sense of similarity.

Activity 17.18 Look-Alike Rectangles

CCSS-M: 6.RP.A.3a; 6.RP.A.1; 7.RP.A.2a; 7.G.A.1; 8.G.A.4

Provide students with cutouts of the rectangles provided in the Look-Alike Rectangles activity page (or have students cut out a set for themselves). Ask students to group the rectangles into three sets that look alike. If your students know the word *similar*, use that term instead of "look alike." Draw three rectangles on the board with two that are similar and one that is clearly dissimilar to clarify the goal of the task:

In small groups, have students sort their rectangles into similar groupings. Afterward, discuss the groups and reasons they classified the rectangles as they did. Next, have students measure and record the sides of each rectangle to the nearest half centimeter on the Look-Alike Rectangles recording sheet. Ask students to explain how the ratios and groupings are related. (If the groups are formed of similar [proportional] rectangles, the ratios within each group will be equivalent.) To provide increased structure for students with special needs in math, you can give them one rectangle from each of the three groups and ask them to find the other two.



Answers: Rectangles A, D, and I have sides in the ratio of 3 to 4.
Rectangles C, F, and H have sides in the ratio of 5 to 8.
Rectangles E, G, and J have sides in the ratio of 1 to 3.
Rectangle B is a square (a special case of a rectangle); its sides are in the ratio of 1 to 1.

Pearson eText Activity Page: Look-Alike Rectangles

Pearson eText Activity Page: Look-Alike Rectangles Recording Sheet

Creating *scale drawings* is an application of similarity. Multiplication *is* scaling (making something three times bigger or one-half the size, for example). Scale drawings, then, are an important way for students to see the connection between multiplicative reasoning and proportional reasoning. Scaling activities ask students to resize a sketch that is similar to one they are given.

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

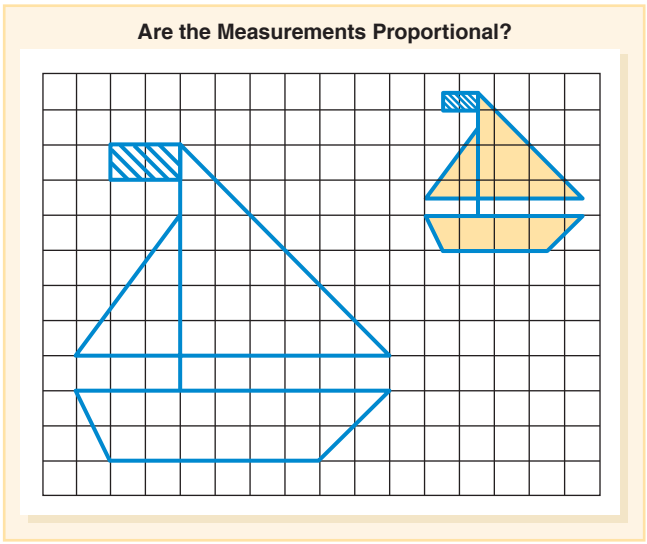


FIGURE 17.11 Scale drawing of a sailboat sketch

Pearson eText
Video Example 17.5
In this video consider how the class engages in creating scale drawings.

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

Activity 17.19 Sketch and Grow

CCSS-M: 6.RP.A.1; 7.RP.A.3a

On grid paper or dot paper, have students draw a relatively simple sketch with endpoints on intersections on the grid paper (or on dots on dot paper) (see Figure 17.11). After students have created their original sketch, collect them and redistribute the sketch to other students. Ask students to double the lengths in the sketch and re-draw it. Once doubled, the two diagrams are returned to the original artist. The artist checks to see that their scaled creature is similar to their original drawing. Have students investigate between and within ratios of the two sketches. With multilingual students, be sure the term *scale* is understood, so that they don't confuse this use of the word with a machine that weighs things or what fish have. To challenge students, the sketches can be scaled to other values. Figure 17.11 shows a second sketch, scaled at $\times$ 0.5.

Pearson eText Blackline Masters: 1-Centimeter Grid Paper

Pearson eText Blackline Masters: Dot Paper

Comparisons of corresponding lengths, areas, and volumes in proportional figures lead to some interesting patterns. If two figures are proportional (similar), then any corresponding linear dimensions will have the same scale factor, but their areas and volumes do not. Imagine you have a square that is 3 by 3 and you create a new square that is 6 by 6. The ratio between the side lengths is 1:2. The ratio between the two areas is 1:4. Why? Try the same idea with the volume of a cube—what is the relationship of the original volume to the new volume when you double the length of the edges? Why?

Scale drawing can and must be connected to real-life contexts. A sampling of ideas is offered here.

Activity 17.20 Interesting Scaling Investigations

CCSS-M: 7.RP.A.2b; 7.RP.A.2c; 7.G.A.1

Choose one of the following authentic contexts; have students find the related data and answer the prompts. As appropriate, have students explain the relationship as equations and as graphs. Interesting investigations involving scale drawings:

- If you want to make a scale model of the solar system and use a ping-pong ball for the earth, how far away should the sun be? How large a ball would you need to represent the sun?
- What scale should be used to draw a scale map of your city (or some region of interest) so that it will fit nicely on a standard piece of poster board?
- Use the scale on a map to estimate the distance and travel time between two points of interest.
- Roll a toy car down a ramp, timing the trip with a stopwatch. How fast was the car traveling in miles per hour? If the speed is proportional to the size of the car, how fast would this have been for a real car?
- Your little sister wants a table and chair for her doll. Her doll is 14 inches tall. How tall should you make the table? The chair?
- Determine the various distances that a 10-speed bike travels in one turn of the pedals. You will need to count the sprocket teeth on the front and back gears. Explore for different gears.

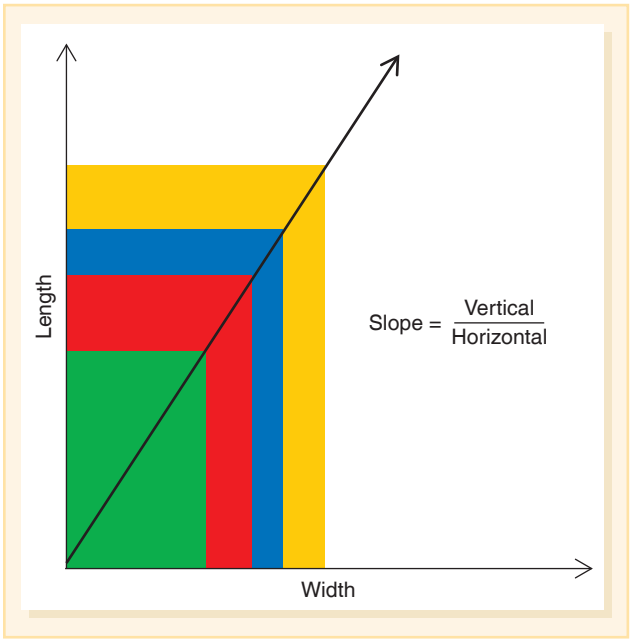


FIGURE 17.12 The slope of a line through a stack of proportional rectangles is equal to the ratio of the two sides

***TECHNOLOGY* Note.** Google Earth (available in online and app formats) is a great resource for doing authentic scaling activities (Roberge & Cooper, 2010). Find a Google Earth diagram that includes something for which the measure is known and ask students to figure out other measures. For example, you know that a standard football field is 100 yards from end zone to end zone (120 yards including end zones). ■

Algebra

Ratio tables, graphs, and equations for proportions are clear connections to algebra. For the look-alike rectangles, ask students to stack them with the largest one on the bottom so that they are aligned at one corner, as illustrated in Figure 17.12. Place a straight edge across the diagonals, and you will see that opposite corners also line up. If the rectangles are placed on a coordinate axis with the common corner at the origin, the slope of the line joining the corners is the ratio of the sides. A great way to connect proportions, geometry, and algebra!

Proportional situations are linear. Graphing equivalent ratios is a powerful way to illustrate this concept. The ratio or rate is the slope of the graph, and the line goes through the origin, so it is represented symbolically as $y = mx$ or sometimes as $y = kx$ where k takes on the specialized meaning of the scale factor.

Activity 17.21 Rectangle Ratios: Graph It!

CCSS-M: 7.RP.A.2a, b, c, d

This activity is connected to Activity 17.18. Have students select one set of similar rectangles, record the measures in the ratio table, and then create three of their own examples of look-alike rectangles, using their knowledge of equivalent ratios. Also see whether students can find a rectangle that has a non-integer side length (e.g., $4\frac{1}{2}$ cm). Next, ask students to graph the data for their six rectangles. Figure 17.13 shows a graph for one of the look-alike groups. After the six rectangles are graphed, challenge students to use their graph to determine a seventh look-alike rectangle. The following questions help students understand the graph:

- What does a point on the line mean (even if it is not one of the rectangles)?
- What is the length for a look-alike rectangle with a width of 1 cm?
- What is the width for a look-alike rectangle with a length of 1 cm?
- If you know the lengths of the short side, how can you find the length of the long side (and vice versa)?

Pearson eText Activity Page: Look-Alike Rectangles: Graph It!

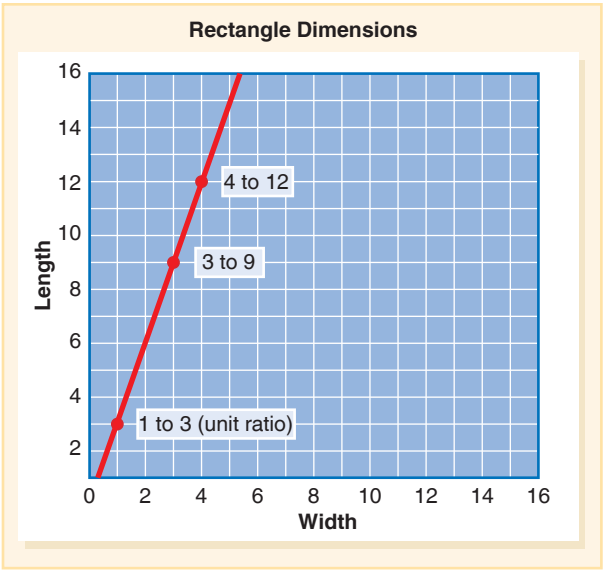


FIGURE 17.13 Graph shows ratios of sides in similar rectangles.

In addition to using geometric illustrations, it is important to use relevant contexts in which students analyze data and represent them in various ways (tables, graphs, equations). For example, a situation like dripping faucets can help students make sense of unit rates and proportional reasoning (Williams, Forrest, & Schnabel, 2006).

Standards for Mathematical Practice
MP4. Model with mathematics

Activity 17.22 Dripping Faucet Problem

CCSS-M: 6.RP.A.1; 6.RP.A.3a, b; 7.RP.A.2b, c

Pose the following problem to students:

> **If you brush your teeth twice a day and leave the water running when you brush, how many gallons of water will you waste in one day? In two days? In a week? A month? Any number of days?**

If possible, have students gather real data, or use the recommended time to brush teeth (e.g., 2 minutes). Ask students to explore the ratio of number of days to water wasted and to illustrate their data using tables, graphs, and equations. Challenge students to consider other such environmental issues, determining two variables to consider, determining what ratio might exist between them, and representing that ratio in multiple ways.

***TECHNOLOGY* Note.** NCTM's Figure This! (figurethis.nctm.org) has dozens of fun problems that are family-friendly and can be assigned to students. In the Math Index, you'll find ratios and proportion activities across the Geometry, Measurement, and Number sections. These worthwhile tasks can provide year-long challenges for your students across the middle school curriculum! ■

Addressing Challenges Related to Ratios and Proportions

Hopefully, one takeaway you have from this chapter is that proportional reasoning (and multiplicative reasoning) is one of the biggest ideas in mathematics; it must be well understood for a myriad of real-life situations as well as a foundation for further mathematics.

Tips for Supporting Ratio and Proportional Reasoning

Considerable research has been conducted to determine how students reason in proportionality tasks (for example, see Lamon, 2020; Lobato et al., 2010; Petit et al., 2020; Riehl & Steinthorsdottir, 2019; Siegler et al., 2010). The findings related to effective teaching to support students' ability to reason about ratios and proportions are shared here as a way to summarize the chapter.

1. Use composed unit and multiplicative comparison ideas in building understanding of ratio. Learning more about multiplicative comparisons leads to an understanding of rate, a strategy to be applied to proportions.
2. Help students distinguish constant, additive, and multiplicative comparisons by providing examples of each and discussing the differences.
3. Provide ratio and proportion tasks in a wide range of contexts, including situations involving measurements, prices, geometric and other visual contexts, and rates of all sorts.
4. Engage students in a variety of strategies for solving proportions, sequencing the instruction from concrete to more abstract. In particular, use ratio tables, visuals (e.g., bar diagrams and double number lines), and equations to solve problems—always expecting students to apply reasoning strategies and understand the strategy they are using.
5. Recognize that the cross-product algorithm, although important to understand and use, is not the goal for teaching proportions—the goal is for students to recognize proportional situations and solve them efficiently (i.e., using any of the strategies described in this chapter).

Addressing Challenges Related to Ratio and Proportional Reasoning

Table 17.3 describes the most common challenges students encounter and offers ways to help students make sense of ratios and proportions.

TABLE 17.3 COMMON CHALLENGES RELATED TO RATIOS AND PROPORTIONS

Common Challenges	What It Looks Like	How to Help
1. Use additive reasoning when the situation requires multiplicative reasoning (Canada et al., 2008; Dougherty et al., 2016)	Student thinks that because a giant's pencil is 15 inches longer than a normal pencil, the giant is 15 inches taller than a normal person (see Activity 17.4).	• See the discussion and tasks in the "Types of Comparing Situations" section. • Use Activities 17.5–17.7.
2. Use phrases that imply confusion between additive and multiplicative reasoning (Rathouz, Cengiz, Krebs, & Rubenstein, 2014)	Student says "is four times *more than*" rather than "is four times *as long as*" for a multiplicative comparison.	• Compare the two statements and their meanings. • Use precise language. • Use revoicing to help students use the correct language.
3. Experience difficulty determining how to graph ratios (Kastberg, D'Ambrosio, Lynch-Davis, Mintos, & Krawczyk, 2014)	Students cannot decide which axes to use for the two measures; students have difficulty interpreting a completed graph.	• Discuss that with ratios, the choice of which variable goes on the *x*-axis and which variable goes on the *y*-axis is arbitrary. • Compare graphs that are done both ways. • Point at values and ask what they mean related to the context.
4. Understand the whole in a ratio. (I, Martinez, & Dougherty, 2018)	In a part-part ratio, such as 5 children to 2 adults ($\frac{5}{2}$), student cannot identify the total number of people.	• Explore more tasks of this type (part-part-whole) [see series of Pet Adoption Center tasks]. • Use tape diagrams (bar models) and other visuals to help students visualize the parts and the whole.

RESOURCES FOR CHAPTER 17

WRITING TO LEARN

1. Describe the idea of a ratio in your own words. Explain how your idea fits with each of the following statements:
a. A fraction is a ratio.
b. Ratios can compare things that are not at all alike.
c. Ratios can compare two parts of the same whole.
d. Rates such as prices or speeds are ratios.
5. Describe a situation in which the comparison involved could be interpreted both additively and multiplicatively. Why is it important to include both types of reasoning when introducing ratios?
6. Use a representation and solve this problem in some way other than cross-products: *The ratio of youths to adults at the YMCA is 7 to 8. If there are 120 members, how many are adults?*
7. Give a geometric and algebraic example of covariation.

LITERATURE CONNECTIONS

Literature brings an exciting dimension to the exploration of proportional reasoning. Many books and stories discuss comparative sizes, concepts of scale as in maps, fantasy characters (giants and miniature), comparative rates (e.g., speed), and so on (see ideas in Table 17.1 (p. 457)).

RECOMMENDED READINGS

Articles

Ercole, L. K., Frantz, M., & Ashline, G. (2011). Multiple ways to solve proportions. *Mathematics Teaching in the Middle School*, *16*(8), 482–490.

This article shares the many ways of reasoning to solve proportions: unit rate, factor of change, building up, ratio tables, and cross-multiplication. Student work is shared throughout.

Rathouz, M., Cengiz, N., Krebs, A., & Rubenstein, R. N. (2014). Tasks to develop language for ratio relationships. *Mathematics Teaching in the Middle School*, *20*(1), 38–44.

These authors provide excellent suggestions for helping students compare additive and multiplicative reasoning while also sharing a well-designed progression of high-quality tasks.

Martinez, R., & I, J. Y. (2018). Four ways to determine equivalent ratios. *Mathematics Teaching in the Middle School*, *24*(1), 48–52.

Student thinking is shared for solving equivalent ratio tasks, along with valuable tips for teaching ratios—very useful!

Books

Lobato, J., Ellis, A. B., Charles, R. I., & Zbiek, R. M. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning: Grades 6–8*. Reston, VA: NCTM.

Olson, T. A., Olson, M., & Slovin, H. (2015). *Putting essential understanding of ratios and proportions into practice in grades 6–8*. Reston, VA: NCTM.

This book series provides a deep understanding of ratios and proportions and a wealth of tasks, instruction, and assessment strategies that will support students developing deep understanding of this topic.

Petit, M. M., Laird, R. E., Wyneken, M. F., Huntoon, F. R., Abele-Austin, M. D., & Sequeira, J. D. (2020). *A focus on ratios and proportions: Bringing mathematics education research to the classroom*. New York, NY: Routledge.

This book provides a deep understanding of teaching ratios and proportions in a developmental manner. Many quality tasks and student work examples are woven throughout.

CHAPTER

18 Developing Measurement Concepts

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **18.1** Describe the meaning and process of measuring.
- **18.2** Explain the role of estimating before measuring.
- **18.3** Demonstrate how to measure the length of objects.
- **18.4** Explain the development of area formulas.
- **18.5** Explain how volume is measured.
- **18.6** Demonstrate strategies for comparing weights of objects.
- **18.7** Explain strategies for teaching angle measurement.
- **18.8** Describe a representation for teaching elapsed time.
- **18.9** Explain strategies for counting a collection of coins.

Measurement is the process of describing a continuous quantity with a numerical value. It is one of the most useful mathematics content strands, as it is a component in everything from occupational tasks to life skills. From gigabytes that measure amounts of information to font size on computers, to miles per gallon, to recipes for a meal, people are surrounded with measurement concepts that apply to many real-world contexts. However, measurement isn't an easy topic for students to understand. Data from international studies consistently indicate that U.S. students are weaker in measurement than in any other topic in the mathematics curriculum (Mullis et al., 2004; Provasnik et al., 2009; Thompson & Preston, 2004).

BIG IDEAS

- Measurement involves comparing an attribute of an item or situation with a unit that has the same attribute. Lengths are compared to units of length, areas to units of area, time to units of time, and so on.
- Estimation of measures and the development of benchmarks or referents for frequently used units of measure help students increase their familiarity with units, preventing errors and aiding in the meaningful use of measurement.

- Measurement tools (e.g., rulers, protractors) group multiple units so that a single unit does not have to be iterated multiple times.
- Area and volume formulas provide a method of measuring these attributes by using only measures of length.
- Area, perimeter, and volume are related. For example, as the shapes of regions or three-dimensional objects change while maintaining the same areas or volumes, there is an effect on the perimeters or surface areas.

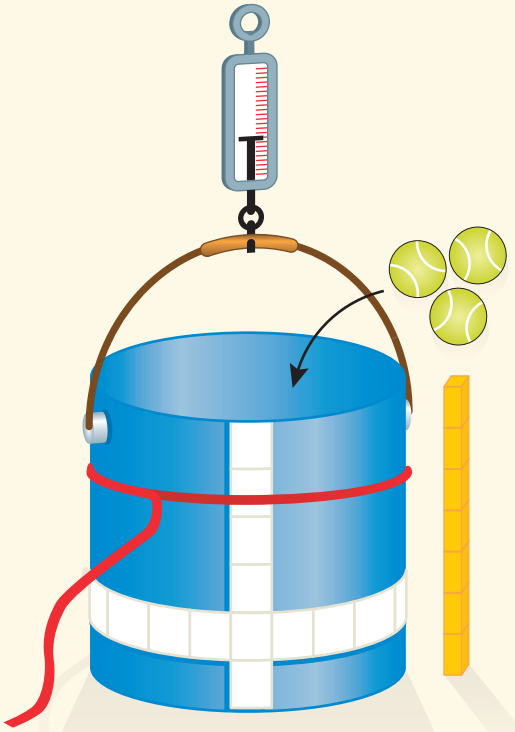
The Meaning and Process of Measuring

Learning Outcome 18.1 **Describe the meaning and process of measuring.**

A *measurement* is a number that indicates a comparison between the attribute of the object (or situation, or event) being measured and a given unit of measure with the same attribute. For example, to measure a length, compare by lining up copies of the unit directly alongside the length being measured. For most attributes measured in schools, we can say that *to measure* means that the attribute being measured is matched or covered or filled with a unit of measure with the same attribute. Let's look at the progression of concepts and skills shown in Table 18.1.

TABLE 18.1 MEASUREMENT INSTRUCTION: A PROGRESSION OF EXPERIENCES

Step	Goal	Type of Activity	Notes
1. Decide on the attribute to be measured.	Identify measurable attributes of an object, situation, or event.	See Figure 18.1 for an example of deciding which attribute(s) will be measured. In this case, is it weight, volume, length, or area?	Make sure students realize that you can measure objects in a variety of ways.
2. Select a unit that has that attribute.	Recognize that the unit must have length if measuring length, if measuring weight the unit must have weight, and so on.	Give students cards with a variety of objects and units and have them match the unit to the measurable attribute.	Recognizing the correct unit is essential in grasping the meaning of measurement.
3. Make comparisons.	Students will understand the *attribute* to be measured.	Make comparisons based on the attribute (for example, longer/shorter, heavier/lighter). Use direct comparisons whenever possible.	When the attribute is understood, there is no further need for comparison activities.
4. Use concrete representations of measuring units.	Students will understand how filling, covering, matching, or making other comparisons of an attribute with measuring units produces a number called a *measure*.	Use concrete representations (think CSA) of individual measuring units to fill, cover, match, or make the desired comparison of the attribute of the object with the unit. The number of units required to match the object is its measure.	Begin with nonstandard units and then progress to the use of standard units.
5. Use measuring tools.	Students will use common measuring tools with understanding and flexibility.	Make measuring tools (grouped units) and compare them with the concrete representations of an individual unit to see how the measurement tool performs the same function.	Without a careful comparison of the nonstandard (or informal) tools with the standard tools, the connection can be lost.
6. Use formulas.	Students should identify the formulas for measuring attributes such as area, perimeter (length), or volume.	After many experiences with individual units, have students identify the patterns of 2 times the length + 2 times the width for finding the perimeter of a rectangle.	Formulas are only presented after students have enough experiences to identify the connections between the formula and the act of matching units.



Attribute: Weight
Units: objects that stretch the spring in the scale
How many units will pull the spring as far as the bucket will?

Attribute: Volume/Capacity
Units: cubes, balls, cups of water
How many units will fill the bucket?

Attribute: Length
Units: rods, toothpicks, straws, string
How many units are as tall as the bucket?
How much string is needed to go around the bucket?

Attribute: Area
Units: index cards, squares of paper, tiles
How many cards will cover the surface of the bucket?

FIGURE 18.1 Measuring different attributes of a bucket

Decide on the Attribute

Suppose you asked your students to measure a bucket, as in Figure 18.1. The first thing they would need to know is *what* about the bucket is to be measured. They might measure the height, depth, diameter, or circumference. All of these are length measures. The surface area of the outside of the bucket could be determined. A bucket also has volume (or capacity) and weight. Each aspect that can be measured is an *attribute* of the bucket.

Activity 18.1 What's the Attribute?

CCSS-M: 3.MD.A.2; 3.MD.C.5; 4.MD.A.1; 5.MD.C.3

Find examples of a variety of measurements in the news, on signs, online, or in other everyday situations. Present the context and measures, but without units, and have students name the attribute(s) to be measured. For example, you may consider ads for carpeting, articles about gas prices, or use the measurement cards. Have students discuss their choices. For students with special needs in math, you may want to provide possible attributes so they can sort the real-world measures into groups (i.e., area, capacity, weight, time, length). For multilingual learners, share a list of attributes with illustrations.

Pearson eText Activity Page: Measurement Cards

Select a Unit

The United States is one of three nations (with Myanmar and Liberia) that have not adopted the International System of Units (SI) a synonym for the metric system. Because of its international importance, U.S. students must know SI (*metric system*) for such things as product design, manufacturing, marketing, and labeling to participate in the global marketplace (National Institute of Standards and Technology [NIST], 2015; NCTM, 2015). Yet, many students are challenged—even at the middle school level—to state how many milliliters were in a liter (Perie, Moran, & Lutkus, 2005). The *customary system* uses inches, feet, and so on. Customary units continue to be important in the U.S. for various careers (e.g., carpentry) and contexts, so students need to develop familiarity with both systems (NCTM, 2015). Interestingly, U.S. students do better with metric units than with customary units (Preston & Thompson, 2004).

The relationships between standard units within either the metric or customary systems are conventions. As such, students must simply be told what the relationships are, and instructional experiences must be devised to reinforce them. In the intermediate grades, knowing basic relationships between the standard units becomes important in making conversions between units in the same system (customary or metric).

Selecting a unit means asking within-system questions like, "Should the room be measured in feet or inches? Should concrete blocks be weighed in grams or kilograms?" The answers to these questions involve knowing how big units are *and* determining the precision needed for the context. If you were measuring a painting in order to make a wood frame, you would need to measure it very precisely. The smallest unit would be an inch or a centimeter, and you would also use small fractional parts.

Pearson eText
Video Example 18.1
In this video Zach Champagne discusses the importance of identifying measurable attributes.

Standards for Mathematical Practice
MP6. Attend to precision

Make Comparisons

In measurement, making comparisons is based on the attribute and includes what is longer/shorter or heavier/lighter. To help make the notion of the unit explicit, use as many copies of the unit as are needed to fill or match the attribute measured. (This action is called *tiling*, and it involves equal partitioning.) Giant footprints (nonstandard) could be used to measure the length of the hallway by placing multiple copies of the footprint end to end. To measure the area of the desktop with an index card (nonstandard) as your unit, you can literally tile (cover) the entire desk with index cards. Somewhat more difficult is to use a single copy of the unit (this action is called *iteration*) and measure the desktop with a single index card by repeatedly moving the card from position to position and keeping track of which areas were already covered.

Sometimes with length, a direct comparison can be made where one object can be lined up and matched to another. But other times, an indirect method, using a third object, must be used. For example, if comparing the volume of one box to another, students must devise an indirect way to compare. They may fill one box with beans and then pour the beans into the other box. When using length, they could use a string to compare the height of a wastebasket to the distance around the top. The string is the intermediary referent, because it is impossible to compare these two lengths directly. When we use a third object to make a comparison, that is called the *transitivity principle for indirect measurement*.

Use precise language when helping students make comparisons. Avoid using ambiguous phrases like "bigger than" and "smaller than"; instead, use more precise language such as "longer than" or "holds more than."

Pearson eText
Video Example 18.2
In this video Jennifer Suh discusses transitivity using indirect measurement for length.

Use Concrete Representations of Measuring Units

For most attributes measured in pre-K–8, it is possible to have concrete representations of the units of measure. Time and temperature are exceptions (as are other attributes not commonly measured in those grades, such as light intensity, speed, and loudness). Concrete representations of units can be found for both nonstandard (sometimes referred to as *informal*) units and standard units. For length, for example, drinking straws (nonstandard) or 1-foot-long paper strips (standard) might be used as units of measurement.

Introduce Nonstandard Units

A common approach is to begin measurement with nonstandard units. This is true not just for inches/feet, but also for measuring area, volume, and angles, and so on. The use of nonstandard units for beginning measurement activities is beneficial at all grade levels because they:

- *Focus on the attribute being measured.* For example, when measuring the area of an irregular shape, units such as square tiles (http://www.didax.com/apps/color-tiles/) or circular counters (http://www.didax.com/apps/two-color-counters/) may be suggested. Each unit covers area, and each will give a different result. The discussion can focus on what it means to measure area.
- *Provide a good rationale for using standard units.* The need for a standard unit gains meaning when your students have measured the same objects with their own collections of nonstandard units and arrived at different and sometimes confusing answers.

The amount of time that should be spent using nonstandard units varies with students' age, level of understanding, and the attribute. Some students need many experiences with a variety of nonstandard units of length, weight, and capacity. Conversely, *fourth* graders may only need to work with nonstandard units for a day or two when they learn to measure angles. When nonstandard units have served their purpose, move on.

Introduce Standard Units

Measurement sense demands that students be familiar with standard measurement units, able to make estimates in these units (i.e., mentally divide a total length into units and then count those units), and able to interpret meaningfully measures given in standard units. Perhaps the

biggest challenge in teaching measurement is students' challenges in: (1) understanding the meaning and technique of measuring a particular attribute, and (2) learning about the standard units commonly used to measure that attribute.

Teaching standard units of measure can be organized around three broad goals:

1. *Familiarity with the unit.* Students should have a basic idea of the size of commonly used units and what attribute they measure. Knowing approximately how much one liter of water is or being able to estimate a bookshelf as five feet long is important.
2. *Ability to select an appropriate unit and an appropriate level of precision.* For a given situation, students should select a reasonable measurement unit and the appropriate level of precision. Would you measure your lawn to purchase grass seed with the same precision as you would use in measuring a window to buy a pane of glass?
3. *Knowledge of relationships between units.* Students should know common unit relationships, such as between inches, feet, and yards or milliliters and liters.

Activity 18.2 focuses on developing familiarity with standard units by helping students locate benchmarks for common units (e.g., one foot).

Activity 18.2 About One Unit

CCSS-M: 1.MD.A.2; 2.MD.A.1; 2.MD.A.3

Give students a concrete representation of a standard unit and have them search for objects with about the same measure. For example, given a 1-meter-long piece of rope, challenge them to find objects in the classroom, on the playground, around the school, or at home that are about the same length (see the About a Meter activity page). You also can have students list things that are about the same or a little shorter (or a little longer), or things that are twice as long (or half as long) as one meter. Include curved or circular lengths.

Pearson eText Activity Page: About a Meter

Families can also help students find local distances that are about 1 mile or about 1 kilometer. Send home a newsletter that suggests families check distances around the neighborhood, to the school or shopping center, or along other frequently traveled paths that are each of these distances. For other options of measuring in the community, go online to Google Earth's Measuring Tools, where distances in centimeters, inches, feet, yards, kilometers, and miles can be made.

By building on developing benchmarks for one unit, students can develop familiarity with units with familiar lengths found on our bodies.

Activity 18.3 Personal Benchmarks

CCSS-M: 2.MD.A.1; 3.MD.B.4; 4.MD.B.4

Students should estimate and then measure the length of their: foot, hand span (stretched or with fingers together), finger width, or arm span (finger to finger and finger to nose). If their foot measures about 10 inches, they can use that as a benchmark to think about how far it is from their desk to the classroom door, for example. Once students have measured their lengths, ask them to estimate larger lengths. Provide sentence frames like:

It is _____ from _____ to _____
#body measure location #1 location #2

Some personal measurements are useful benchmarks for singles or multiples of standard units. The average child's fingernail width is about 1 cm, and most people can find a 10-cm length on their hands. As a challenge or extension, have students find relationships between their body measurements. There are wonderful proportional relationships to be found. For example, compare total height to the circumference of your head (estimate first, and you will be amazed).

Pearson eText Activity Page: Personal Benchmarks

Students need experiences selecting appropriate units and knowing the relationship between units. Activity 18.4 attends to both.

Activity 18.4 Measuring Favorite Things

CCSS-M: 1.MD.A.2; 2.MD.A.1; 3.MD.B.4

Read *Measuring Penny* (Leedy, 2000) to think about the many things that can be measured. In this book, the author bridges between nonstandard units (e.g., dog biscuits, swabs, etc.) and standard units (inches, centimeters, etc.) to measure Penny, the pet dog. Have students find something at home (or in class) to measure in as many ways as they can think of, using standard units. Ask students to explore how their measurements in inches compared to the measurements in feet (of the same objects). Help students see the pattern that for every 12 inches, they have 1 foot.

Discuss with the students which is a better standard unit for _____ (item), inches or feet? Why? Ask students to find favorite things that are a size that would be measured in inches and favorite things that would be measured in yards or miles (replace or repeat with metric measures). For a challenge or, depending on the grade level, help, measures can be rounded to whole numbers or can include fractional units. Discuss in class the familiar items chosen and their measures so that different ideas and benchmarks are shared. See also Lee and Francis (2019) for more classroom-tested ideas.

You can repeat this activity, using different units and/or measuring different objects. For each different-sized unit, estimate the measure (using benchmarks) and discuss the reasonableness of the estimate afterward. Ask students to explain patterns they notice related to how many units match the object and the size of the unit. Students will observe that smaller units produce larger numeric measures, and vice versa. Students mentally construct this inverse relationship by estimating, then experimenting, and finally reflecting on the measurements. These experiences help students develop a strong foundation for using standard units.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Use Measuring Tools

Research shows that many students are challenged to give the correct measure of an object that wasn't aligned with the end of a scale, as in Figure 18.2 (Blume et al., 2007; Kloosterman et al., 2009b). Students also experienced difficulty when the increments on the tool were not one unit (i.e., two units or half a unit). These results point to the difference between using a measuring tool and understanding how it works.

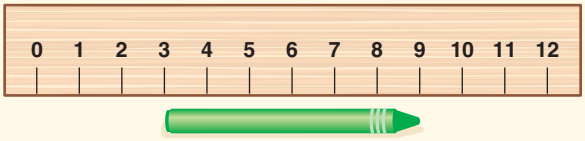


FIGURE 18.2 How long is this crayon?

When students construct simple measuring tools, they are more likely to understand how a tool measures. A ruler is a good example. If students line up individual physical units along a strip of card stock and mark them off, they can see that it is the *spaces* on the scale and not the tick marks or numbers that are the measure. It is essential that students discuss how measurement with iterating individual units compares with measurement using a tool. Without this comparison and discussion, students may not understand that these two methods are essentially the same.

***TECHNOLOGY* Note.** SplashLearn (https://www.splashlearn.com/s/math-games/measure-lengths-on-a-ruler-by-subtracting) has several games (e.g., Measure Lengths on a Ruler by Subtracting) for students to find the measure of an object that is not aligned with the end of the ruler. Additionally, we must keep in mind the power of telephone and computer applications as measuring tools. The Measure app on an iPhone or iPad uses technology to turn the device into a tape measure that can identify dimensions and save a photo of the measurement. Android has Google Measure, which does the same thing. Protractor Smart on iOS or Android measures angles by using visual and sonic feedback! ■

Use Formulas

Measurement formulas include general rules such as area = base × height. When students *develop* formulas, rather than just being told the formula, they gain conceptual understanding of the ideas and relationships involved, and they engage in doing mathematics. Students form general relationships when they notice how all area formulas are related to one idea: length of the base times the height. And students who have a conceptual understanding of where formulas come from tend to remember them or can derive them, which reinforces the idea that mathematics makes sense.

Estimation and Approximation

Learning Outcome 18.2 **Explain the role of estimating before measuring.**

Measurement estimation is the process of using mental and visual information to measure or make comparisons without using measuring tools. People use this practical skill almost every day. Do I have enough sugar to make cookies? Can I throw the ball 15 meters? Is this suitcase over the weight or size limit? Will the car fit into that parking space? Here are reasons for including estimation in measurement activities:

- *Helps students focus on the attribute and the measuring process.* Think how you would estimate the area of the cover of this book, using playing cards as the unit. To do so, you have to think about the meaning of area and how units might be placed on the book cover.
- *Provides intrinsic motivation.* It is interesting to see how close you can come in your estimate to the actual measure.
- *Develops familiarity with standard units.* If you estimate the height of the door in meters before measuring, you must think about the size of a meter.
- *Promotes multiplicative reasoning through using a benchmark.* The width of the building is about one-fourth of the length of a football field—perhaps 25 yards.

Approximation of a quantity is one that is close to but not exactly the same as the quantity. All measurements are approximations, because it is always possible to measure with more precision. Therefore, emphasize the use of approximate language, because measurements are not exact. The desk is about 15 orange rods long. The front of the math book is covered by *slightly fewer than* 8 index cards. Approximate language is very useful because many measurements do not result in whole numbers. As students begin to measure with fractional units, they will continue to see that they could be more precise, which is an opportunity to develop the idea that all measurements include some error.

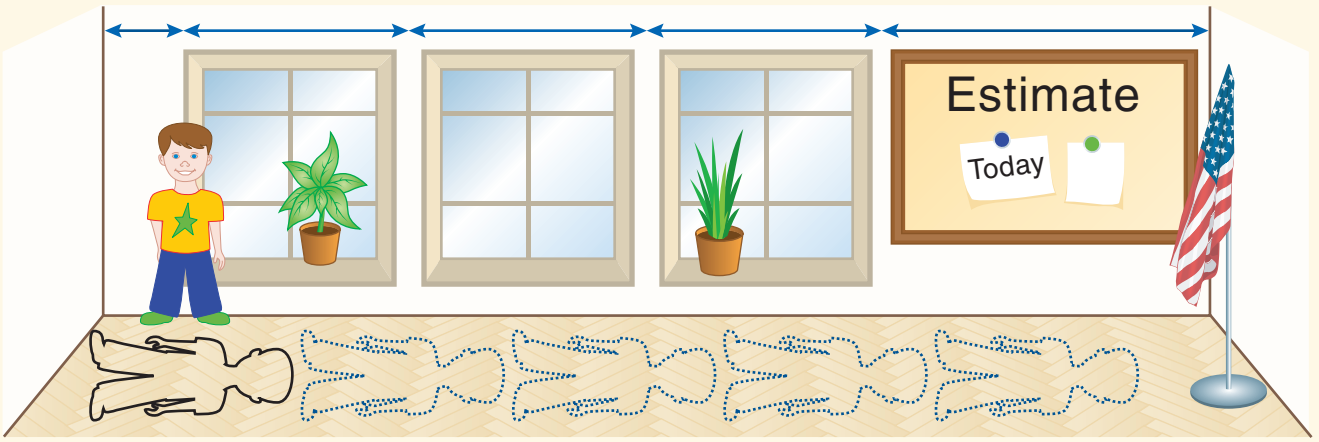
Each smaller unit or subdivision produces a greater degree of precision, reducing measurement error. A length measure can never be more than one-half unit in error. For example, suppose you are measuring a length of ribbon with a ruler that shows quarter inches as the smallest unit. If the length of ribbon is between $3\frac{3}{4}$ and 4 inches, we would usually round to whichever number is closer. If more than halfway toward the 4-inch mark, we would say it's 4 inches long or, if less than halfway, we would say $3\frac{3}{4}$ inches long. In either case, we are within $\frac{1}{8}$ of an inch (one-half of the unit) and are essentially ignoring the difference that constitutes our error. If we need more precision, we use smaller units. Yet even with the smaller units, the resulting number is an approximation of what is being measured. Because, mathematically, there is no smallest unit, there is always some error, so avoid saying a measurement is exact.

Standards for Mathematical Practice
MP6. Attend to precision

Strategies for Estimating Measurements

Begin measurement activities with students making an estimate. Just as for computational estimation, students need specific strategies to estimate measures and should compare those strategies (McAlister & Marshman, 2019). Here are options:

1. *Develop benchmarks or referents.* Students who use mental images of benchmarks or reference points for measurements (for single units and for useful multiples of standard units) *and* practice using them in class activities are much better estimators than students who do



Estimate the room length.

Use: windows, bulletin board, and spaces between as "chunks."
Use: mental benchmark—"My height is about 5 feet long. I could get 5 kids lying down in here plus maybe 3 more feet. Say, 28 feet."

FIGURE 18.3 Estimating measures, using benchmarks and chunking

not (Joram & Gabriele, 2016). Students must pay attention to the size of the unit to estimate well (Towers & Hunter, 2010). Referents should be things that students can easily visualize. One example is the height of a child (see Figure 18.3). Other everyday objects can also help: A doorway is a bit more than 2 m high, and a doorknob is about 1 m from the floor. A paper clip weighs about a gram and is about 1 cm wide. A pineapple or liter of water weighs about 1 kilogram.

2. *Use chunking.* Chunking involves subdividing a measurement into components to estimate the amount better. Figure 18.3 shows an example of chunking by using windows, bulletin boards, and spaces between as chunks making the estimation easier. The weight of a stack of books is easier to estimate if some referent is given for the weight of an average book. But if the wall length to be estimated has no useful chunks, it can be mentally subdivided in half and then in fourths or even eighths by repeated subdivisions until a manageable length is found. Length, volume, area, and surface area measurements all lend themselves to the estimation strategy of chunking.
3. *Iterate units.* For length, area, and volume, it is sometimes easy to mark off single units mentally (visualizing) or physically. You might use your hands or make marks to keep track as you work. For example, if you know your stride is about $\frac{3}{4}$ meter, you can walk off a length and then multiply to get an estimate. Hand and finger widths are also useful.

Standards for Mathematical Practice
MP7. Look for and make sense of structure

Keep the following tips in mind:

1. *Explicitly teach each strategy.* After learning and practicing each strategy, students can choose the one that works best in a particular situation.
2. *Discuss how different students made their estimates.* These conversations will confirm that there is no single right way to estimate while reminding students of other approaches.
3. *Create a list of benchmarks.* Record and post on a chart benchmarks for common measures.
4. *Accept a range of estimates.* Think in relative terms about good estimates. Within 10 percent for length is reasonable. Even 30 percent off may be reasonable for weights or volumes.
5. *Do not promote a winning estimate.* An emphasis on a contest discourages estimation and promotes only seeking the exact answer.
6. *Encourage students to give a range of estimates that they believe includes the actual measure.* For example, the door is between 7 and 8 feet tall. This focus on reasonable minimum and maximum values not only is a practical real-life approach but also helps focus on the approximate nature of estimation.

7. *Make measurement estimation an ongoing activity.* Post a daily/weekly measurement to be estimated. Students can record their estimates and discuss them for five minutes. Invite a student or a team of students to select measurements to estimate.
8. *Be precise with your language.* Do not use the word *measure* interchangeably with the word *estimate* (Towers & Hunter, 2010). Randomly substituting one word for the other will cause students uncertainty and possibly confusion.

Measurement Estimation Activities

Any measurement activity can have an estimate-first component. Here are two suggestions.

Activity 18.5 Estimation Exploration

CCSS-M: K.MD.A.1; K.MD.A.2; 1.MD.A.1; 1.MD.A.2; 2.MD.A.3; 3.MD.A.2; 5.MD.C.4; 6.G.A.1; 6.G.A.2; 7.G.B.4; 8.G.C.9

Select a single object such as a box, a pumpkin, a painting on the wall of the school, or even the principal. Each day, select a different attribute or dimension to estimate. For a pumpkin, for example, students can estimate its height, circumference, weight, volume, and surface area. For multilingual learners, include metric measures or help students connect the centimeter to the inch.

Pearson eText
Video Example 18.3
In this video kindergarteners use a pumpkin to discuss various measurements including height, weight, and circumference.

https://www.youtube.com/watch?v=ygeIbA4TJNk

Activity 18.6 Estimation Scavenger Hunt

CCSS-M: 1.MD.A.2; 2.MD.A.1; 2.MD.A.3; 4.MD.A.1

Conduct estimation scavenger hunts by giving teams a list of either nonstandard or standard measurements and having them find things that are close to those measurements. At first, do not permit the use of measuring tools. Use the Estimation Scavenger Hunt activity page for possible ideas and add your own. Let students suggest how to judge results in terms of accuracy. Students with special needs in math may benefit from having a reference such as an example of 1 square inch or 1 milliliter. Multilingual learners are likely to have stronger metric measurement sense and native U.S. students of customary measures, so group heterogeneously to build on the strengths of both.

Pearson eText Activity Page: Estimation Scavenger Hunt

FORMATIVE ASSESSMENT Notes. Use an observation look-for tool to record students' strategies for estimating and measuring a variety of real objects, using, weight, area, volume, length, and so on. Prompt students to explain their thinking. Only looking at their numeric estimate can mask a lack of understanding and will not give you the evidence you need to provide next instructional steps. ■

Pearson eText Teacher Resource: Observation Look-For Tool

Length

Learning Outcome 18.3 **Demonstrate how to measure the length of objects.**

Length is commonly the first attribute students learn to measure. Length is found by locating two endpoints and examining how far it is between those points. We measure lengths by selecting a unit that has length and repeatedly matching that unit to the object. Understanding length is a gateway to understanding perimeter, area, and volume (Tan Sisman & Aksu, 2016).

Comparison Activities

At the pre-K and kindergarten level, students begin with direct comparisons of two or more lengths and then move to indirect comparisons by the first grade. Seek opportunities for students to compare lengths directly, as in the next activity.

Activity 18.7 Length (or Unit) Hunt

CCSS-M: K.MD.A.2; 1.MD.A.1; 1.MD.A.2; 2.MD.A.3

Show pairs of students a strip of card stock, a Popsicle stick, a length of rope, or some other concrete representation with an obvious length dimension to serve as a target unit. The task is to find five things in the room that are shorter than, longer than, or about the same length as the target unit. Pairs can record what they find in pictures or words. Allow students with special needs in math to have a copy of the target unit to make actual comparisons. By making the target length a standard unit (e.g., a meter stick), the activity can be repeated to provide familiarity with important standard units. This activity can be adapted to learning stations, too.

Students also need to consider indirect comparisons, which means using another object to make the measure (particularly for curved lengths), as in the next activity.

Activity 18.8 Crooked Paths

CCSS-M: 1.MD.A.1; 2.MD.A.3; 2.MD.A.4

Make crooked or curvy paths on the floor with painters' tape or outside with chalk. Ask student pairs to determine which path is longest, next longest, and so on. Some students may initially think about the straight line distance (as the crow flies) from the beginning of the curved path to its end (Battista, 2012). Have students suggest ways to measure the paths so that they can be compared. For a hint, provide a long piece of string (at first make it longer than the path). For students with special needs in math, you may need to tape the end of the string to the beginning of the path and help them mark the final measurement on the string with a marker. Use another string for the other path in the same way. Then compare the string lengths. Students who argue that the straight path "looks longer" than the more compact crooked path may need an example of these two paths on the floor by walking on each to see which takes longer to walk. See Expanded Lesson: Crooked Paths for an enhanced version of this activity.

Pearson eText Expanded Lesson: Crooked Paths

Use Concrete Representations

There are four important principles of iterating length units, whether nonstandard or standard (Dietiker et al., 2010). Units must be:

- Equal in length, or you cannot iterate them by counting.
- Aligned with the length being measured, or a different quantity is measured.
- Placed without gaps, or part of the length is not measured.
- Placed without overlaps, or the length has portions that are measured more than one time.

Students can begin to measure length, using a variety of nonstandard units, including:

- *Giant footprints:* Cut out a large footprint about $1\frac{1}{2}$ to 2 feet long from poster board for each pair of students.
- *Measuring ropes:* Cut rope into 1-meter lengths. These ropes can measure the perimeter and the circumference of objects such as the teacher's desk, a tree trunk, or a pumpkin.

- *Drinking straws:* Straws can easily be cut into smaller units and can be linked with string. The string of straws can be a bridge to understanding a ruler or measuring tape.
- *Short units:* Connecting cubes, toothpicks, and paper clips are useful nonstandard units for measuring shorter lengths. Cuisenaire rods are easy to place end to end and are also metric (centimeters).

The following activity encourages students to develop their own approach to measuring lengths.

Activity 18.9 How Long Is the Teacher?

CCSS-M: 1.MD.A.2; 2.MD.A.1; 2.MD.A.2; 2.MD.A.3

Explain that you have received an important request from the principal. She needs to know how tall each teacher in the building is. The students should decide how to measure the teachers and then write a note to the principal explaining how tall their teacher is and detailing the process used. If you wish to give hints, ask, "Would it help if I lie down?" and have students make marks at your feet and head and draw a straight line between these marks.

Put students in pairs and allow them to select one nonstandard or standard unit with which to measure. For each option, supply enough units to more than cover the length. Ask students to estimate first and then use their unit to measure.

After students measure, follow up with "How did you get your measurement?" "Did students who measured with the same unit get the same answers? If not, why not?" "How could the principal make a line that was as long as the teacher?" Focus on the value of carefully lining units up end to end. Discuss what happens if you overlap units, have a gap in the units, or don't follow a straight line. For a challenge, have students measure twice with two different units.

The following activity highlights estimation.

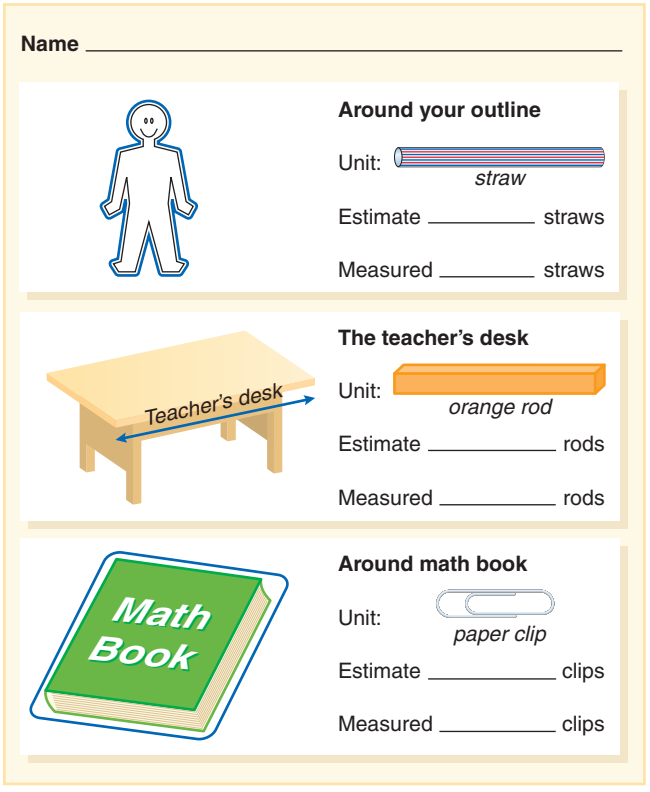


FIGURE 18.4 Estimating and measuring with nonstandard length units

Activity 18.10 Estimate and Measure

CCSS-M: 1.MD.A.2; 2.MD.A.1; 2.MD.A.3

Make lists of items in the room to measure (see Figure 18.4) or use the Estimating and Measuring with Nonstandard Units Recording Sheet (which also can be done with standard units). Run a piece of painters' tape along the dimension of objects to be measured. Include curves or other distances that are not straight lines. Have students estimate before they measure. Students with special needs in math may need a list of possible strategies. For example, cluster a row or chain of 10 units as a referent to help them visualize. Then they lay 10 units against the object and make their estimate.

Pearson eText Activity Page: Estimating and Measuring with Nonstandard Units Recording Sheet

FORMATIVE ASSESSMENT Notes. Tasks that can also be used as diagnostic interviews are:

- Ask students to draw a line or mark off a distance of a prescribed number of units. Observe whether they know to align the units in a straight line without overlaps or gaps.
- Demonstrate how a fictitious student used a ruler to measure the length of an object. Display errors such as

making gaps, overlaps, using units of different sizes in the same measure, and making a wavy line in the placement of the ruler. The students' task is to explain whether these measurements are accurate. If not, why?

- Have students measure two different objects. Ask, "How much longer is the longer object?" Observe whether students use the two measurements they have to answer or whether they need to make a third measurement to find the difference.
- Have students measure a length with small paper clips and then again with large paper clips. Can they identify the inverse relationship between the measures they find and the size of the units?

If your assessment indicates some confusion about how length is measured, use the class discussion of these results to help students self-assess and come to a deeper understanding. ■

Make and Use Rulers

The jump from measuring with actual units to using standard rulers is challenging. One method to help students understand rulers is to have them make their own rulers.

Activity 18.11 Make Your Own Ruler

CCSS-M: 1.MD.A.2; 2.MD.A.1; 2.MD.A.3

Use two colors of cardstock to print ruler units (1 inch by 1 inch units). Discuss how the units could be used to measure by laying them end to end. Then provide long ruler strips and have students glue the units in alternating colors onto the card stock as shown at the top of Figure 18.5.

Have students use their new rulers to measure items on a list that you provide. Include lengths that are longer than their rulers. Discuss the results. There might be discrepancies due to rulers that were not made properly or students who need more discussion on how a ruler works.

Rather than glue units together, students can cut out strips of inch or centimeter grid paper, coloring every other square on a row to create a ruler.

Pearson eText Activity Page: Ruler Units
Pearson eText Activity Page: Ruler Strips

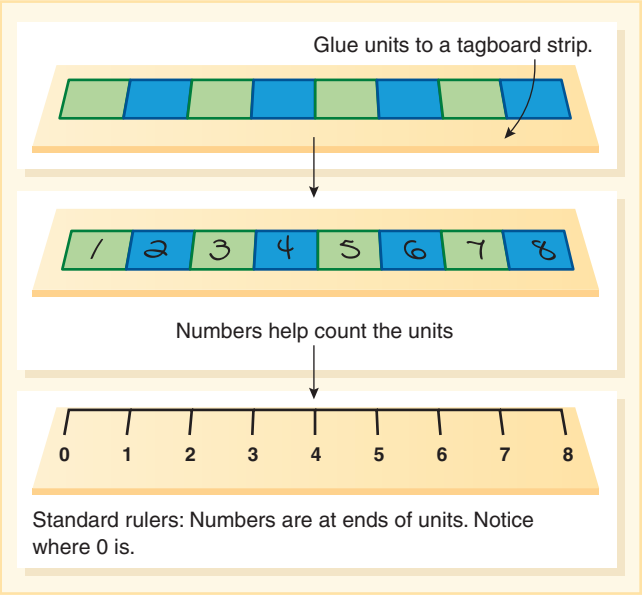

Glue units to a tagboard strip.
1 2 3 4 5 6 7 8
Numbers help count the units
0 1 2 3 4 5 6 7 8
Standard rulers: Numbers are at ends of units. Notice where 0 is.

FIGURE 18.5 Give meaning to numbers on rulers

Challenge students to find more than one way to measure a length with a ruler. Do you have to begin at the end? What if you begin at another unit in the center? Students can eventually put numbers on their handmade rulers, as shown in Figure 18.5. In the beginning, numbers can be written in the center of each unit to show that the numbers are a way of pre-counting units. When numbers are written in the standard way, at the ends of the units, the ruler becomes a number line.

Make an explicit connection from the handmade rulers to standard rulers. Give students a standard ruler and have them identify and discuss how the self-made ruler and the standard rulers are alike and how they are different. What are the units? What do the numbers mean? What are the tick marks for? Where do the units begin?

FORMATIVE ASSESSMENT Notes. Students sometimes incorrectly believe that the numbers on a ruler are counting the tick marks rather than the units or spaces between the marks. Provide students a ruler with tick marks but no numbers. Have students use the ruler to measure an item that is shorter than the ruler. Use an observation look-for tool to record whether

students count spaces between the tick marks or count the tick marks. Repeat by holding the object in the middle of the ruler. Finally, have students measure an object that is longer than the ruler.

Pearson eText Teacher Resource: Observation Look-For Tool

Another excellent assessment of ruler understanding is to have students measure with a broken ruler, as in Activity 18.12.

Activity 18.12 Broken Rulers

CCSS-M: 1 MD.A.2; 2.MD.A.1; 2.MD.A.3

Provide pairs of students with paper rulers (in centimeters or inches) with one or more of the first units broken off. Give the pairs Cuisenaire rods to measure in centimeters or other lengths that they can measure in inches. Use Broken Ruler Activity Page 1 for measures to the nearest inch or Broken Ruler Activity Page 2, which includes fractional units. Note whether students initially say that it is impossible to measure with such a ruler because there is no starting point. Note also students who match and count the units meaningfully.

Pearson eText Pearson eText Activity Page: Broken Ruler 1

Pearson eText Pearson eText Activity Page: Broken Ruler 2

***TECHNOLOGY* Note.** GeoGebra (www.geogebra.org) offers a couple of interactive applets in which students can measure different lengths, using a broken ruler (search "broken rulers" under materials). ■

Pearson eText
Video Example 18.4
In this video the late professor Janet Caldwell discusses converting units.

Conversion

The customary system has few patterns or generalizable rules to guide students in converting length units. In contrast, the metric system was systematically created around powers of ten. Understanding of the role of the decimal point as indicating the units position is a powerful concept for making metric conversions (see Figure 16.4). As students grasp the structure of decimal notation, develop the metric system with all seven places: three prefixes for smaller units (*deci-*, *centi-*, *milli-*) and three for larger units (*deka-*, *hecto-*, *kilo-*). Avoid mechanical rules that might be misused, misunderstood, or forgotten, such as, "To convert centimeters to meters, move the decimal point two places to the left." Instead, create conceptual, meaningful methods for conversions.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Students in the second grade begin to consider the relationship between the size of a unit and the resulting measure. And fourth and fifth graders must be able to convert measures in the same system to larger or smaller units. However, students can find it a challenge to understand that larger units will produce a smaller measure and vice versa. Engage students in activities that emphasize this relationship like the following:

Activity 18.13 Changing Units

CCSS-M: 2.MD.A.2; 2.MD.A.3; 4.MD.A.1; 5.MD.A.1

Have students measure a length with a specified unit (e.g., Cuisenaire rods) and use the Changing Units recording sheet. Then provide a different unit that is either twice as long or half as long as the original unit. Their task is to predict the measure of the same length, using the new unit. Students should write down and discuss how they made their estimates and then determine the actual measurement. Challenge students with units that are more difficult multiples of the original unit.

Pearson eText Activity Page: Changing Units Recording Sheet

In this activity, you reinforce the basic idea that when the unit is longer, the measure is smaller and when the unit is smaller, the measure is larger, which is useful for introducing unit conversion with standard units, and an excellent proportional reasoning task for middle school students.

After completing activities like the one that follows, encourage students to notice patterns that help them conclude that you multiply to convert a larger unit to a smaller unit. What do you predict is the process to go from a smaller unit to a larger unit?

Activity 18.14 Conversion Please

CCSS-M: 4.MD.A.1; 5.MD.A.1

Give students the two-column conversion table. Select several items around the classroom and have the students measure them in feet and then again in inches. Have them complete only half of the table. Ask them to describe the relationship between the two measurements. They are likely to notice that the longer the unit chosen, the fewer units needed and vice versa. For the second half of the table, give them the item to measure only with feet and have them convert to find how many inches. When complete, students should confirm their answers by measuring with inches. Try other units.

Pearson eText Activity Page: Two-Column Conversion Table

Other measures of length, such as perimeter and circumference, will be discussed in subsequent sections.

Area

Learning Outcome 18.4 **Explain the development of area formulas.**

Area is the measure of two-dimensional space inside a region. As with other attributes, students must first understand the attribute of area before measuring. Estimating and measuring area begins in third grade, as students connect to multiplication by using arrays, and continues in grade 4 with finding the area of rectangles by using formulas. In fifth-grade, students explore area problems with fractional measures and use area to find volumes of three-dimensional shapes. By grade 6, students explore the areas of a wide range of polygons and learn surface area. In grade 7, students explore the area of circles.

Comparison Activities

Comparing area measures is more of a conceptual challenge than comparing length measures, because areas come in different shapes. Comparison activities with areas should help students distinguish between size (i.e., area) and shape, length, and other dimensions. A long, skinny rectangle may have less area than a triangle with shorter sides. Many students do not understand that rearranging areas into different shapes does not affect the amount of area (although the perimeter can change).

Direct comparison of two areas is possible only when the shapes involved have some common dimension or property. For example, two rectangles with the same width can be compared directly, as in this Rectangle Comparison activity page, as can any two circles. Comparison of these special shapes, however, does not challenge students to think about the attribute of area. Instead, try activities in which one area is rearranged (conservation of area).

Pearson eText Activity Page: Rectangle Comparison

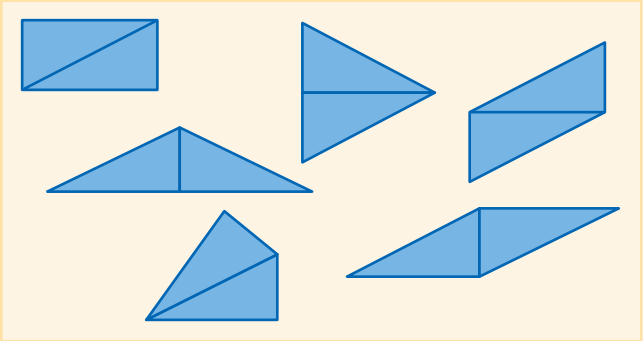


FIGURE 18.6 Different shapes, same area

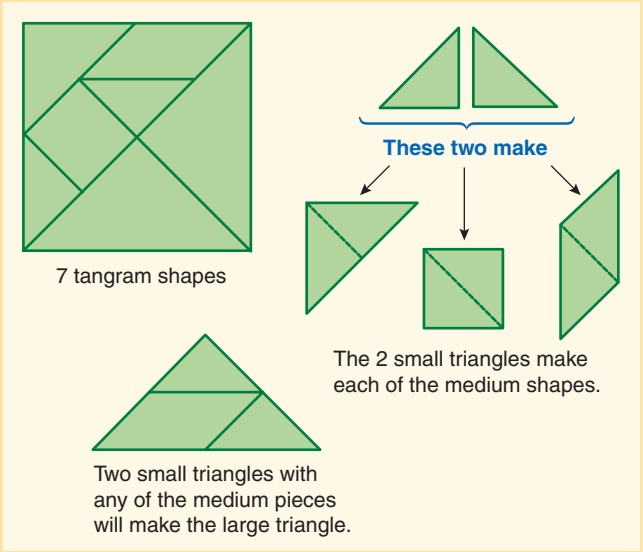

7 tangram shapes
These two make
The 2 small triangles make each of the medium shapes.
Two small triangles with any of the medium pieces will make the large triangle.

FIGURE 18.7 Tangrams provide opportunities to investigate area

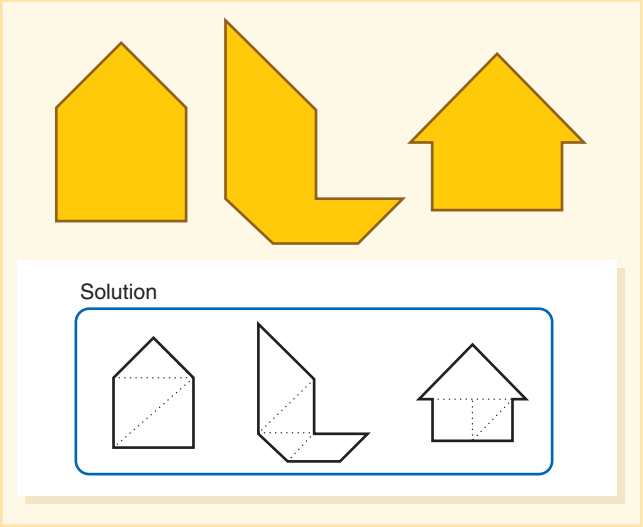


FIGURE 18.8 Compare the area of shapes with tangram pieces

Activity 18.15 Two-Piece Shapes

CCSS-M: 3.MD.C.5

Cut out rectangles from the Rectangles with the Same Area activity page, giving each pair of students six different rectangles. Have them fold and cut the rectangles on the diagonal, making two identical triangles. Next, have them rearrange the triangles into different shapes, including back into the original rectangle. The rule is that only sides of the same length can be matched, and they must be matched precisely. Have students work in pairs to find all the shapes that can be made this way, gluing the triangles on paper as a record (see Figure 18.6). Discuss the area of each. Does one shape have a greater area than the rest? How do you know? Help students conclude that although each figure is a different shape, all the figures have the same *area*. Students, particularly multilingual learners, may find that this cutout and reposition approach helps them appreciate the connections between area and space (Fernandes, Kahn, & Civil, 2017).

Pearson eText Activity Page: Rectangles with the Same Area

Tangrams, an ancient puzzle, can also be used for the same purpose (https://mathigon.org/polypad#tangram). The standard set of seven tangram pieces is cut from a square, as shown in Figure 18.7 or in the online GeoGebra version (www.geogebra.org—search "tangram"). The two small triangles can be used to make the parallelogram, the square, and the medium triangle. This composing permits a similar discussion about the pieces having the same area but different shapes.

Pearson eText Blackline Master: Tangram Pieces

Activity 18.16 Tangram Areas

CCSS-M: 3.MD.C.5; 3.MD.C.6

Give students outlines of tangram shapes and a set of tangram pieces, as in Figure 18.8. Ask groups to estimate which shape has the largest (or smallest) area. Then students use tangram pieces to decide which shapes are the same size, which are larger, and which are smaller in area. Ask students to justify their conclusions. Use the animal shapes from *Grandfather Tang's Story* (Tompert, 1997) to see if they have the same or different areas. Also, try the tangram game at PBS Kid's Cyberchase (http://pbskids.org/cyberchase) where players use different pieces to form rabbits, ducks, space ships, sailboats, etc. Toy Theater (https://www.mathplayground.com/tangram_puzzles.html) has a user-friendly collection of various tangram puzzles to solve—not just animals.

Pearson eText Activity Page: Outlines of Tangram Shapes
Pearson eText Blackline Master: Tangram Pieces

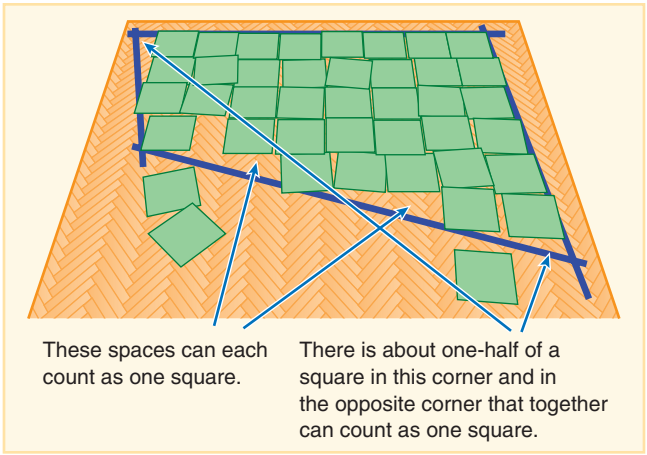

These spaces can each count as one square.
There is about one-half of a square in this corner and in the opposite corner that together can count as one square.

FIGURE 18.9 Combining partial units to measure the area of an irregular figure

Use Concrete Representations

Students need multiple opportunities to cover the surface of two-dimensional shapes to develop understanding of the attribute of area.

Tiles. Tiles can be any shape that easily fills up a plane region, although square tiles are the most commonly used concrete representations for exploring area (Bittman & Moomaw, 2017). For example, sticky notes, sheets of newspaper, carpet tiles, or playing cards can be used. Students can choose units to measure surfaces in the room (i.e., desktops or book covers). Large regions can be outlined with tape on the floor. Small regions can be duplicated on geoboards or grid paper so that students can work at stations (https://www.mathlearningcenter.org/apps/geoboard or https://www.geogebra.org/m/fdaTY58r).

Tiles may only partially fit in an area; thus, students must decide whether to count a tile or leave a gap. This is approximation with area measures and encourages flexible strategy use (Hourigan & Leavy, 2021). In the intermediate grades (as they learn about fractions and decimals), students begin to wrestle with partial units and mentally put together two or more partial units to count as one unit (see Figure 18.9).

The following activity is a good starting point.

Pearson eText
Video Example 18.5
In this video the late professor Janet Caldwell describes ideas of area.

Activity 18.17 Two Rectangles, a Parallelogram, a Trapezoid, and a Blob

CCSS-M: 3.MD.C.5; 3.MD.C.6; 4.MD.A.3

Give students the Two Rectangles, a Parallelogram, a Trapezoid, and a Blob activity page, in which the areas are not the same but with no area that is clearly largest or smallest. Ask students to predict which shape has the smallest and the largest areas and then use strategies to compare the areas. Brainstorm and create a list of options to support students with special needs in math. Students may trace or glue the same two-dimensional unit on the shapes, place tiles on them, or cut the shapes out and place them on grid paper.

Pearson eText Activity Page: Two Rectangles, a Parallelogram, a Trapezoid, and a Blob

Your objective in initial instruction on area is to develop the idea that area is measured by covering or tiling. Do not introduce formulas yet. Groups will likely come up with different measures for the same region. Discuss these differences and point to the difficulties involved in making estimates around the edges.

With rectangles, students benefit from associating multiplication arrays to the area (Hong & Runnalls, 2019). This connection requires that students develop the ability to see a rectangular region as rows and columns. The following comparison activity is a step in that direction.

Activity 18.18 Rectangle Comparison: Square Units

CCSS-M: 3.MD.C.5; 3.MD.C.6; 3.MD.C.7; 5.NF.B.4b

Give students the Rectangle Comparison activity page, which includes four rectangles with a similar area, a concrete representation of a single square unit, and a ruler that measures the unit. Students are not permitted to cut out the rectangles, but they may draw on them if they wish. The task is to use their rulers to determine, in any way that they can, which rectangle is larger or whether they have the same area. They should use words, pictures, and numbers to explain their conclusions. Some suggested pairs are as follows:

4×10 and 5×8 $\quad$ 5×10 and 7×7

4×6 and 5×5

Some students with special needs in math may need to have modified worksheets of the figures on grid paper that matches the square units used.

Pearson eText Activity Page: Rectangle Comparison

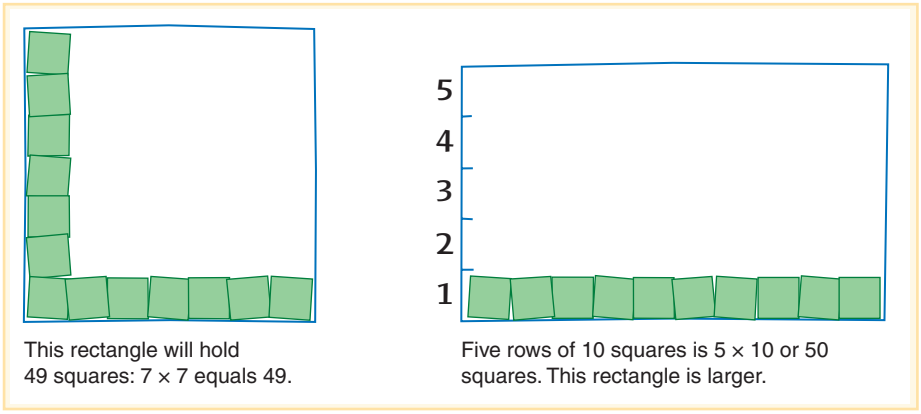

5
4
3
2
1
This rectangle will hold 49 squares: 7 × 7 equals 49.
Five rows of 10 squares is 5 × 10 or 50 squares. This rectangle is larger.

FIGURE 18.10 Students multiply to find the total number of square units

The goal of this activity is to apply students' developing concept of multiplication to the area of rectangles without using a formula. To count a single row of squares along one edge and then multiply that by the length of the other edge, the first row must be thought of as a single unit that is then replicated to fill in the rectangle (Outhred & Mitchelmore, 2004). Students may attempt to draw in all the squares. However, some may use rulers to determine the number of squares that fit along each side and, from that, use multiplication to determine the total area (see Figure 18.10). These multiplicative reasoning strategies lead into the formula for the area of rectangles.

Grids. Grids of various types can be thought of as area rulers. A square grid does for area what a ruler does for length: It clusters and lays out the units for you. Grids of various units (2-Centimeter grid paper, 1-Centimeter grid paper, and 0.5-Centimeter grid paper) can be copied on clear transparency sheets and placed over a region to count the square units inside. An alternative method is to trace around a region on grid paper.

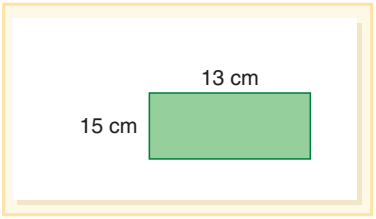
Pearson eText Blackline Master: 2-Centimeter Grid Paper

Pearson eText Blackline Master: 1-Centimeter Grid Paper

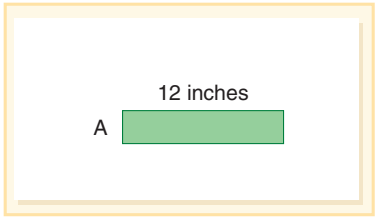
Pearson eText Blackline Master: 0.5-Centimeter Grid Paper

The Relationship between Area and Perimeter

Perimeter is a length measure of the distance around a region and, as such, it is additive. We include it in the section on area because it is used to measure rectangles and other shapes. Students should be able to calculate perimeter, given side measures, as well as identify missing side lengths, as in Figure 18.11.



(a) What is the perimeter of this figure?



(b) The perimeter of this rectangle is 32. What is the length of the side marked A?

FIGURE 18.11 Common perimeter problems

Pearson eText
Video Example 18.6
In this video the late professor Janet Caldwell discusses ideas of perimeter.

A hint for helping students remember the definition of perimeter is that the word *rim* is in the word per*rim*eter, as is reinforced by the next activity.

Activity 18.19 What's the Rim?

CCSS-M: 3.MD.D.8

Have students select objects with a perimeter that they would like the whole class to measure. First, students estimate the perimeter on the What's the Rim? activity page. They also need to choose tools strategically (rulers, cash register tape, or non-stretching string) and measure the perimeter, noting the unit. The discussion should include class comparisons of at least one common item that will provide a basis for exploring any measurement differences. Students should describe how they measured objects that were larger than the tool they were using and how they knew when to use a flexible measuring tool (i.e., string or cash register tape). For students with special needs in math, have them trace the perimeter with their finger prior to measuring. For a challenge, students can measure curved perimeters such as the top edge of a wastebasket.

Pearson eText Activity Page: What's the Rim?

Perhaps because both measures relate to shapes, students often confuse area and perimeter. For example, a common error for 6th graders given an illustration like the one in Figure 18.11(a) was to use the area formula instead (Tan-Sisman & Aksu, 2016). Also, teaching these two concepts within a close time frame is particularly challenging for students with special needs in math (Parmar et al., 2011). Beyond confusing the two is understanding the relationship between them. Can a shape with a given area have different perimeters (and vice versa)? The next three activities help students reason about the relationships between area and perimeter.

Pause & Reflect

How are perimeter and area related? When the perimeter is fixed at 24 cm, will the areas be the same? Why or why not? Which rectangle creates the largest area? The smallest area? When areas are fixed at 36 cm squared, will all of the perimeters be the same? If not, which rectangle has the longest or shortest perimeters? ●

Activity 18.20 Fixed Perimeters and Fixed Areas

CCSS-M: 3.MD.C.6; 3.MD.C.7; 3.MD.D.8; 4.MD.A.3

Here is a three-part, small-group activity that is well worth the time, because it may take more than one class session.

Part 1: Give a loop of non-stretching string that is 24 centimeters in circumference and 1-centimeter grid paper, or just use the grid paper alone. The task is to decide what different-sized rectangular gardens can be made using a fenced perimeter of 24 feet (1 ft = 1 cm). Each different rectangular garden can be recorded on the grid paper, with the area noted inside the sketch of the garden ($A = 20$ cm^2). Then record all of the results on the Fixed Perimeter recording sheet.

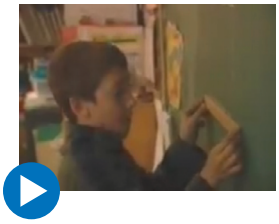
Part 2: Provide 1-centimeter grid paper. The task is to see how many different rectangular gardens can be made with an area of 36 square feet—that is, not just borders. Sketch each new garden on the grid paper (including the dimensions) and record the perimeter inside the figure ($p = 24$ cm). Record all the results on the Fixed Area recording sheet. Refer to Expanded Lesson: Fixed Areas for this part of the activity.

Part 3: Student groups now cut out all the rectangles. Have two charts, labeled "Fixed Perimeter" and "Fixed Area." Teams place their rectangles (left to right) from smallest perimeter (or area) to largest perimeter (or area) on the appropriate chart. Ask students to write down observations, make conjectures, and draw conclusions.

Pearson eText Blackline Master: 1-Centimeter Grid Paper
Pearson eText Activity Page: Fixed Perimeter Recording Sheet
Pearson eText Activity Page: Fixed Area Recording Sheet
Pearson eText Expanded Lesson: Fixed Areas

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Students may be surprised to find out that rectangles having the same areas do not necessarily have the same perimeters and vice versa. And, of course, this fact is not restricted to rectangles. They will notice interesting relationships like when the area is fixed, the shape with the smallest perimeter is square-like, as is the rectangle with the largest area when the perimeter is fixed. If you allowed for any shape whatsoever, the shape with the smallest perimeter for a fixed area is a circle. Relatedly, the fatter a shape, the shorter its perimeter; the skinnier a shape, the longer its perimeter. These relationships are also true in three dimensions—replace perimeter with surface area and area with volume.



Pearson eText
Video Example 18.7
In this video consider how this class engages with the concepts of perimeter.

Develop Formulas for Perimeter and Area

Formulas are taught only after many experiences with concrete representations and measuring tools, and the formulas as connected to these conceptual investigations.

Formulas for Perimeter. As students find the perimeter of rectangles, they will begin to notice patterns. Have students generalize ways that perimeter problems can be solved. As in Figure 18.11(a), it is common for students to be given a rectangle in which only one length and one width are written. Some students may only add these two numbers, rather than the four length measures. A good way to address this common error explicitly is to give students both answers (18 and 36) and debate which is correct. Once students see that they are adding four sides, engage them in how to write the general formula by using variables, connecting to the measures on a rectangle. Options include:

$p = l + w + l + w$ [going around the rectangle and adding each side]
$p = 2(l + w)$ [adding one length and width and then doubling]
$p = 2l + 2w$ [doubling the length and the width and then adding]

When you have triangles or more complex shapes, the perimeter formula is adjusted to reflect the number of sides.

***TECHNOLOGY* Note.** Math Playground (https://www.mathplayground.com) has a game called Area Blocks, in which students create closed polygon shapes that must match given perimeters and areas. The goal is to cover more area than your opponent before the game ends. Students can play against the computer or challenge a classmate. ■

Formulas for Area. Finding the area of shapes must be understood. Teaching formulas too early with little or no conceptual background experiences results in two major challenges:

1. *Confusing linear and square units.* Students may have difficulty as they move from length measurement to the more abstract measurement of area. Students may think of area as the length of two sides (length × width), rather than the measure of a surface (Putrawangsa et al., 2013). It's initially challenging for students to think about multiplying two lengths and getting an area (Kamii & Kysh, 2006). A focus on the formula and the use of a ruler to measure the *sides* conflates the unit with the tool for measuring area (because the use of the ruler is indirect). This causes some students to believe that if there are no sides to measure (a blob shape), the shape doesn't have an area (Zacharos, 2006).

   The tasks in Figure 18.12 cannot be solved with simple formulas; they require an understanding of concepts and how formulas work. "Length times width" is not a definition of area; instead, area is a measure of a two-dimensional surface enclosed by a boundary.

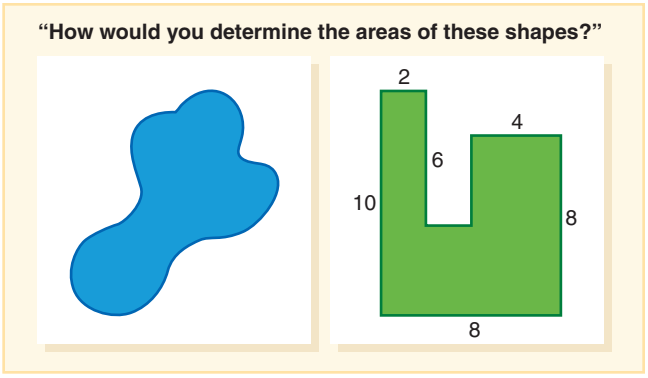


FIGURE 18.12 Understanding the attribute of area

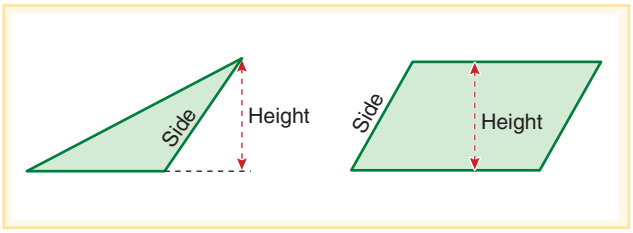


FIGURE 18.13 Heights of two-dimensional figures are not always measured along an edge

2. *Difficulty in conceptualizing the meaning of height and base.* The shapes in Figure 18.13 each have a slanted side and a height given. Students can confuse these two lengths. Any side of a figure can be called a *base*. For each base that a figure has, there is a corresponding height. If the figure were to slide into a room on a selected base, the *height* would be the height of the shortest door it could pass through without tipping—that is, the perpendicular distance to the base. Confusion may arise because students have focused on experiences with the $L \times W$ formula for rectangles, in which the height is the same as the length of a side.

The formula for the area of a rectangle is one of the first that is developed and is usually given as $A = L \times W$, or "area equals length times width." An equivalent but more unifying idea is $A = B \times H$, or "area equals *base* times *height*." The base-times-height formulation can be generalized to all parallelograms (not just rectangles) and is useful in developing area formulas for triangles and trapezoids. Furthermore, the same approach can be extended to three dimensions—volumes of cylinders are given in terms of the *area of the base* times the height. Therefore, base times height connects a large family of formulas that otherwise would be accomplished independently.

Area of Rectangles. Using concrete representations supports students' movement from counting squares inside of a rectangle to a conceptual development of a formula. Battista (2003) found that students often try to fill in rectangles with drawings of squares (often unequal units) and then count the result one square at a time. Students who do this have not yet thought about a row of squares as a single unit that can be iterated.

To make this connection, review multiplication as seen in arrays with an emphasis on the structure of rows and columns of square units. When we multiply a length times a width, we are not multiplying squares times squares. Rather, the *length* of one side indicates how many squares will fit on that side. If this set of squares is taken as a unit, then the *length* of the other side (not a number of squares) will determine how many of these *rows of squares* can fit in the rectangular region. Then the number of square units covering the rectangle is the product of the length of a row and the number of rows: Rows × Column = Area.

Standards for Mathematical Practice
MP7. Look for and make sense of structure

Measuring one side tells how many squares fit in a row along that side, and it is called the *base* of the rectangle even though some people call it the *length* or the *width*. Measuring the other

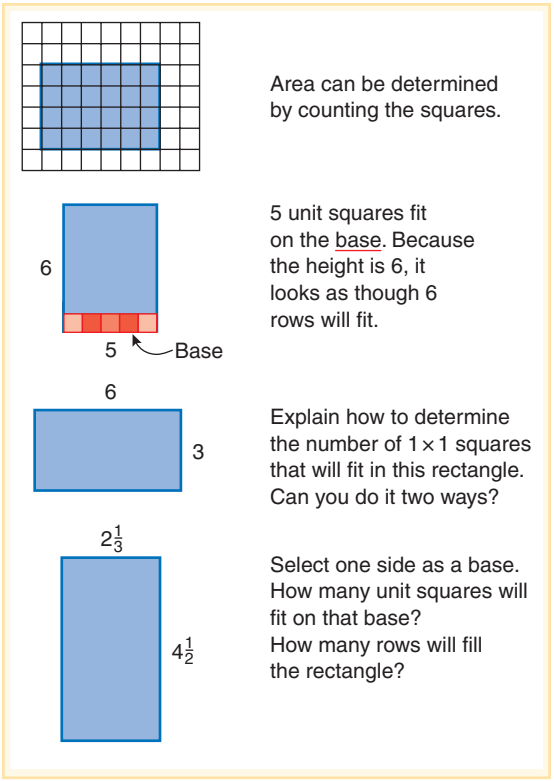

Area can be determined by counting the squares.
6
5
Base
5 unit squares fit on the base. Because the height is 6, it looks as though 6 rows will fit.
6
3
Explain how to determine the number of 1 × 1 squares that will fit in this rectangle. Can you do it two ways?
$2\frac{1}{3}$
$4\frac{1}{2}$
Select one side as a base. How many unit squares will fit on that base? How many rows will fill the rectangle?

FIGURE 18.14 Developing the formula for the area of a rectangle

side tells how many rows, or the *height*. As students rotate rectangles, they notice that either side could be the base and, therefore, the formula $A = B \times H$ generates the same area. Figure 18.14 offers tasks that focus on this idea.

Areas of Other Parallelograms. Once students understand $A = B \times H$ as the formula for rectangles, the next step is to determine the areas of other parallelograms. Use the following activity, in which students devise their own formula, building on what they know about rectangles.

Activity 18.21 Area of Parallelograms

CCSS-M: 6.G.A.1

Give students the Grid of Parallelograms activity page or, for a slightly harder challenge, other parallelograms drawn on plain paper with all dimensions—the lengths of all four sides and the height. Ask students to use what they know about the area of rectangles to find the areas of these parallelograms by counting squares in the grid or other means. Students should eventually find a method that will work for any parallelogram, even if not drawn on a grid.

Pearson eText Activity Page: Grid of Parallelograms

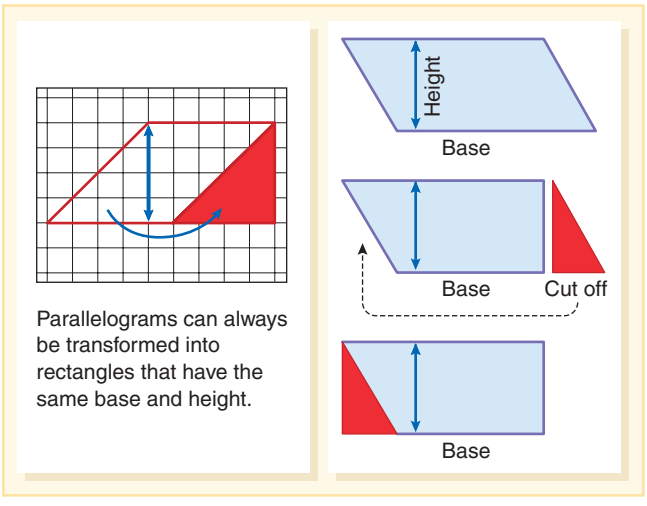


FIGURE 18.15 Transforming a parallelogram into a rectangle

If students need support, ask them to examine ways that the parallelogram is like a rectangle or how it can be changed into a rectangle by decomposing and recomposing the shape. As shown in Figure 18.15, a parallelogram can always be transformed into a rectangle with the same base, the same height, and the same area. Therefore, the formula for the area of a parallelogram is the same as for a rectangle: *base* times *height*.

Areas of Triangles. Knowing the relationship between a rectangle and a parallelogram, how can the area of a triangle logically follow? Engage students in a problem-based approach, as in the next activity.

As shown, two congruent triangles can always be arranged to form a parallelogram with the same base and the same height as the triangle. The area of the triangle will, therefore, be one-half that of the parallelogram, or $\frac{1}{2}$Base × Height Sound familiar? You may want to see if you can get students to break the formula to test whether it always works, or create their own formula (Gilbertson, 2020, p. 1015).

Pause & Reflect

There are three possible parallelograms, one for each triangle side that serves as a base. Will the computed areas always be the same? How might you engage students in this exploration? ●

Activity 18.22 Area of a Triangle

CCSS-M: 6.G.A.1

Provide students with the Grid of Triangles activity page. Challenge students to use what they know about the area of parallelograms to find the area of each of the triangles and to develop a method that will work for any triangle. For students with special needs in math, ask, "Can you find a parallelogram that is related to your triangle?" Then suggest that they fold a piece of paper in half, draw a triangle, and cut it out, making two identical copies. Use the triangles together to make a parallelogram (Figure 18.16). This composing of shapes provides a nice visual of how a triangle is related to a parallelogram.

Pearson eText Activity Page: Grid of Triangles

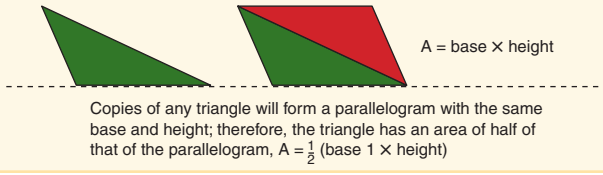

A = base × height
Copies of any triangle will form a parallelogram with the same base and height; therefore, the triangle has an area of half of that of the parallelogram, A = $\frac{1}{2}$ (base 1 × height)

FIGURE 18.16 Two congruent triangles always form a parallelogram

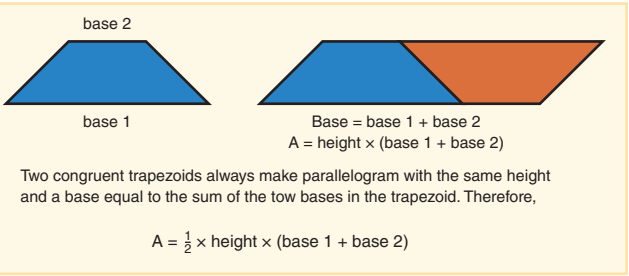

base 2
base 1
Base = base 1 + base 2
A = height × (base 1 + base 2)
Two congruent trapezoids always make parallelogram with the same height and a base equal to the sum of the tow bases in the trapezoid. Therefore,
A = $\frac{1}{2}$ × height × (base 1 + base 2)

FIGURE 18.17 Two congruent trapezoids always form a parallelogram

Areas of Trapezoids. After developing formulas for parallelograms and triangles, students are ready to develop the formula for the area of a trapezoid. There are many methods they can use, each related to decomposing the trapezoid into a simpler shape or combining shapes where ways to find these areas are already known and can be summed. One method uses the same approach that was used for triangles, except students work with two identical trapezoids. Figure 18.17 shows how this method results in the formula.

Here are a Grid of Trapezoids activity page and different approaches to finding the area of a trapezoid:

- Make a parallelogram inside the given trapezoid, using three of the sides.
- Make a parallelogram, using three sides that surround the trapezoid.
- Draw a diagonal, forming two triangles.
- Draw a line through the midpoints of the nonparallel sides. The length of that line is the average of the lengths of the two parallel sides.
- Draw a rectangle inside the trapezoid, leaving two triangles; put those two triangles together.
- Enclose the trapezoid in a larger shape (Manizade & Mason, 2014).

Pearson eText Activity Page: Grid of Trapezoids

***TECHNOLOGY* Note.** The relationship among the areas of rectangles, parallelograms, and triangles can be dramatically illustrated using dynamic geometry software such GeoGebra (www.geogebra.org). Draw two congruent segments on two parallel lines, as shown in Figure 18.18. Connect the segment end points to form a parallelogram and two triangles. The height is indicated by a segment perpendicular to the parallel lines. Either of the two line segments can be dragged left or right to change the slant of the parallelogram and triangle without changing the base or height. All area measures remain fixed! Also, explore the area of parallelograms and trapezoids app at NCTM Illuminations to test how changes in the base and height of these shapes affect the area. ■

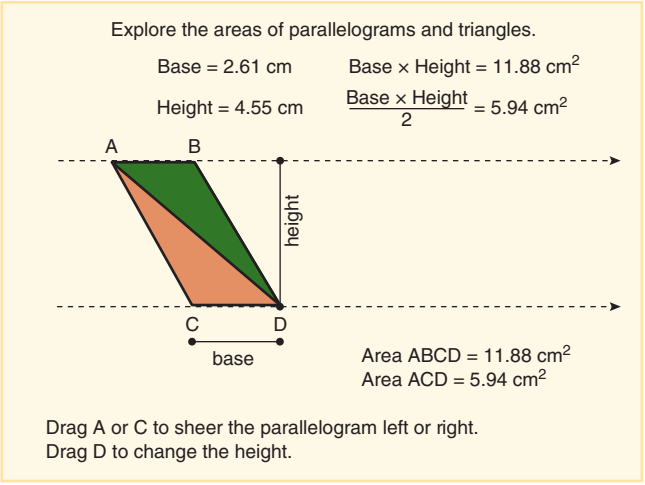

Explore the areas of parallelograms and triangles.
Base = 2.61 cm
Height = 4.55 cm
Base × Height = 11.88 cm²
Base × Height / 2 = 5.94 cm²
A
B
C
D
height
base
Area ABCD = 11.88 cm²
Area ACD = 5.94 cm²
Drag A or C to sheer the parallelogram left or right.
Drag D to change the height.

FIGURE 18.18 Figures with the same base and height maintain the same area

Surface Area. After learning to find the area of two-dimensional shapes, students are ready to explore the surface area of three-dimensional shapes. Sixth-grade students begin a study of surface area by exploring shapes with rectangular faces, which extends to shapes with faces of a variety of shapes in seventh grade.

Build on the knowledge students have of the areas of two-dimensional figures with *nets*, the two-dimensional faces of the shape (like an unfolded box). Students find the area of each face and then add them (https://www.nctm.org/Classroom-Resources/Illuminations/Interactives/Cubes/). An excellent way to do this is to create rectangular prisms, cubes, or cylinders with card stock sides held together by pieces of Velcro. In this way, students can pull apart the components of the net of the figure and calculate the area of the individual faces and add. For virtual materials that can unfold three-dimensional solids into nets, explore the Polypad feature of Mathigon (https://mathigon.org/polypad) or GeoGebra at (https://www.geogebra.org/m/n6EjQDw8).

Activity 18.23 Making Boxes

CCSS-M: 6.G.A.4; 7.G.B.6

Joan's Dine & Dessert Shop orders take-out boxes in three sizes (dinner, pie, and cake), and Joan wants an estimate of how much cardboard is used for each so that she can decide whether she should make her own boxes. Find how much cardboard is used for these boxes (assume they are taped and there is no overlap of cardboard).

- Dinner box: 7 in. × 7 in. × 3 in.
- Pie box: 5 in. × 4 in. × 3 in.
- Cake box: 8 in. × 8 in. × 5 in.

Use 2-Centimeter grid paper for students to create nets for each box. Then they calculate the surface areas. They may have picked different bases, and the illustrations may look different, but this opportunity can spur a discussion about why these different nets have equal surface areas. For multilingual learners (and others), the term *net* may be confused with the Internet or a net used in fishing, so have visual supports available.

Pearson eText Blackline Master: 2-Centimeter Grid Paper

Circumference. The relationship between the *circumference* of a circle (the distance around or the perimeter) and the length of the *diameter* (a line through the center joining two points on the circle) is one of the most interesting that students can discover and is an expectation for seventh grade in most states. The circumference of every circle is about 3.14 times as long as the diameter. The ratio is an irrational number close to 3.14 and is represented by the Greek letter π. So $\pi = \frac{C}{D}$, the circumference divided by the diameter. In equivalent forms, $C = \pi D$ or $C = 2\pi r$. (Activity 19.19 in Chapter 19 will discuss the concept of π and how students can discover this universal ratio.)

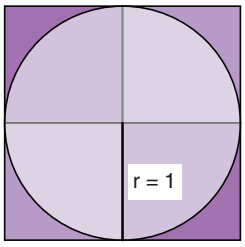


FIGURE 18.19 Radius square approach

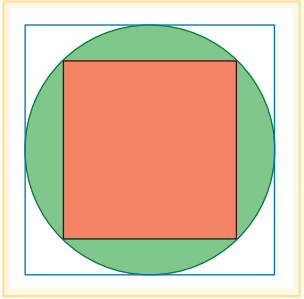


FIGURE 18.20 Using inscribed and superscribed squares to estimate the area of a circle

Area of Circles. As with polygons, students should investigate the area formula for circles rather than just be given the formula. Here are ways to explore area of circles that build conceptual understanding:

1. *Cover a circle with tiles.* Place 1-inch square tiles on the circle or cut out squares that can be glued on. The advantage of cutouts is that students can cut squares that are only partially inside the circle and place the extra pieces somewhere else. Then students can get a measure of the radius from the sides of the tiles.
2. *Use radius squares.* Draw a circle and draw a radius. Set the radius as the unit and use it to make squares—we will call them *radius squares*. Form a grid that is 2×2 radius squares and draw a circle inside (see Figure 18.19). By observation, you notice that the circle inscribed in this grid has an area of less than 4 radius squares (because much of the grid is outside the circle). Students need two identical copies of this picture—one to cut out the radius squares to see how many it takes to cover the circle in the other picture. Students can estimate that the number of radius squares needed is about 3.1 or 3.2. This technique reinforces the geometric idea that r^2 is a square with a side of r.
3. *Draw inscribed and superscribed squares.* This approach is similar to the method Archimedes used to approximate π. Give students or have students draw a circle on grid paper with a given radius. They draw a square inside the circle and outside the circle (Figure 18.20). Find the areas of the two squares, and average them to

find the area of the circle. Archimedes started using polygons that got closer to the shape of the circle (see the Archimedes Circles Activity Page for students to see how the estimates become closer). See the NCTM "Discovering the Area Formula for Circles" from NCTM's Classroom Resources (https://www.nctm.org/Classroom-Resources/ARCs/Measures-of-Circles/Circles-Lesson-2).

Pearson eText Activity Page: Archimedes Circles

4. *Cut to make a parallelogram.* Cut a circle apart into sectors and rearrange them to look like a parallelogram (see Figure 18.21). You may need to help them notice that the smaller the size of the sectors used, the closer the arrangement gets to a rectangle. This approach presents a common development of the area formula $A = \pi r^2$.

***TECHNOLOGY* Note.** Explore a simulation tool at Visnos (https://www.visnos.com/demos/pi) that not only shows the relationship of pi to the circumference of the circle but also shows the area of a circle, using slices that are animated.

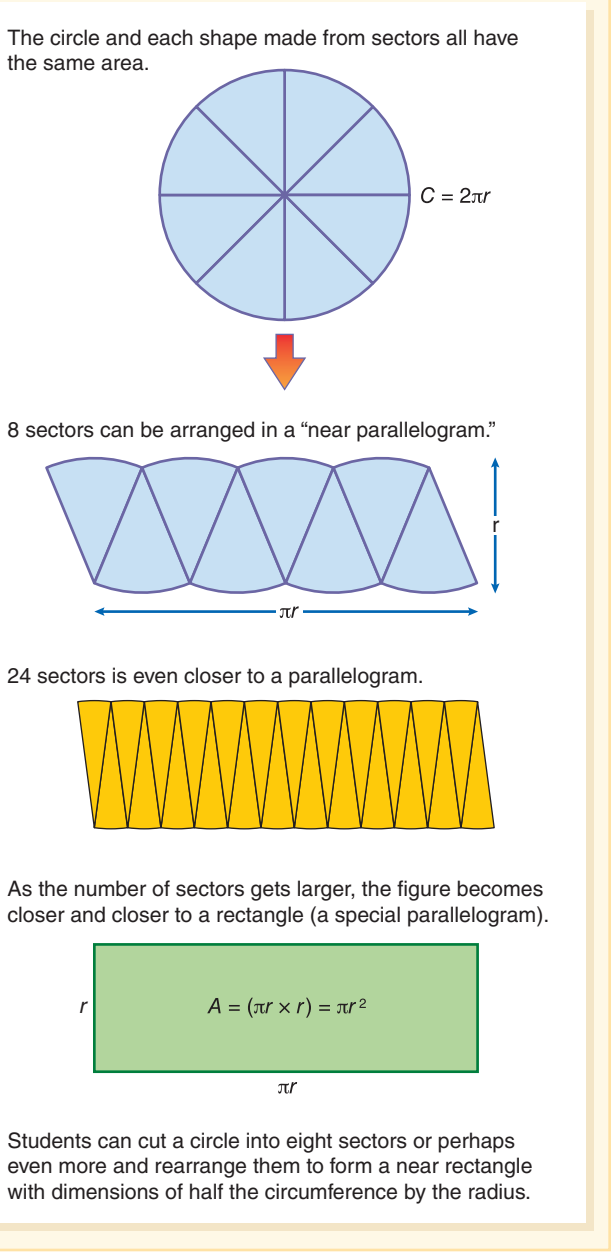

The circle and each shape made from sectors all have the same area.
C = 2πr
8 sectors can be arranged in a "near parallelogram."
r
πr
24 sectors is even closer to a parallelogram.
As the number of sectors gets larger, the figure becomes closer and closer to a rectangle (a special parallelogram).
r
A = (πr × r) = πr²
πr
Students can cut a circle into eight sectors or perhaps even more and rearrange them to form a near rectangle with dimensions of half the circumference by the radius.

FIGURE 18.21 Development of the area of a circle formula

Volume and Capacity

Learning Outcome 18.5 **Explain how volume is measured.**

Volume and *capacity* are terms for measures of the size of three-dimensional regions—a topic beginning in the fifth grade with rectangular prisms, with continuing emphasis in grades 6 and 8 with cylinders, cones, and spheres. The term *liquid volume* or *capacity* is generally used to refer to the amount that a container will hold. Standard units of capacity include quarts, gallons, liters, and milliliters. The term *volume* can be used to refer to the capacity of a container but is also used for the amount of space occupied by three-dimensional objects. Standard units of volume of solid figures are expressed in terms of cubic units such as cubic inches or cubic centimeters.

Comparison Activities

Comparing capacity focuses on how many of one unit (container) fills another unit (container). A method of comparing *capacity* is to fill one container with water and then pour that amount into the comparison container. In primary grades, most students understand the concept of "holds more" with reference to containers. Young children should compare the capacities of different containers, as in the following activity.

Another method of comparing volumes of solids (e.g., a ball and an apple), is displacement—placing each solid in water and seeing how much the water increased. In both cases, engage students in estimating and then comparing.

Pearson eText Video Example 18.8

In this video the late professor Janet Caldwell discusses appropriate units and tools for measuring volume.

Activity 18.24 Capacity Sort

CCSS-M: K.MD.A.2; 3.MD.A.2; 5.MD.C.3

Provide a variety of containers, with one marked as the target. Ask students to sort the collection into those that hold more than, less than, or about the same amount as the target container. Then use the Capacity Sort activity page to record the estimate. Provide a filler (such as beans, rice, water, or popcorn), scoops, and funnels. Working in pairs, have students measure and record results on the recording sheet. Discuss what students noticed (e.g., that fatter/rounder shapes hold more).

Pearson eText Activity Page: Capacity Sort

Use Concrete Representations

Standards for Mathematical Practice
MP8.

Two types of units can be used to measure volume and capacity: solid units and liquid units in containers. Solid units are objects like wooden cubes that can be used to fill the container being measured (https://www.nctm.org/Classroom-Resources/Illuminations/Interactives/Cubes/). The other unit is a small container that is filled with liquid, for example, and poured repeatedly into the container being measured (eventually using containers with scales). In each case, we count the units to quantify the amount. You might want to collect: liquid medicine cups, plastic containers, same-sized wooden cubes, or packing peanuts to provide fill.

The following activity explores the volumes of two boxes.

Activity 18.25 Box Comparison: Cubic Units

CCSS-M: 5.MD.C.3, 5.MD.C.4, 5.MD.C.5; 6.G.A.2; 7.G.B.6

Provide students with (or have them make) a pair of small card stock boxes. Draw a square on each corner of a rectangular piece of paper (see Figure 18.22). Cut on the solid lines, fold the box up, wrap the corner squares to the outside, and tape them to the sides. Use unit dimensions that match the cubes that you have for units. Students are also given a corresponding ruler. (If you use 2-centimeter cubes, make a ruler with 2-centimeter units.) Ask students to decide which box has the greater volume or if they have the same volume.

Here are some suggested box dimensions ($L \times W \times H$):

$$6 \times 3 \times 4 \qquad 5 \times 4 \times 4 \qquad 3 \times 9 \times 3 \qquad 6 \times 6 \times 2 \qquad 5 \times 5 \times 3$$

Students should use words, drawings, and numbers to explain their conclusions. Repeat with boxes with fractional values. For example, ask students to estimate and then determine which of the following shipping boxes has the greatest and least volume:

$$\tfrac{1}{4}\text{ ft} \times 3\text{ ft} \times 2\tfrac{1}{2}\text{ ft} \qquad 4\text{ ft} \times \tfrac{3}{4}\text{ ft} \times \tfrac{1}{2}\text{ ft} \qquad \tfrac{5}{12}\text{ ft} \times 2\text{ ft} \times \tfrac{3}{4}\text{ ft} \qquad 2\text{ ft} \times \tfrac{1}{4}\text{ ft} \times 3\text{ ft}$$

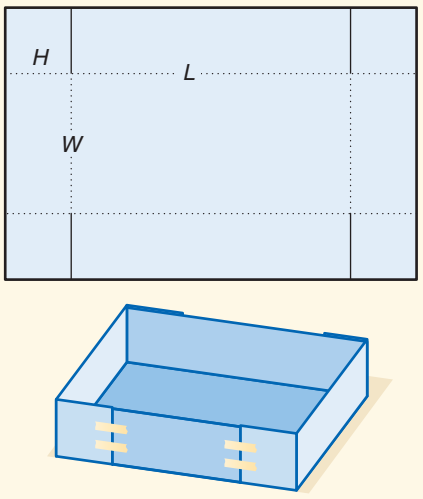


FIGURE 18.22 Making boxes to fill

A useful hint in the preceding activity was first to figure out how many cubes will fit on the bottom of the box. Some students may find a pattern leading to a multiplicative formula for the volume. The boxes can be completely filled with cubes to confirm conjectures. Also found to be very successful for solving volume problems is a gridded version of the 3D shape on paper that illustrates the individual cube structure (Vasilyeva et al., 2013).

Tools for measuring capacity are generally used for small amounts of liquids or pourable materials such as rice or water. These tools are commonly found in kitchens and laboratories. Students should use measuring cups to explore recipes such those in the *Better Homes and Gardens New Junior Cookbook* (2018), which provide student-friendly opportunities to use units of capacity.

The following two small group activities focus on liquid volume.

Activity 18.26 That's Cool

CCSS-M: 3.MD.A.2

Give students beakers marked in milliliters. Tell the groups they will receive three ice cubes. First, they must estimate how many milliliters of water will be in the beaker when the ice cubes melt. Then the ice is placed in each team's container, and students wait until the ice turns to water. What was the difference between their estimates and their actual answers? Use a dot plot or line plot to record and then discuss the different measures.

Activity 18.27 Squeeze Play

CCSS-M: 3.MD.A.2

Give each team a beaker marked in milliliters. Have several stations set up with buckets and different-sized sponges. Have students first estimate how much water they can squeeze from each sponge, using the hand they do not write with. Does a sponge that is two times larger than another sponge squeeze out two times the water? What do the students notice?

Developing Formulas for Volume

The relationships between the formulas for volume are completely analogous to those for area. A common challenge that repeats from two- to three-dimensional shapes is that students can confuse the meaning of height and base in their use of formulas. Note that the shapes in Figure 18.23 each have a slanted side and a given height. As mentioned, the base of the figure can be any flat surface of a figure, and to visualize the height, visualize the figure sliding under a doorway. Keep this visual in mind as you use precise language to develop formulas for volume.

Notice similarities between rectangles and prisms, between parallelograms and slanted (oblique) prisms, and between triangles and pyramids as you read the next sections. Not only are the formulas related, but the processes for developing the formulas are similar.

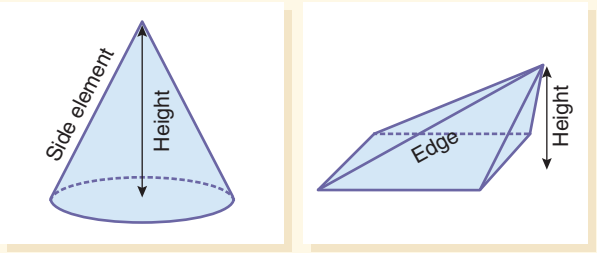


FIGURE 18.23 Heights are not always measured along an edge or surface

Volume of Cylinders.

A *cylinder* is a solid with two congruent parallel bases and sides with parallel elements that join corresponding points on the bases. There are several special classes of cylinders, including *prisms* (with polygons for bases), *right prisms*, *rectangular prisms*, and *cubes* (Zwillinger, 2011). Interestingly, all of these solids have the same volume formula, and one formula is analogous to the area formula for rectangles and other parallelograms.

Review Activity 18.25, comparing how the development of the volume formula is parallel to the development of the area of a rectangle, as shown in Figure 18.24. The *area* of the base (instead of *length* of the base for rectangles) determines how many *cubes* can cover the base, forming a single unit—a layer of cubes. The *height* of the box then determines how many of these *layers* will fill the box, just as the height of the rectangle determined how many *rows* of square units would fill the rectangle.

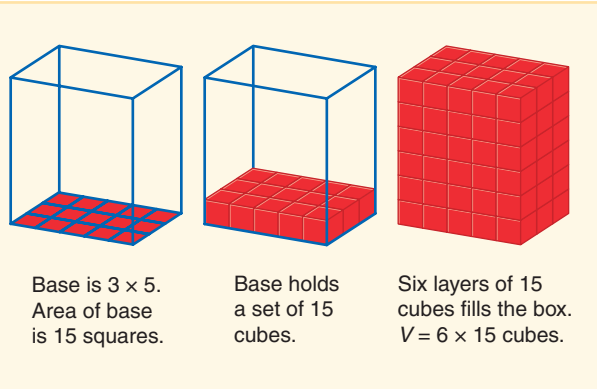


FIGURE 18.24 Developing the formula for volume of a right prism

Recall that some parallelograms can be thought of as a slanted rectangle. Show students a stack of three or four decks of cards (or a stack of books). When stacked straight, they form a rectangular solid. The volume, as just discussed, is $V = A \times H$, with A equal to the area of one playing card (the base). Now, if the stack is slanted to one side, as shown in Figure 18.25,

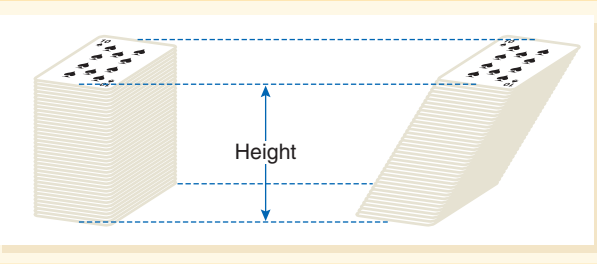
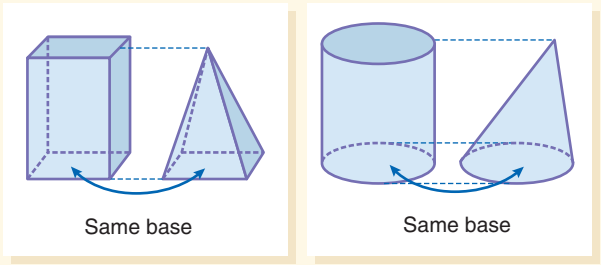


FIGURE 18.25 Prisms with the same base and height have the same volume



The volume of a pyramid or cone is one-third the volume of a prism or cylinder with the same base and same height.

FIGURE 18.26 Comparing volumes of pyramids to prisms and cones to cylinders

what will the volume be? Students should be able to argue that this figure has the same volume (and same volume formula) as the original stack.

What if the playing cards were another shape? If they were circular (cylinder), the volume would still be the area of the base times the height; if they were triangular (triangular prism), still the same. The conclusion is that the volume of *any* cylinder is equal to *Area of the Base* × *Height*.

Volumes of Cones, Pyramids, and Spheres. Understanding the formula for the volumes of these shapes is expected in eighth grade. Recall that when parallelograms and triangles have the same base and height, the areas are in a 2-to-1 relationship. Interestingly, the relationship between the volumes of cylinders and cones with the same base and height is 3 to 1. That is, *area* is to *two*-dimensional figures what *volume* is to *three*-dimensional figures. Furthermore, triangles are to parallelograms as cones are to cylinders.

Standards for Mathematical Practice MP2.

To investigate this relationship, use translucent plastic solids and have students estimate the number of times the volume of the pyramid will fit into the prism. Have students test their predictions by filling the pyramid with water and emptying it into the prism. They will discover that the volume of three pyramids will precisely fill a prism with the same base and height (see Figure 18.26), making the volume of a pyramid (or cone) one-third the volume of the corresponding cylinder.

Pearson eText
Video Example 18.9
In this video consider how this class engages with the displacement volume method.

Using the idea of area of the base times height, it is possible to explore the volume of a sphere as $\frac{2}{3}$ of the volume of a cylinder with the same height and base as identified through pouring water into a cylinder with the same height and base. The height of the matching cylinder is the sphere's radius doubled ($2r$). The volume of the cylinder is the area of the base $(\pi r^2) \times \text{height}(2r)$, so we find that the volume of the corresponding sphere is $\frac{2}{3}(2\pi^3)$ or $\frac{4}{3}\pi r^3$.

The connectedness of mathematical ideas is beautifully illustrated with the connections of all of these volume formulas as the single concept of *base times height*. A conceptual approach to the development of formulas helps students understand that they are meaningful and efficient ways to measure different attributes of the objects around us. After developing formulas conceptually, students can derive formulas from what they already know. Mathematics makes sense!

The Relationship between Volume and Surface Area

Just as there is an important relationship between area and perimeter, there is a relationship between volume and surface area.

Activity 18.28 Fixed Volume: Comparing Prisms

CCSS-M: 5.MD.C.3; 5.MD.C.4; 5.MD.C.5; 6.C.A.2

Give each pair of students a supply of centimeter or inch cubes and a Comparing Prisms recording sheet. For multilingual learners, provide a visual of a rectangular prism, labeling key words such as *length, width, height, surface area, cube, volume,* and *side*. Ask students to use 36 (or 64) cubes to build different rectangular prisms with that volume and record in a table the surface area for each. Ask students to describe any patterns that they notice as they compare the dimensions of each prism to its surface area. What happens as the prism becomes less like a tall, skinny box and more like a cube? Also explore the Expanded Lesson: Fixed Volume.

Pearson eText Activity Page: Comparing Prisms Recording Sheet
Pearson eText Expanded Lesson: Fixed Volume

Try the following activity yourself first. What do you think?

Activity 18.29 Which Silo Holds More?

CCSS-M: 5.MD.C.3; 6.G.A.4; 7.G.B.7

Give pairs of students two sheets of equal-sized paper. With one sheet, they make a tube shape (cylinder) by taping the two long edges together. They make a shorter, fatter cylinder from the other sheet by taping the short edges together. Then ask, "If these were two silos, would they hold the same amount, or would one hold more than the other?" To test the conjectures, use a filler such as beans, popcorn, or pasta. Place the skinny cylinder inside the fat one. Fill the skinny tube and then lift it up, allowing the filler to empty into the fat cylinder. What do they find?

The eventual goal is for students to realize that surface area does not dictate volume and to recognize that there is a pattern between surface area and volume in the same way as there is between area and perimeter. Namely, prisms that are more cube-like have less surface area than prisms with the same volume that are long and narrow.

Weight and Mass

Learning Outcome 18.6 **Demonstrate strategies for comparing weights of objects.**

Weight is a measure of the pull or force of gravity on an object. *Mass* is the amount of matter in an object and a measure of the force needed to accelerate it. On the moon, where gravity is much less than on Earth, an object has a smaller weight than on Earth but the identical mass. For practical purposes, on Earth, the measures of mass and weight will be about the same. In this discussion, the terms *weight* and *mass* will be used interchangeably.

Although the concept of heavier and lighter begins to be explored in kindergarten, the notion of units of weight or mass is introduced in third grade. At any grade level, experiences with informal unit weights are good preparation for standard units and scales.

Comparison Activities

The most conceptual way to compare weights of two objects is to hold one in each hand, extend your arms, and experience the relative downward pull on each, which effectively communicates to a pre-K–1 student what heavier or weighs more means. This personal experience can then be transferred to one of two basic types of scales—balances and spring scales.

When students place two objects in the two pans of a balance, the pan that moves downward is understood to hold the heavier object. As discussed in Chapter 13, the pan balance therefore is an excellent tool for exploring the equal sign and inequalities. Even a relatively simple balance detects small differences. If two objects are placed one at a time in a spring scale, the heavier object pulls the pan down farther. Visit https://www.teacherled.com/2015/02/04/reading-scales-mass/ to explore reading scales to levels of precision. Both balances and spring scales have real value in the classroom. (Technically, spring scales measure weight and balance scales measure mass. Why?)

Pearson eText
Video Example 18.10
In this video consider how students can put items in both hands to make a comparison of weights.
https://www.youtube.com/watch?v=fqG4EDClyRM

Activity 18.30 Which Weighs More?

CCSS-M: K.MD.A.2

Give a pan balance to groups of three students or set as a learning station. Use the Which Weighs More? 1 activity page and have students first estimate and then test with the actual items which collection of objects in each row weighs more. The second activity page is blank, and students can be challenged to draw their own problems for others.

Pearson eText Activity Page: Which Weighs More? 1
Pearson eText Activity Page: Which Weighs More? 2

Use Concrete Representations

Any collection of uniform objects with the same mass can serve as nonstandard weight units—that is, for light objects, large paper clips, wooden blocks, or plastic cubes, or coins (U.S. nickels weigh 5 grams and pennies weigh 2.5 grams). Large metal washers from home improvement stores can weigh slightly heavier objects. Standard weights are needed to weigh things as heavy as a kilogram or more.

Weight cannot be measured directly. With a balance scale, place an object in one pan and weights in the other pan until they balance. In a spring scale, first place the object in and mark the position of the pan on a piece of paper taped behind the pan. Remove the object and place just enough weights in the pan to pull it down to the same level. Discuss how equal weights will pull the spring with the same force.

Angles

Learning Outcome 18.7 **Explain strategies for teaching angle measurement.**

Angle units are based on an angle formed by rays extending from the center of a circle. The attribute of angle size might be called the "spread of the angle's rays" or the measure of rotation between two lines that meet at a point (Bustang et al., 2013). Angles are composed of two rays that are infinite in length with a common vertex and are measured by how widely or narrowly the two rays are spread apart or rotated around that vertex. An angle with n turns of $\frac{1}{360}$ is an angle of n degrees.

Understanding the concept (or attribute) of angle and measuring angles with degrees begins at fourth grade and develops in seventh grade as students compare supplementary, complementary, vertical, and adjacent angles. Just like other measurements, angle measures must be developed conceptually using comparisons and concrete tools. Do not begin with protractors until students understand the concept of angle measures.

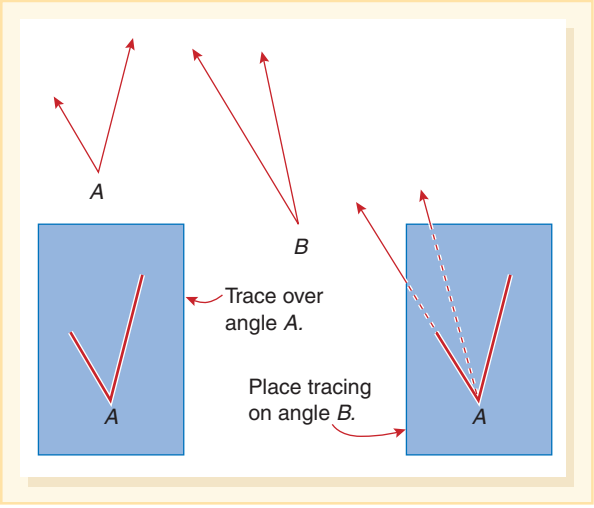


FIGURE 18.27 Which angle is larger?

Comparison Activities

To help students conceptualize the attribute of the spread of the rays, two angles can be directly compared by tracing one and placing it over the other (see Figure 18.27). Be sure to have students compare angles with rays of different lengths. A student may mistakenly think a wide angle with short rays is smaller than a narrow angle with long rays (Sari & Tertemiz, 2021). As soon as students can differentiate between a large angle and a small one, regardless of the length of the rays, you can move to measuring angles.

Use Concrete Representations

A unit for measuring an angle must be an angle. Nothing else has the same attribute of spread that we want to measure. Contrary to what many people think, you do not need to use degrees to measure angles.

Activity 18.31 A Unit Angle

CCSS-M: 4.MD.C.5; 7.G.B.5

Give each student an index card. Have them draw a narrow angle on the card, using a straightedge, and cut it out or use the Wedges activity page. The resulting wedge can then be used as a unit of angular measure by counting the number of wedges that will fit in a given angle (see Figure 18.28). Distribute the Angles activity page and have students use their angle unit to measure the angles. Because students made different unit angles, the results will differ and can be discussed in terms of unit size. Multilingual learners may need to know the meaning of the word *degree* in this context.

Pearson eText Activity Page: Degrees and Wedges
Pearson eText Activity Page: Angles

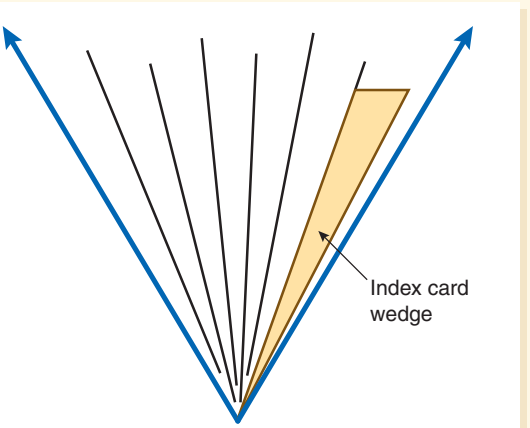
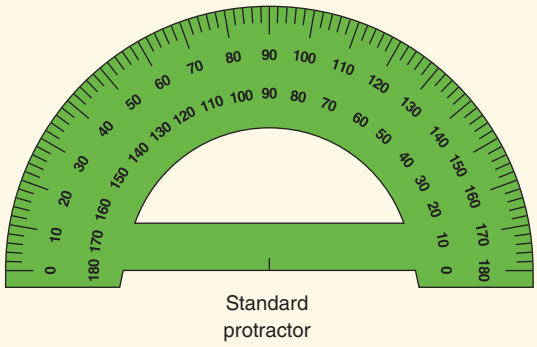


FIGURE 18.28 This angle measures about $7\frac{1}{2}$ wedges



The marks on a plastic protractor are the rays of degrees. A degree is just a very small angle.

FIGURE 18.29 A protractor measures angles

Activity 18.31 illustrates that measuring an angle is the same as measuring area; unit angles are used to cover the spread of an angle just as square units cover an area. Once this concept is understood, move on to the use of measuring tools.

Use Protractors

The tool commonly used for measuring angles in classrooms is the protractor (Figure 18.29), although phones have apps that may take over in the real world. Yet, the protractor is often poorly understood and used improperly (AMTE, 2017). Part of the difficulty is that the units (degrees) are so small. It would be almost impossible for students to cut out and use a single degree to measure an angle accurately. In addition, the numbers on protractors run clockwise and counterclockwise along the edge, making the scale hard to interpret without a strong conceptual foundation. Notice that the units of degrees are based on an angle where the vertex of the rays is located at the midpoint of a circle, creating an arc. A one-degree angle is where the arc is $\frac{1}{360}$ of a circle, as illustrated on the Degrees and Wedges page.

Pearson eText Activity Page: Degrees and Wedges

To introduce the protractor, students can make non-standard waxed-paper protractors (see Figure 18.30) or try funky protractors (Hardison & Lee, 2020). Another idea is to use an angle maker as in Activity 18.32. Each of these ideas helps students develop approximate mental images of angle sizes, helping students estimate angle measures as well as make sense of the scale on the protractor.

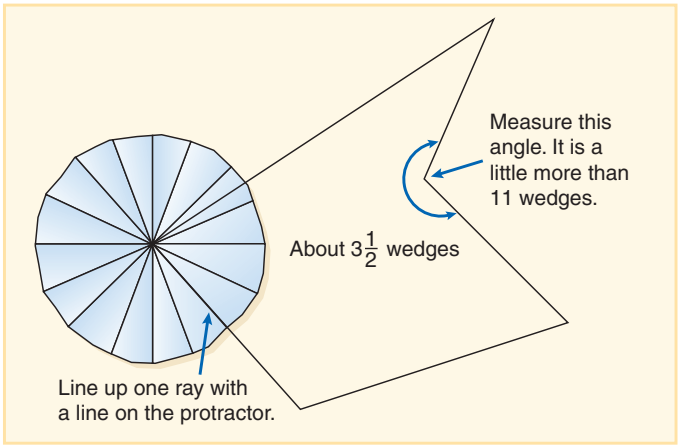


FIGURE 18.30 Measuring angles with a waxed-paper protractor

Activity 18.32 Angle Maker

CCSS-M: 4.MD.C.5; 7.G.B.5

Cut and merge two different-colored paper plates in the same way as the Rational Number Wheel in Figure 16.7. Have students rotate the plates to match angles observed or to estimate benchmark angles such as 30, 45, 60, 90, 135, 180, and 270 degrees and hold up their plate. If students have a strong grasp of the approximate sizes of angles, this angle sense will give them the background needed to use standard measuring tools effectively.

Pearson eText Blackline Master: Rational Number Wheel

Activity 18.33 Angle Relationships

CCSS-M: 4.MD.C.5; 7.G.B.5

To explore supplementary, complementary, vertical, and adjacent angles, invite students to discover these relationships by measuring angles and looking for patterns. Have students:

- Draw two intersecting lines (not perpendicular) on paper.
- Measure the four angles created and record the angles in degrees.
- Notice and record any relationships.
- Discuss as a whole class whether these relationships are always true.
- Link the findings to the vocabulary of supplementary, vertical, and adjacent angles.

Time

Learning Outcome 18.8 **Describe a representation for teaching elapsed time.**

Time is different from most other attributes commonly measured in school because it cannot be seen or physically manipulated. This difference makes it more difficult for students to comprehend units of time or how those units are matched against a given time period or duration.

Comparison Activities

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Time can be thought of as the duration of an event from its beginning to its end. For students to understand the attribute of time adequately, they should make comparisons of events that have different durations. If two events begin at the same time, the shorter duration will end first and the other will last longer—for example, which fidget spinner spins longer? However, this form of comparison focuses on the ending of the duration rather than on the duration itself. To think of time as something that can be measured, it is helpful to compare two events that do not start at the same time, which requires the use of some form of time measurement.

As students learn about seconds, minutes, and hours, they develop some concept of how long these units are. Have students time familiar events in their daily lives: brushing teeth, eating dinner, riding to school, doing homework. Point out the duration of short and long events during the school day. Timing events of $\frac{1}{2}$ a minute to 2 minutes can be engaging, as in the following activity.

Activity 18.34 Ready for the Bell

CCSS-M: 1.MD.B.3

Give students a Clock Faces recording sheet. Secretly set a timer to go off at the hour, half hour, or minute. When the timer rings, students should look up and record the time on the clock face and in numerals on the recording sheet. This activity motivates students not only to think about telling time but also to consider the relationship between analog clock reading and digital recording. Elapsed time can also be explored by discussing the time between timer rings.

Pearson eText Blackline Master: Clock Faces

Pearson eText
Video Example 18.11
In this video Jennifer Suh highlights key components in teaching the telling of time.

Reading Clocks

The common tool for measuring time is the clock (https://apps.mathlearningcenter.org/math-clock/). However, learning to tell time has little to do with time measurement and more to do with the skills of learning to read a scale. Clock reading may be a difficult skill to teach. Starting in first grade, students are usually taught to read clocks to the hour then the half hour and, in second and third grades, to 5- and 1-minute intervals.

Digital clocks permit students to read times easily, but they do not relate to benchmark times. To know that a digital reading of 7:58 is nearly 8:00, the student must know that there are 60 minutes in an hour, that 58 is close to 60, and that 2 minutes is not a very long time. Analog clocks show close-to times visually, without the need to know how many minutes in an hour.

Here are ways to help students focus on the actions and functions of the minute and hour hands.

1. Begin with a one-handed clock by breaking off the minute hand from a regular clock. Use lots of approximate language: "It's about 7 o'clock." "It's a little past 9 o'clock." "It's halfway between 2 o'clock and 3 o'clock" (see Figure 18.31).
2. Discuss what happens to the big hand as the little hand goes from one hour to the next. When the minute hand is at 12, the hour hand is pointing precisely to a number. If the hour hand is about halfway between numbers, about where would the minute hand be? If the hour hand is a little past or before an hour (10 to 15 minutes), where would the minute hand be?
3. Use two clocks, one with only an hour hand and one with two hands. Cover the two-handed clock. Periodically during the day, direct attention to the one-handed clock. Have students predict where the minute hand should be. Uncover the other clock and discuss.
4. After step 3, teach 5-minute intervals by counting by fives, going around the clock. Instead of saying the minute hand is "pointing at the 4," transition to the language of "It is about 20 minutes after the hour." As skills develop, suggest that students look first at the hour hand to learn approximately what time it is and then focus on the minute hand for precision.
5. Predict the reading on a digital clock when shown an analog clock and set an analog clock when shown a digital clock. See the interactive teaching clock at Visnos Mathematics (http://www.visnos.com/demos/clock).
6. Relate the time after the hour to the time before the next hour. This approach is helpful for telling time and developing number sense.
7. Finally, discuss a.m. and p.m.

FIGURE 18.31 Approximate time with one-handed clocks

Standards for Mathematical Practice
MP7. Look for and make sense of structure

The following activity assesses students' ability to read an analog clock.

Activity 18.35 One-Handed Clocks

CCSS-M: 1.MD.B.3; 2.MD.C.7

Prepare a page of clock faces by drawing an hour hand on each. Include placements that are approximately a quarter past the hour, a quarter before the hour, half past the hour, and some that are close to but not on the hour. For each, ask students to write the digital time and draw the corresponding minute hand. For multilingual learners, note that telling time is done differently in different cultures. For example, in Spanish, any time past 30 minutes is stated as the next hour minus the time until that hour. For example, 10:45 is 15 minutes before 11, or 11 minus a quarter. Be explicit that in English, it can be said either way: "10:45" or "a quarter until 11."

Pearson eText Blackline Master: Clock Faces

Elapsed Time

Determining combinations and comparisons of time intervals in minutes is a skill required, starting in grade 3. If given the digital time or the time after the hour, students should be able to tell how many minutes to the next hour. The mental process used is counting on in multiples of 5 minutes. Avoid having students use pencil and paper to subtract 25 from 60; possibly using an analog clock face to support the skip-counting at the beginning will help. This counting-on approach provides a foundation for problems involving elapsed time, which can be a challenge when the period of time spans noon or midnight.

Figuring the time from, say, 8:15 a.m. to 11:45 a.m. is a multistep task that requires deciding what to do first and keeping track of the intermediate steps (https://www.teacherled.com/2015/01/27/time-change/). In this case, you could count on hours from 8:15 to 11:15 and add on 30 minutes. But then what do students do if the end points are 8:45 and 11:15?

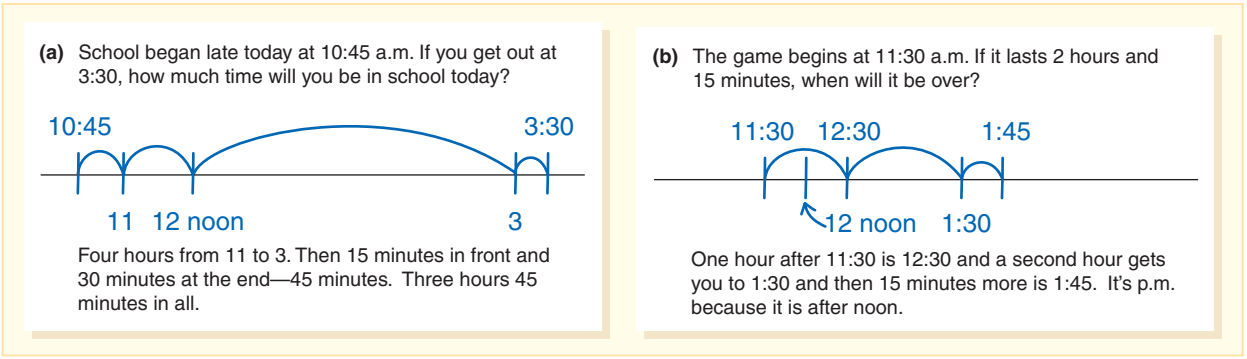

(a) School began late today at 10:45 a.m. If you get out at 3:30, how much time will you be in school today?
10:45
3:30
11
12 noon
3
Four hours from 11 to 3. Then 15 minutes in front and 30 minutes at the end—45 minutes. Three hours 45 minutes in all.
(b) The game begins at 11:30 a.m. If it lasts 2 hours and 15 minutes, when will it be over?
11:30
12:30
1:45
12 noon
1:30
One hour after 11:30 is 12:30 and a second hour gets you to 1:30 and then 15 minutes more is 1:45. It's p.m. because it is after noon.

FIGURE 18.32 Using open timelines for solving elapsed-time problems

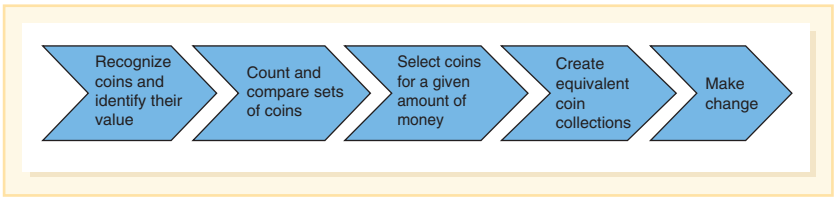
There is also the task of finding the end time, given the start time and elapsed time, or finding the start time, given the end time and the elapsed time. In keeping with the spirit of problem solving and the use of semi-concrete representations (CSA), consider having students sketch an open timeline (similar to the open number line, discussed for computation). It is important to focus on flexibility in using the timeline, because there are various approaches (Dixon, 2008). For example, in Figure 18.32, a student might count by full hours from 10:45 (11:45, 12:45, 1:45, 2:45, 3:45) and then subtract 15 minutes. Another student might count 15 minutes to 11:00 and then count by full hours to 3:00 and then add on 30 minutes.

Money

Learning Outcome 18.9 **Explain strategies for counting a collection of coins.**

Here is a progression of the money ideas and skills required in the primary grades. Throughout this process, students should experience story problems involving money situations.

Pearson eText
Video Example 18.12
In this video consider how the student solves a story problem involving money. Should the teacher ask other questions that link the amounts to the money, such as, "What is 100 cents also equal to?" in an effort to get the student to use money denominations such as one dollar or four quarters? Why is that flexibility important?


Recognize coins and identify their value
Count and compare sets of coins
Select coins for a given amount of money
Create equivalent coin collections
Make change

Recognize Coins and Identify Their Values. The names and values of our coins are conventions of our social system. Students learn these names the same way that they learn the names of physical objects in their daily environment—through exposure and repetition. For these values to make sense, students must understand 5, 10, and 25 (https://apps.mathlearningcenter.org/money-pieces/) and, through experiences, eventually think of these quantities without seeing countable objects. They need to interpret "This is 5" as a coin value while we are pointing to a single nickel—a challenge for students who are still counting objects by ones. Coin value lessons should focus on purchase power—a dime can buy the same thing that 10 pennies can buy.

Count and Compare Sets of Coins. Naming the total value of a group of coins is the same as mentally adding their values. Second-grade students can be asked to do the mental math required in counting a collection of different coins.

Students often sort their coins and start counting from the highest values. The next activity is a preparation for counting money.

Activity 18.36 Money Skip-Counting

CCSS-M: 2.MD.C.8

Have students start skip-counting by one number and, at your signal, shift to a count by a different number. Use any two of these amounts: 100, 50, 25, 10, 5, and 1—for example, 25 and 10—and write them on the board. Always start with the larger number (25) and have students skip-count by 25 from that number. After three or more counts, raise your hand to indicate a pause in the counting. Then point to 10. Students continue the skip-count from where they left off but now skip-count by tens. For multilingual learners, invite them to share the coins from their country and see how their values compare to our coins.

Activity 18.37 Hundred Chart Money Count

CCSS-M: 2.MD.C.8

Give students a bottom-up hundred chart and a collection of play money. Begin with two different coins—for example, a quarter and a dime. Use place value to represent the 25 cents in the same way students have previously used the chart (count two rows up and over five spaces to the right). Place the quarter on the 25 space and then count 10 more (up one row) and place the dime on 35. The total of the two coins is 35 cents. Use other coin collections and what students already know about patterns on the hundred chart to calculate the total value.

Pearson eText Blackline Master: Bottom-Up Hundred Chart

Select Coins for a Given Amount of Money. Rather than count coins to find out how much, this concept involves getting a total and figuring out which coins can make that total. Still using the Bottom-Up hundred chart, point at a particular number (e.g., 57), and have students find one way to use coins to equal that amount.

Create Equivalent Coin Collections. Continue the exploration and ask students to find at least three ways to reach 57. This can become a routine in which an amount of money is posted and students are challenged to make that quantity in at least two ways. Students will notice patterns like whenever they used a dime, another option is use two nickels.

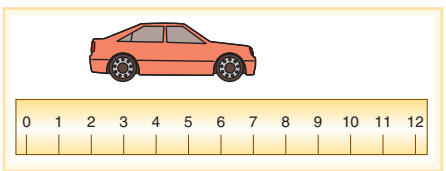
Make Change. The bottom-up hundred chart can also be used to make change. With a marker on 57, ask students to figure out the change from $1.00. Students may count down from 100 to 57 or may count up from 57 to 100. Because adding on to find a difference is such a valuable skill, it makes sense to give students experiences with think-addition to make change. As students become more skillful at adding on, they can see the process of making change as an extension of a skill already acquired.

There are numerous money games and activities at SplashLearn (https://www.splashlearn.com/counting-money-games) that will help your students practice the ideas in this section.

Common Challenges in Measurement

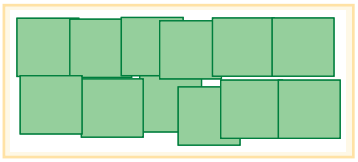
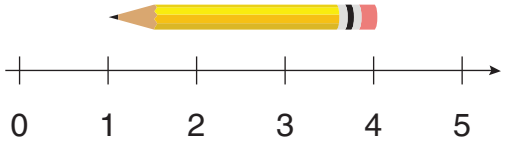
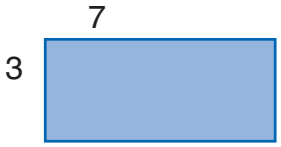
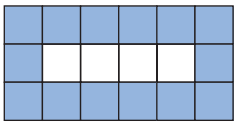
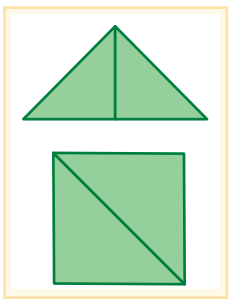
See Table 18.2 for examples of common challenges about measurement and ways to assist.

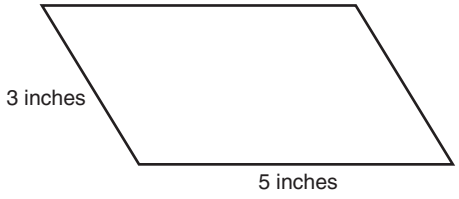
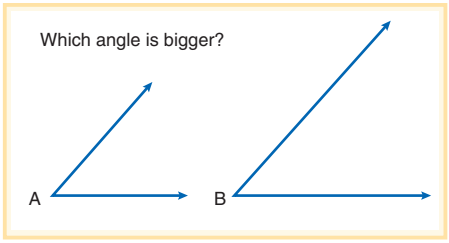
TABLE 18.2 COMMON CHALLENGES IN MEASUREMENT AND HOW TO HELP

Common Challenge	What It Looks Like	How to Help
1. Focus on the ending point of a ruler rather than the length.	Students read the number on the ruler that aligns with the object's right edge. 	• Move items along a ruler and ask students if they believe the item has changed size. • Iterate units to show how long the item is and then show how those units match the units on the ruler. • Give students experiences with the Broken Ruler.

(Continued)

TABLE 18.2 ***(CONTINUED)***

Common Challenge	What It Looks Like	How to Help
2. Place multiple individual units with gaps between the units or with some of the units overlapping; or students use unequal units.	Students have gaps between units or use unequal units when measuring length: Students overlap units when measuring area: 	• Ask students to estimate the measure before measuring the object and explain why it is a reasonable estimate. • If students use unequal units, ask what the number refers to (e.g., the number of white rods or brown rods). • If students overlap length units, have them use linking cubes to measure length. • If students overlap area units, use a unit with some thickness, making it difficult to overlap the pieces (e.g., square tiles or pattern blocks).
3. Count tick marks rather than units.	It's four units.  	• Have students iterate units to see how a ruler is made. Then emphasize that units are what we are counting and how the number or tick mark indicates the end of the unit. • Have students make their own ruler (see Activity 18.11).
4. Bigger units result in a larger measure.	Student measures an object's length using small paperclips as the unit. When measuring the same object using paperclips twice the size, the student estimates the measure will be a larger number.	• Provide many experiences where the students measure the same object's attribute with two different-sized units (where one of the larger units is equal to several of the smaller units). Have the student predict how the measures will relate to the size of the units.
5. Measure perimeter by counting two sides.	The perimeter is 10.  	• This is an artifact of problems having only two sides labeled with numbers. Have students draw a line around the rim or highlight the sides to show what is being measured. Then suggest they label the length of each side before adding.
6. Measure perimeter by counting square units rather than side lengths.	Students count the number of squares around the border of the figure rather than the length of the sides of the squares. This results in a measure that is 4 units less than the actual perimeter. 	• Focus on perimeter being a length rather than an area to cover. Show with gestures how the corner needs two units (of fence, for example).
7. If you know the area of a figure, you know the perimeter and vice versa (Amore and Pinilla, 2006).	Students will say as the area increases so does the perimeter. Students use the formula for area to find the perimeter.	• Avoid teaching the topics of area and perimeter at the same time. • Focus on the similarities, differences, and relationships between the two measurements. • Review Activity 18.20.
8. Cutting a shape into parts and rearranging the pieces changes its area.	The first shape has more area than the second shape. 	• Have students use the same set of pattern blocks to create 3 to 4 designs, tracing around each design on paper. Discuss how they can compare each area. They may need to iterate one pattern block (i.e., green triangle), recording how many are used for each shape.

Common Challenge	What It Looks Like	How to Help
9. Area formula can be stated but students cannot describe the meaning of area.	When given a blob shape, students will say it has no area as it doesn't have a length and width.	• Go back to the use of nonstandard units to focus on area as a process of covering two-dimensional space. • Use Activity 18.17.
10. As the side length of the figure doubles, the area doubles.	When asked what happens to the area of a two-by-two-inch square if the sides double to four by four inches, students say the area doubles.	• Act this problem out. Because the figure has two dimensions, this is a change in square units, not a change in lengths alone. Students should place tiles down on the new figure to prove that the area is four times as large—not doubled.
11. Uses $L \times W$ for the area of other figures.	Students say the area for this figure is 15 square inches.  	• Make a parallelogram scaled to precisely 3 inches by 5 inches. Have students use one inch squares to cover the area. What do students notice?
12. Estimates of measures must be exact and will be graded on how close you are to the right answer (Muir, 2005).	Students constantly try to change their estimate to reach the correct answer.	• Rather than give prizes or promote winning estimates, give a range of estimates that is reasonable. • Avoid using the terms *measure* and *estimate* interchangeably. • Measurements always have some error, so using the word *exact* should be avoided.
13. The larger the rays of the angle, the larger the angle.	Given these two angles, students will say B is the larger angle.  	• Use the classroom door to act out angle sizes. Show how the angle is not affected by the size of the door—it is only about the spread of the space between the door and the wall of the classroom.
14. The height of the solid figure or container indicates its volume.	Students select the tallest container as having the greatest volume or capacity.	• Use a collection of containers (where the tallest is not the largest) and have students estimate the order from smallest to largest. Test to show the actual volume.
15. The length or size of an object is an indication of its weight.	Students select the largest object as the heaviest object.	• Use a collection of items (where the longest or largest are not the heaviest) and have students estimate the order from lightest to heaviest. Test to show the actual weights.
16. Confusion about the minute and hour hands on an analog clock, or student reads the number closest to the hour hand as the hour.	When shown 6:45 on an analog clock, the student states one of the following times: 6:09 9:06 9:30 7:45	• This is due to analog clocks having two scales (hours and minutes) (there are also seconds if reading a clock to greater precision). Start with only one scale, using Activity 18.35. • Pair an analog clock with a digital clock so the student can compare the time on each.
17. The size of a coin equates to its value.	The student thinks the dime is worth less than the penny or nickel because it is smaller in size.	• Glue coins to connecting cube towers that represent their values or to a ten-frame that shows each coin's value (i.e., a penny next to a ten-frame with only one space filled; a nickel to a ten-frame with five spaces filled).

RESOURCES FOR CHAPTER 18

WRITING TO LEARN

Assess your understanding and application of chapter content.

1. Explain the measuring process. Does your explanation work equally well for length, area, weight, volume, and time?
2. What are three reasons for starting with nonstandard units instead of standard units in instructional activities? Which reason seems most important to you and why?
3. What is the connection between the area formulas for rectangles, parallelograms, triangles, and trapezoids? Draw sketches and provide explanations.
4. Explain how the area of a circle can be determined using the basic formula for the area of a parallelogram.

LITERATURE CONNECTIONS

There are numerous books that link well to lessons on the measurement attributes discussed earlier. The teacher resource here includes 100 children's books linked to measurement concepts. You will find great options to add to your classroom or school library!

Pearson eText Teacher Resource: 103
Children's Literature Books on Measurement

Here is just one book highlighted from the list of 103 that presents multiple attributes (see that same heading on the list for other suggestions in this category).

I Am a Book. I Am a Portal to the Universe

Posavec (2021)

This book is a design masterpiece that presents a multitude of measures. There is encouragement to move the book and you will find such observations as that the book weighs 465 grams and is 20 cm in both length and width. This engaging read could inspire many other follow-up activities involving measuring.

RECOMMENDED

Wickstrom, M. H., Carr, R., & Lackey, D. (2017). Exploring Yellowstone National Park with mathematical modeling. *Mathematics Teaching in the Middle School*, *22*(8), 462–470.

This article discusses ways to explore national parks through a look at area and perimeter. By the use of maps and concrete approaches, students were able to reason about their use of tools to achieve deeper thinking.

Wickstrom, M., & Jurczak, L. (2016). Inch by inch, we measure. *Teaching Children Mathematics*, *22*(8), 468–475.

Using the book Inch by Inch *(Lionni, 1960), the authors link inchworms to building the idea of a length unit. Through estimation and the use of multiple strategies, young students began to make sense of how rulers work.*

Developing Geometric Thinking and Geometric Concepts

CHAPTER 19

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **19.1** Describe the van Hiele levels of geometric thinking.
- **19.2** Analyze strategies for teaching about shapes and properties.
- **19.3** Identify approaches to teaching geometric transformations.
- **19.4** Explore ways to engage students in thinking about location.
- **19.5** Illustrate approaches that develop students' mental imagery and visualization skills.

Geometry is a "network of concepts, ways of reasoning and representation systems" used to explore and analyze shape and space (Battista, 2007, p. 843). This critical area of mathematics has many real-world applications, such as global positioning systems and computer animation. But for too long, the geometry curriculum merely emphasized learning terminology and tasks such as naming shapes or angle categories like acute or obtuse. Geometry is much more. Geometry is at the core of early mathematics, alongside the strand of number (NCTM, 2000). For example, students begin to learn about fractions by partitioning shapes. In many state standards, geometry concepts span K–8. As such, we will describe standards by grades in different sections of this chapter.

First, geometry involves developing and applying *spatial sense*, which is an intuition about shapes and the relationships between them. Like number sense, it is considered a core area of mathematical study (Casey & Fell, 2018; Sarama & Clements, 2009). Spatial sense includes the ability to visualize objects and spatial relationships mentally —to turn things around in one's mind: mental rotation. Spatial sense also includes how students use diagrams and sketches as they develop concepts; compose and decompose shapes, maps, and orientation; and visualize mathematical ideas by using mental imagery (Mulligan, 2015). People with well-developed spatial sense appreciate geometric forms in art, nature, and architecture, and they use geometric ideas to describe and analyze their world. Students who have shown exceptional ability with mentally manipulating two-dimensional and three-dimensional objects have greater creative achievements and academic success more than thirty years later, especially in STEM fields (Kell et al., 2013; Stieff & Uttal, 2015). The best part is that spatial sense and reasoning can be enhanced by carefully planned instruction (Uttal et al., 2013).

Second, geometry has significant content strands that apply to all grade levels. This chapter is organized around these four major geometry strands: shapes and properties, transformation, location, and visualization. You will notice that more attention is devoted to the shapes and properties strand to align with the emphasis identified in state standards for grades K–8. However, notice that experiences targeted toward each of the strands have potential to enhance students' understanding of other strands.

BIG IDEAS

- What makes shapes alike and different can be determined by geometric properties (i.e., defining attributes). Shapes can be classified into a hierarchy of categories according to shared properties.
- Transformations help students think about the ways properties change or do not change when a shape is moved in a plane or in space. These changes can be described in terms of translations, reflections, rotations, dilations, the study of symmetries, and the concept of similarity.
- Shapes can be described in terms of their location in a plane or in space. Young children use language such as *above, below*, and *next to*, whereas older students use coordinate systems to describe these locations more precisely. The coordinate view also offers ways to understand certain properties of shapes and can be used to measure distance, an important application of the Pythagorean theorem.
- Visualization includes the recognition of shapes in the environment, developing relationships between two-and three-dimensional objects, the ability to draw and recognize objects from different viewpoints, and to change the orientation and size of shapes mentally.

A Progression for Developing Geometric Thinking

Learning Outcome 19.1 **Describe the van Hiele levels of geometric thinking.**

All learners are capable of growing and developing the ability to think and reason in geometric contexts, but this ability requires ongoing and significant experiences across a developmental progression. The research of two Dutch educators, Pierre van Hiele and Dina van Hiele-Geldof, provides insights into students' development of geometric thinking through the descriptions of different levels of thought (see Figure 19.1).

This well-known progression, the van Hiele theory (1984), significantly influences geometry curricula worldwide. When teachers learned about the van Hiele levels, it helped them

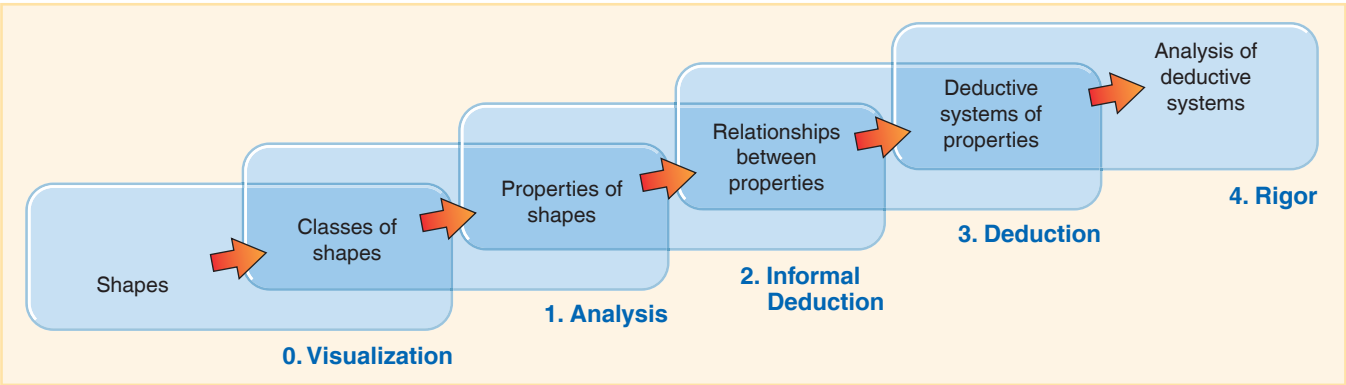


FIGURE 19.1 The van Hiele theory's levels of geometric thinking

understand next steps for their students' geometry instruction but also, importantly, improved their knowledge of students' geometric thinking (Yi, Flores, & Wang, 2020). These researchers went on to say that teachers' knowledge of the van Hiele levels is a critical type of geometry knowledge for teaching. Here are some characteristics of the levels in Table 19.1:

TABLE 19.1 CHARACTERISTICS OF THE VAN HIELE LEVELS

Characteristic	Implication
1. Sequential	To arrive at a level, students must move through all prior levels. The outcomes at each level are the same as the focus at the next level (see Figure 19.1).
2. Developmental	When instruction (or the language used) is at a level higher than that of the students', students need additional experiences to understand the concept being developed. A student can, for example, memorize a fact (such as all squares are rectangles) but not yet understand the relationship of how the properties of a square and rectangle are related.
3. Age-independent	A third-grade student, a high school student, or an adult could be at the lowest level (0).
4. Experience-dependent	Advancement through the levels requires geometric *experiences*. Students should have instructional activities where they explore, talk about, and interact with content at the next level while increasing experiences at their current level.

The van Hiele Levels of Geometric Thought

The van Hiele model is a five-level hierarchy of understanding spatial ideas in which each level describes what types of geometric ideas we focus on and think about (called *objects of thought*) and the outcome or what students can do with the information (*products of thought*). These levels are important for you to know because as you work with students at one level, you can use your understanding of where they need to move to on the next level and help support the connections that should be developed. Table 19.2 can serve as a reference for teaching geometry.

Implications for Instruction

The rich collection of geometric experiences you provide are the single most important factor in moving students up this developmental ladder to higher levels of geometric thought! In this section, we put the van Hiele levels into action by presenting a quintessential example of an activity that matches the guiding features of each level. There are also ideas for how to bridge from one level to the next. Many activities can be modified to span two levels of thinking, helping students link one level to the next.

Pearson eText
Video Example 19.1
In this video late author John Van de Walle discusses instructional activities that correspond with early levels of van Hiele's model of geometric thinking.

FORMATIVE ASSESSMENT Notes. How do you discover students' van Hiele level so you can select the right activities? Listen to the types of observations that students make and record them on an Observation Look-For Tool, noting whether they:

- Talk about shapes as classes—referring to rectangles—or do they base their discussion on a particular rectangle.
- Understand that shapes do not change when the orientation changes.
- Describe all features of the group of quadrilaterals? Triangles?
- Compare groups of shapes (e.g., rectangles) to other groups of shapes (e.g., squares).

With simple observations such as these, you will be able to begin to distinguish between levels 0, 1, and 2. If students are not able to understand logical arguments, are not comfortable with conjectures, and are unsure of if–then reasoning, they are likely still at level 1 or below and will need more experiences to reach level 2. The geometry in high school is primarily at level 3, so a priority in middle school is that students move to be at least strong level 2 thinkers. ■

Pearson eText Teacher Resource: Observation Look-For Tool

TABLE 19.2 VAN HIELE LEVELS CONNECTED TO LEARNING AND TEACHING

Level	Definition	Students' Characteristic Thinking at This Level	Instructional Emphasis
Level 0: Visualization	*Students move from considering shapes and what they look like to developing classes or groupings of shapes that seem to be alike.*	Students recognize and name shapes based on global visual characteristics. For example, a square is defined by a level 0 student as a square "because it looks like a square." They may see a square with sides that are not horizontal or vertical (appears tilted) and believe it is a diamond (not a mathematical term for a shape) and no longer a square. Students sort and classify shapes based on their appearance—"I put these shapes together because they are all pointy."	Explore shapes that students can observe, feel, build (compose), or take apart (decompose). Investigate how shapes are alike and different and use these ideas to create classes of shapes (both physically and mentally). Present the names of some of these classes—rectangles, triangles, prisms, cylinders, and so on. Properties of shapes, (i.e., parallel sides, symmetry, right angles) are included at this level but only in an informal, observational manner.
Level 1: Analysis	*Students move from considering classes of shapes rather than individual shapes by focusing on the properties of shapes.*	Students at level 1 consider shapes within a class rather than just the single shape on their desk. They talk about properties of all rectangles instead of just the one they have. They are able to think about what makes a rectangle a rectangle (i.e., four sides, opposite sides parallel). Irrelevant features (e.g., size or orientation) fade into the background. They understand that if a shape belongs to a particular class such as cubes, it has the corresponding properties of that class: "All cubes have six congruent faces, and each of those faces is a square." Students begin to see that some shapes are subclasses of one another (e.g., all squares are rectangles, all rectangles are parallelograms).	Provide concrete models and semi-concrete drawings of shapes, because students see these individual shapes as examples of members in classes of shapes. Build and refine students' understanding of identifying the properties of shapes.
Level 2: Informal Deduction	*Students move from considering the properties of shapes to exploring the relationships between properties of geometric objects.*	Students begin to think about properties of geometric objects and develop relationships between these properties. "If all four angles are right angles, the shape must be a rectangle. If it is a square, all angles are right angles. If it is a square, it must be a rectangle."	Engage students in if–then reasoning, so they can classify shapes using only a minimum set of defining characteristics. For example, four congruent sides and at least one right angle are sufficient to define a square, or rectangles are parallelograms with a right angle. Have students focus on logical arguments about the properties and support them in following informal deductive arguments about shapes and their properties. Proofs may be more intuitive than deductive.
Level 3: Deduction (beyond the scope of this book—high school geometry)	*Students move from considering the relationships between properties of geometric objects to developing deductive axiomatic systems for geometry.*	Students analyze informal arguments; the structure of a system complete with axioms, definitions, theorems, corollaries, and postulates begins to develop and can be appreciated as the necessary means of establishing geometric truth.	Give students abstract statements about geometric properties and ask them to make conclusions based on logic. Encourage students to develop a proof from a series of sound deductive arguments.
Level 4: Rigor (beyond the scope of this book—college mathematics major)	*Students move from considering deductive axiomatic systems to comparing and contrasting different axiomatic systems.*	Students understand deductive proof.	Focus on axiomatic systems, not just on the deductions within a system.

Level 0: Visualization. Clements and Battista (1992) proposed a level prior to level 0 called Pre-Recognition. Clements and Sarama (2020) recommend activities in which examples and non-examples that look alike are paired so students focus attention on a shape's defining attributes. An attribute applies to only particular shapes (two sides the same length), but a property would apply to all shapes in a category—like triangles have three sides. Activity 19.1 provides a pre-recognition experience.

Activity 19.1 Tricky Shapes

CCSS-M: K.G.A.2; K.G.B.4; 1.G.A.1

Provide pairs of students with Tricky Shapes for Triangles or Tricky Shapes for Rectangles activity pages that have several examples and non-examples of a target shape (see Figure 19.2). Students should identify examples that are like the target shape and non-examples that are not like the target shape—and state why in each case. Facilitate a whole class discussion and summarize by reviewing the shape's relevant features.

Pearson eText Activity Page: Tricky Shapes for Triangles
Pearson eText Activity Page: Tricky Shapes for Rectangles

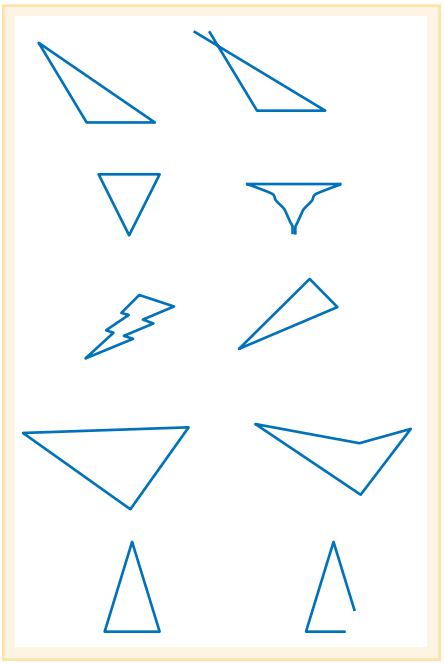


FIGURE 19.2 Examples and non-examples help students attend to critical properties

Then students move to level 0, where they focus on how shapes are alike and different, as in the following activity.

Activity 19.2 Shape Sorts

CCSS-M: K.G.B.4; 1.G.A.1; 2.G.A.1; 3.G.A.1

Group students and provide a set of two-dimensional shapes (see Figure 19.3) for them to carry out the following activities in order:

- Randomly select two shapes and try to find something that is alike about the two shapes and something that is different.
- Choose one target shape at random and place it in the center of the workspace. Find all other shapes that are like the target shape according to the same property. For example, say, "This shape is like the target shape because it has a curved side and a straight side," and then all other shapes put in the collection must have that property. As a challenge, make a second sort with the same target shape but using a different property.
- Do a secret sort. Create a group of about three to four shapes that fit a secret rule. Leave other shapes that belong in that set in the larger pile. Find additional pieces that belong to the set and guess the secret rule. For example, students could sort the shapes so that all have at least one right angle or square corner. When students discover the rule, if they haven't already used the terms at that time, help them name the property, "right angle."
- Explore the Expanded Lesson: Shape Sorts for more details.

Pearson eText Activity Page: Two-Dimensional Shapes
Pearson eText Expanded Lesson: Shape Sorts

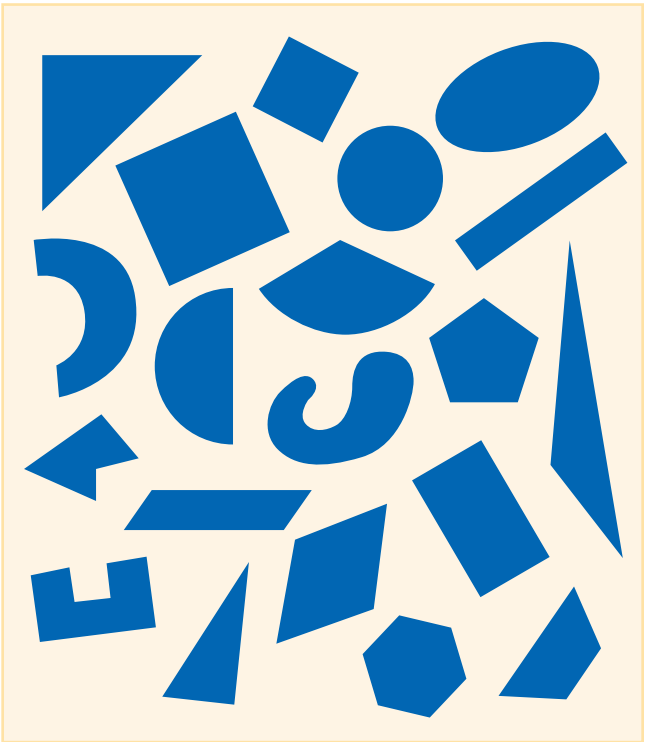


FIGURE 19.3 A collection of shapes for sorting

These find-a-rule activities elicit a wide variety of ideas. Students may initially describe shapes with ideas such as "curvy" or "looks like a rocket" rather than typical geometric properties, but as they notice more complex properties, you can attach appropriate terminology to them. For example, students may notice some shapes have corners "like a square" (explain that those are called *right angles*) or that "these shapes are the same on both sides" (explain that that is called *line symmetry*). For young learners, consider using Venn diagrams as an organizational tool for the sorting (Howse & Howse, 2014/2015).

What clearly makes this activity level 0 is that students are operating on the shapes that they see in front of them and are beginning to see ways they are alike and different. By forming groups of shapes, they begin to imagine shapes belonging to the same class that are not present in their current collection.

Moving from Level 0 to Level 1. Instructional considerations that support students moving from level 0 to level 1 are as follows:

- *Focus on the properties of shapes rather than on simple identification.* As new geometric concepts are learned, students should be challenged to use these attributes to classify shapes. Otherwise, they may rely on mental prototypes and say, "that isn't a parallelogram because it has to have slanted sides" (it doesn't).
- *Challenge students to test ideas about shapes by using a variety of examples from a particular category.* Say, "Let's see if that is true for other rectangles," or "Can you draw a triangle that does not have a right angle?" Question students to see whether observations made about a particular shape apply to other shapes of the same kind.
- *Provide ample opportunities to draw, build, make, put together (compose), and take apart (decompose) shapes in both two and three dimensions.* Build these activities around understanding and using specific attributes or properties.
- *Apply ideas to entire classes of figures (such as all rectangles or all prisms) rather than to individual models.* For example, find ways to sort all possible triangles into groups. From these groups, define classes of triangles.

Level 1: Analysis. In the following activity, students use the properties of shapes such as symmetry, angle classification (right, obtuse, acute), parallel and perpendicular, and the concept of congruent line segments and angles.

Activity 19.3 Property Lists for Quadrilaterals

CCSS-M: 2.G.A.1; 3.G.A.1; 4.G.A.2; 4.G.A.3

Prepare parallelograms, rhombuses, rectangles, and squares (see Figure 19.4) for each group of three. Assign the group to work with one type of quadrilateral. (For multilingual learners and students with special needs in math, post labeled shapes as a reference.) Ask students to list as many properties as they can that apply to all of the shapes on their sheet. They will need tools such as index cards (to check right angles, compare side lengths, and draw straight lines), mirrors (to check line symmetry), and tracing paper (for identifying angle congruence). Encourage students to use the words *exactly, at least*, and *at most* when describing how many of something; for example, "rectangles have at least two lines of symmetry," because squares—included in the rectangles category—have four. Notice that students must assess whether the properties apply to all shapes in the category. If they are working on the squares, for example, their observations should as accurately apply to a square mile as to a square centimeter.

Have students prepare their Property Lists under these headings: Sides, Angles, Diagonals, and Symmetries. Groups then share their lists with the class and eventually develop a class list for each category of shape. For multilingual learners, having students say the words aloud and point to the word as you say it are ways to reinforce meaning and support participation in discussions.

Pearson eText Activity Page: Property List for Parallelograms
Pearson eText Activity Page: Property List for Rhombuses
Pearson eText Activity Page: Property List for Rectangles
Pearson eText Activity Page: Property List for Squares
Pearson eText Activity Page: Property List for Sides, Angles, Diagonals, Symmetries

Moving from Level 1 to Level 2. Level 2 thinking is expected to begin at a minimum by grade 5, when students classify shapes based on their properties in categories and subcategories. Instructional considerations supporting students' movement from level 1 to level 2 are:

- *Challenge students to explore and test examples.* Ask questions that involve reasoning, such as "If the sides of a four-sided shape are all congruent, will you always have a square?" and "Can you find a counterexample?"
- *Encourage the making and testing of hypotheses or conjectures.* "Do you think that grouping will work all the time?" "Is that true for all triangles or just for equilateral ones?"
- *Examine properties of shapes to determine necessary and sufficient conditions for a shape to be a particular shape.* "What properties must diagonals have to guarantee that a quadrilateral with these diagonals will be a square?"
- *Use the language of informal deduction.* Say: *all*, *some*, *none*, *if… then*, *why*, *what if*, and so on.
- *Encourage students to attempt informal proofs.* Engage students in making sense of informal proofs that you or other students have suggested.

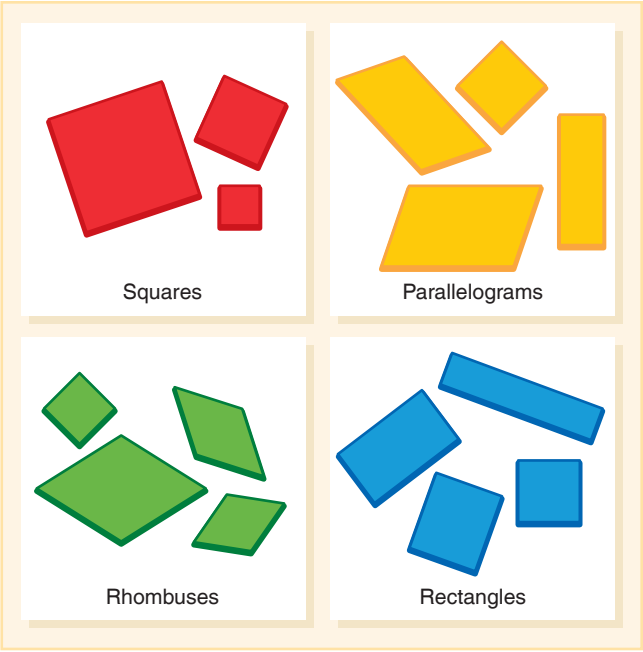


FIGURE 19.4 Shapes for the Property Lists for Quadrilaterals activity

Level 2: Informal Deduction. The signature characteristic of a level 2 activity is the inclusion of informal logical reasoning. Because students have developed an understanding of various properties of shapes, it is now time to encourage conjecture and to ask "Why?" or "What if?"

Activity 19.4 Minimal Defining Lists

CCSS-M: 4.G.A.2; 5.G.B.3

This activity is a sequel to Activity 19.3, "Property Lists for Quadrilaterals." Once the class has agreed upon property lists for the parallelogram, rhombus, rectangle, and square (and possibly the kite and trapezoid), post the lists. Have students work in groups to find minimal defining lists, or MDLs, which is a subset of the properties for a shape that is defining and minimal. The term *defining* here means that any shape with all the properties on the MDL must be that shape. The term *minimal* means that if any single property is removed from the list, it is no longer defining. For example, one MDL for a square is a quadrilateral with (1) four congruent sides and (2) four right angles (two items on the list). If a shape has these two properties, it must be a square. Another minimal defining list for a square is (1) it has four sides of the same length and (2) perpendicular diagonals. Challenge students to find more than one MDL for their shape. A proposed list is not defining if a counterexample—a shape other than one being described—can be produced by using only the properties on the list.

Students can also make guessing puzzles, using the clues from their list and basic computer coding such as from *Scratch*, a free platform developed at MIT (Bolognese, 2016).

Pearson eText
Video Example 19.2
In this video consider how this class engages with the identifying Properties of Shapes Activity.

The hallmark of this activity and other level 2 tasks is the emphasis on logical reasoning: "*If* a quadrilateral has these properties, *then* it must be a square." Logic is involved in proving that a proposed list is faulty—either not minimal or not defining through the use of counterexamples. Students will come to realize that any minimal defining list (MDL) is a potential definition. The other aspect of this activity that involves level 2 thinking is that students focus on analyzing the relationships between properties (e.g., if a quadrilateral has four right angles, it also has diagonals of the same length).

The remainder of this chapter is organized by the four content goals (shapes and properties, location, transformations, and visualization). Understand that these content areas develop based on the levels just discussed and that they also overlap and build on each other. In other words, activities in one section help develop geometric thinking in another.

Shapes and Properties

Learning Outcome 19.2 **Analyze strategies for teaching about shapes and properties.**

Shapes and properties is the content area most often associated with geometry in pre-K–8 classrooms, when young students begin to "perceive, say, describe/discuss and construct objects in 2-D space" (National Research Council Committee, 2009, p. 177).

Consider Clements and Sarama's (2020) four features of effective early geometry instruction:

1. Show a variety of shapes and have students compare both examples and non-examples with a focus on critical characteristics.
2. Facilitate student discussions about the properties of shapes, having them develop essential language along the way.
3. Encourage the examination of classes of shapes that goes beyond the traditional, allowing students to explore relationships and recognize different categories, orientations, and sizes.
4. Provide students with a range of geometric experiences at every level, having them use concrete materials, semi-concrete drawings, and technology.

Students need experiences with a wide variety of two- and three-dimensional shapes. Triangles should be shown in more than just equilateral forms and not always with the vertex at the top or the base horizontal to the bottom of the paper on which it appears. Shapes should have curved sides, straight sides, and combinations of these. As students describe the shape or property, the terminology can be introduced, as in the next activity.

Activity 19.5 Shape Show and Hunt

CCSS-M: K.G.A.1; K.G.A.2; K.G.B.4

Use painter's tape to create a large target shape that students can walk around on the community rug or floor. Ask students if they can name the shape or any attributes (e.g., straight or curved sides, number of sides, right angles, etc.). Confirm that students, especially multilingual learners, understand the terms you use. You can also have students draw the shape in the air or on paper. Ask, "Can you find one or two items in the classroom, on the playground, or somewhere else at school that have this shape?" Take photos of the examples students find so they can be used later for discussion. Ask, "How do you know the example is like the target shape?" For students with special needs in math, provide a cutout of the target shape that they can take with them as they search for an example.

Concrete and Virtual Tools

Students of all ages benefit from exploring concrete shapes. Each activity here is described with the use of such tools. Additionally, more and more interactive apps are emerging, and these sites offer various advantages. For example, virtual geoboards avoid the possibility of breaking rubber bands. Something built virtually can also be saved, providing evidence of student thinking, although if students' initial experiences are with a virtual geoboard, they don't have the context for what a geoboard represents. So begin with concrete tools and continue to use both concrete and virtual tools. Table 19.3 offers sites with interactive virtual geometry tools.

Sorting and Classifying

As young students work at classification of shapes, be prepared for them to notice features that you do not consider to be real geometric attributes, such as "dented" or "looks like a tree." Students at this level will also assign shapes with ideas that are not part of the shape, such as "points up" or "has a side that is the same as the edge of the board."

TABLE 19.3 SITES FOR VIRTUAL GEOMETRY TOOLS

Source	URL	Geometry Tools
Brainingcamp (subscription-based)	http://Brainingcamp.com	Geoboard, pattern blocks, coordinate graph
Math Learning Center	https://www.mathlearningcenter.org/apps	Geoboard, pattern blocks
Toy Theater Teacher Tools	https://toytheater.com/category/teacher-tools/	Geoboard, tangrams, pattern blocks, shape counters, coordinate graph, cubes (building 3-D shapes)
Didax	https://www.didax.com/apps	Square color tiles, pattern blocks, geoboard
Mathigon	https://mathigon.org/	Polygons, tangrams, polyominoes (pentominoes, tetrominoes), tiles, 3D solids, measuring tools, tantrix tiles
GeoGebra	https://www.geogebra.org	Geoboard, properties of polygons, area-finding feature, 3D calculator
NCTM	https://Illuminations.nctm.org	Pattern blocks, geometric solids, tessellations, transformations, Dynamic Paper
Math Playground	https://www.mathplayground.com	Geoboard, pattern blocks, tangrams, 3D builder, protractor

In any sorting and classifying activity, the students—not the teacher—should decide how to group the shapes. By listening to the kinds of attributes that they use in their sorting, you will be able to tell what properties they know and use and how they think about shapes. Figure 19.5 illustrates a few of the possible ways a set might be sorted.

The following activity also uses a variety of two-dimensional shapes.

Activity 19.6 What's My Shape?

CCSS-M: 1.G.A.1; 2.G.A.1; 3.G.A.1; 4.G.A.2; 7.G.A.2

Cut out a double set of two-dimensional assorted shapes on card stock. Glue one of each shape in the set to the inside of a file folder to make secret-shape folders. Glue the other set of shapes on cards and place them in view for reference. Designate one student as the leader who holds the secret-shape folder. The other students are to identify the shape in the folder by asking the leader yes-or-no questions about the attributes of the shape. The group eliminates shapes (turning over the cards) as they get answers about the attributes that narrow the choices to one shape. Students cannot point to a piece and ask, "Is it this shape?" Instead, they ask, "Does it have all straight sides?" Check the final shape card against the one in the folder. Provide students with special needs in math a list of possible properties and attributes (such as number of sides) to help support their question asking. Also, adapt this activity for three-dimensional shapes. For older students, cover the mystery shape with a paper curtain and slowly reveal portions or add questions such as "Is it concave?" At each reveal, let students speculate on what the shape could be and what it couldn't be until it is uncovered (Ronau, Meyer, & Crites, 2015).

Pearson eText Activity Page: Assorted Shapes

The difficulty of Activity 19.6 largely depends on how many shapes in the collection share attributes with the secret shape. The more shared, the more difficult the task. For virtual shapes, explore 3D Shapes in Polypad, a feature of the Mathigon site (https://mathigon.org/polypad). You can do many cool things with this online tool.

FORMATIVE ASSESSMENT Notes. Adapt "Shape Sorts" (Activity 19.2) for a diagnostic interview. Make sure you have a collection that has a lot of variation (curved surfaces, etc.) such as commercially available collections of three-dimensional shapes or real objects such as cans, boxes, containers, and balls. Figure 19.6 illustrates some classifications of solids. ■

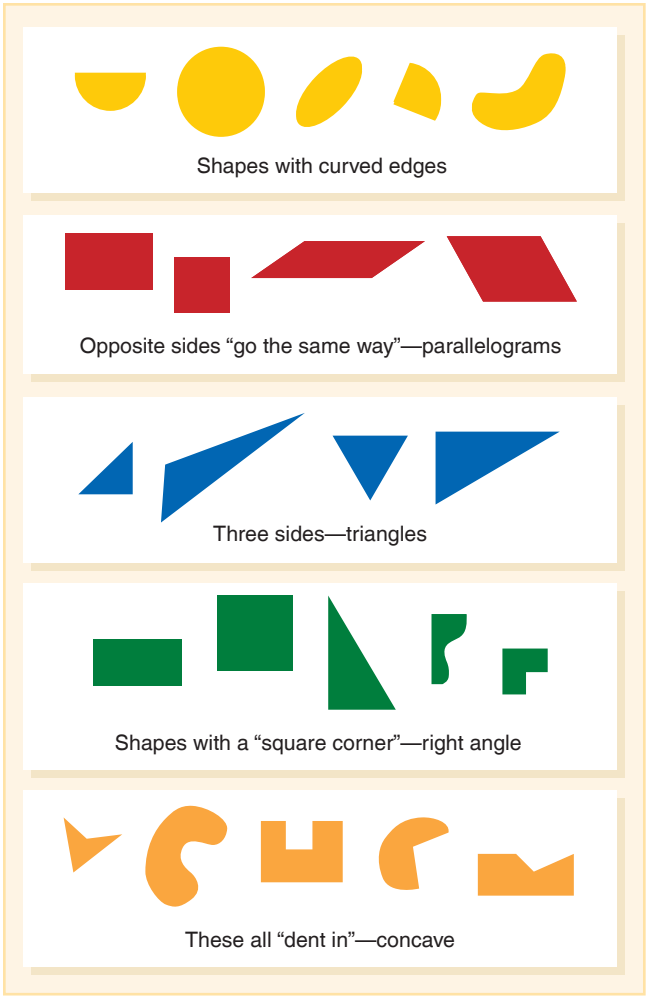

Shapes with curved edges
Opposite sides "go the same way"—parallelograms
Three sides—triangles
Shapes with a "square corner"—right angle
These all "dent in"—concave

FIGURE 19.5 By sorting shapes, students begin to recognize properties

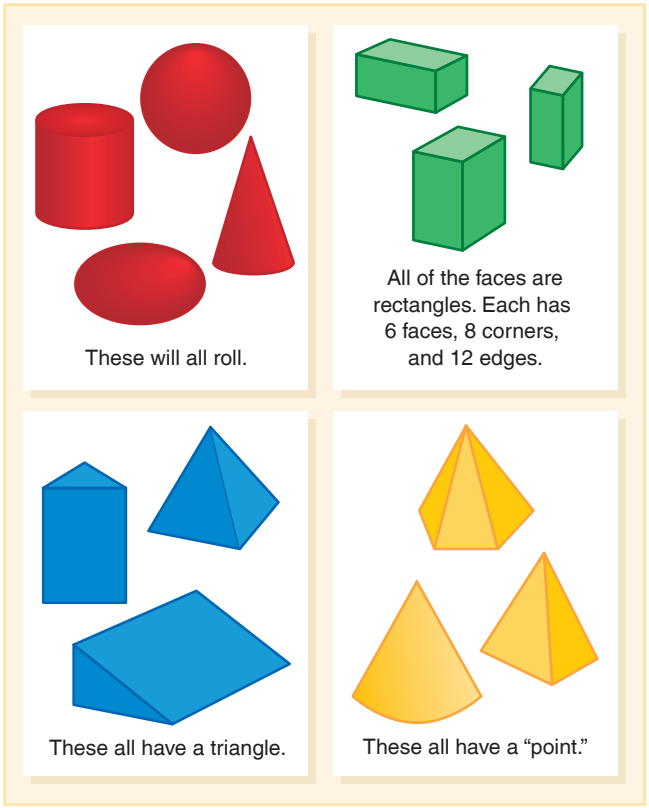

These will all roll.
All of the faces are rectangles. Each has 6 faces, 8 corners, and 12 edges.
These all have a triangle.
These all have a "point."

FIGURE 19.6 Classifications of three-dimensional shapes

The ways students describe these three-dimensional shapes provides evidence of their level of thinking. The classifications made by level 0 thinkers are generally limited to the shapes that they have in front of them. Level 1 thinkers will begin to create categories based on attributes, and their language will indicate that they know there are more shapes in the group than those in the present collection. Students may say things like, "These shapes have square corners sort of like rectangles," or "These shapes look like boxes. All the boxes have square [rectangular] sides."

Activity 19.7 Can You Make It?

CCSS-M: 1.G.A.1; 2.G.A.1; 3.G.A.1; 4.G.A.2

Print out the Can You Make It? activity page for each student, or project the word clue from the activity page, such as "a shape with two pairs of parallel lines," one item at a time for the whole class. Students use the description to create a corresponding shape on the geoboard. (Warning: two of these tasks are impossible.) Encourage students to create challenges for others to try. If the class keeps track of solutions to the challenges in the preceding activity, there is an added possibility of creating classes of shapes possessing certain attributes that may result in definitions of new classes of shapes.

Pearson eText Activity Page: Can You Make It?

Composing and Decomposing Shapes

Students need to explore freely how shapes fit together to form larger shapes (compose) and how larger shapes can be made of smaller shapes (decompose). In addition, in the middle grades, students compose (draw) shapes by hand, with tools such as rulers or with technology support, to study the attributes of the shape with given specifications. This ability to compose and decompose shapes starts in prekindergarten (Copley, 2017) and eventually supports geometric measurement such as finding the area, surface area, and volume of irregular shapes.

Among two-dimensional shapes for these activities, pattern blocks and tangrams are the best known. Pierre van Hiele (1999) also describes an interesting set of tiles he named the Mosaic Puzzle (see Figure 19.7).

The geoboard is one of the best tools for composing and decomposing two-dimensional shapes. Explicitly teach students how to record their designs on geoboard recording sheets. To support this transfer, suggest that they first mark the dots for the corners of their shape ("second row, end peg") on the paper. With the corners identified, they can draw lines between the corners to make the shape. For any of the activities suggested for geoboards, allow students to choose the tool (geoboard, grid paper, or dot paper) that best supports their thinking on a given problem. There are also excellent digital versions of the geoboard. One is found at the Math Learning Center (see Table 19.3), which allows students to make discoveries about angles, area, congruence, perimeter, and more.

Here are several activities appropriate for thinking about constructing shapes using a geoboard.

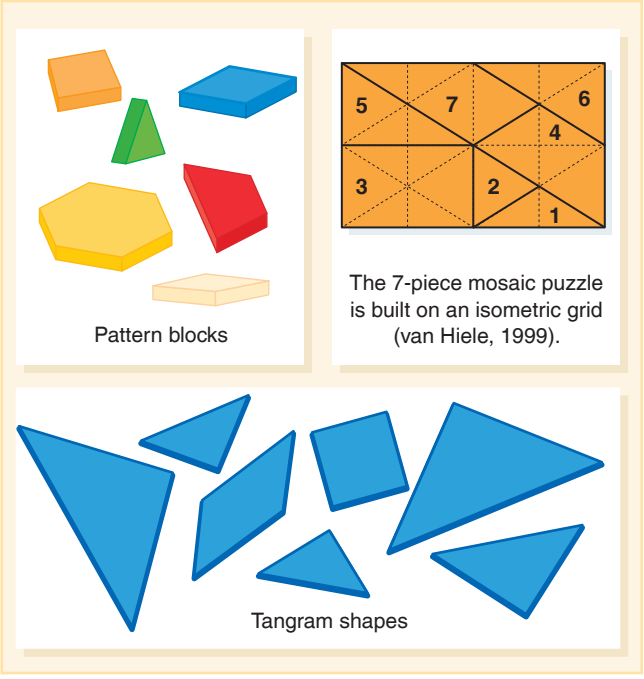


FIGURE 19.7 Materials for composing and decomposing shapes

Activity 19.8 Tangram Puzzles

CCSS-M: K.G.B.6; 1.G.A.2; 2.G.A.1

Using a set of tangram pieces or virtual versions, have students explore the Tangram Puzzles activity page, where they compose shapes to create a larger figure. Also notice how the creation of these picture puzzles can also focus on area. (If all seven pieces are used, the areas of the figures are the same!) The ABCya! website also has a tangram game (http://www.abcya.com/tangrams.htm) with different puzzle figures that students can unlock within each of the levels: easy, medium, and hard.

Pearson eText Blackline Master: Tangram Pieces
Pearson eText Activity Page: Tangram Puzzles

Activity 19.9 Mosaic Puzzle

CCSS-M: K.G.B.6; 1.G.A.2; 2.G.A.1; 3.G.A.1; 4.G.A.2; 7.G.A.2

Give pairs of students the Mosaic Puzzle to cut out and have them use the Mosaic Puzzle Questions activity page to explore using what they know about a shape's attributes to compose and decompose shapes. The value of the mosaic puzzle is that the set contains five angles (which could lead to discussions of types of informal angle comparisons and groupings of angle measures such as right, acute, and obtuse).

Pearson eText Activity Page: Mosaic Puzzle
Pearson eText Activity Page: Mosaic Puzzle Questions

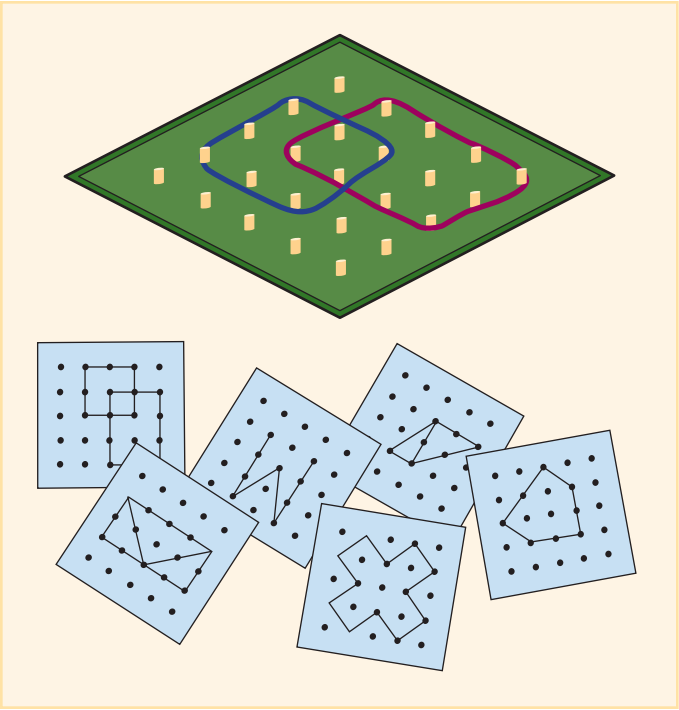


FIGURE 19.8 Shapes on geoboards

Activity 19.10 Geoboard Copy

CCSS-M: 1.G.A.2; 2.G.A.1; 4.G.A.1; 4.G.A.2; 4.G.A.3

Prepare geoboard design cards (see Figure 19.8). Project the semi-concrete versions of the shapes onto a screen one at a time and have students copy them on geoboards (or on geoboard patterns or geoboard recording sheets). Begin with designs that can be made using one band, and then create more complex designs, including those that show a shape composed of smaller shapes. Depending on the grade level, discuss attributes such as number of sides, parallel lines, or line symmetry. Students with special needs in math may need to have a copy of the design card at their desk for closer reference. Have students also model real shapes on their geoboards, such as houses, letters of the alphabet, and so on.

Pearson eText Activity Page: Geoboard Design Cards
Pearson eText Blackline Master: Geoboard Patterns (10 by 10)
Pearson eText Blackline Master: Geoboard Recording Sheets (10 by 10)

Geoboard Copy has an element of proportional thinking in it, just as any puzzles in which students work from small designs to make larger versions. This concept of scaling (or resizing) is particularly important for fifth graders and beyond in state standards. In addition, the use of the geoboard supports the study of area, particularly of compound shapes. Students also begin to understand that rectangles can be decomposed into equal rows or columns, which connects multiplication to the area formula.

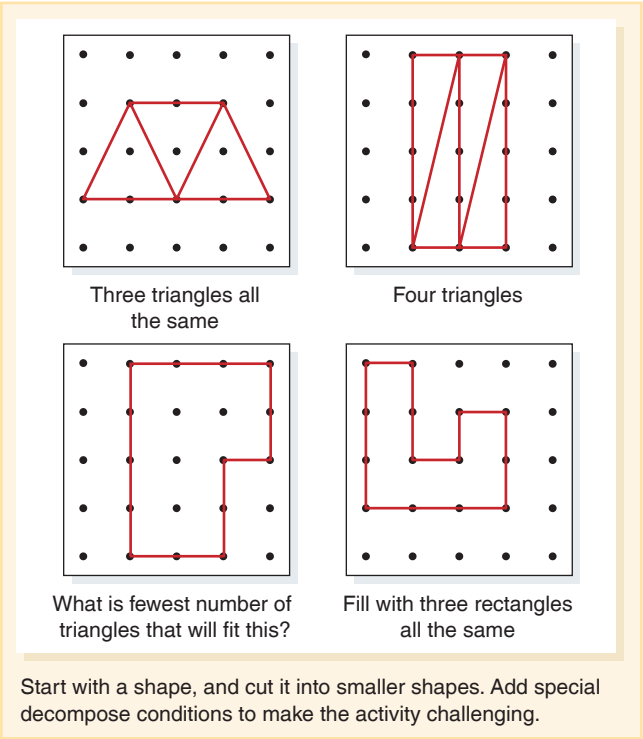

Three triangles all the same
Four triangles
What is fewest number of triangles that will fit this?
Fill with three rectangles all the same
Start with a shape, and cut it into smaller shapes. Add special decompose conditions to make the activity challenging.

FIGURE 19.9 Decomposing shapes

Activity 19.11 Decomposing Shapes

CCSS-M: 1.G.A.2; 1.G.A.3; 2.G.A.1; 2.G.A.3; 3.G.A.2; 3.NF.A.1

Show students a shape from the Decomposing Shapes activity page and ask them to copy it on their geoboards or geoboard recording sheet. Specify the number of smaller shapes they should decompose each large shape into, as in Figure 19.9. Specify whether the smaller shapes are all to be congruent or simply of the same type (all triangles). In the case of the congruent decomposition, connect geometry to fractions by discussing the equal parts and writing the fraction on each part within the composed whole.

Pearson eText Activity Page: Decomposing Shapes
Pearson eText Blackline Master: Geoboard Recording Sheet (10 by 10)

Categories of Two- and Three-Dimensional Shapes

As students begin to notice attributes of shapes (moving to level 1 thinking), the important definitions of two- and three-dimensional shapes support the development of identifying the relationships between shapes.

Two-Dimensional Shapes. Table 19.4 lists some important categories of two-dimensional shapes. Examples of these shapes can be found in Figure 19.10.

By third grade, students must think about subcategories of quadrilaterals, and by fifth grade, they must grasp relationships such as that a square is a rectangle, and a rhombus and all parallelograms are trapezoids, but not all trapezoids are parallelograms.* At first, students may have difficulty seeing this type of subcategory. They may quite correctly list all the attributes of a square, a rhombus, and a rectangle and still might not classify a square as a rhombus or a rectangle. To help learners think about these overlapping groupings, suggest that a four-legged pet can be an animal and a canine. A square is an example of a quadrilateral that belongs to two other categories.

TABLE 19.4 CATEGORIES OF TWO-DIMENSIONAL SHAPES

Shape	Description
Simple Closed Curves	
Concave, convex	An intuitive definition of concave might be "having a dent in it." If a simple closed curve is not concave, it is convex.
Symmetrical, nonsymmetrical	Shapes may have one or more lines of symmetry.
Regular	All sides and all angles are congruent.
Polygons	Simple closed curves with all straight sides.
Triangles	
Triangles	Polygons with exactly three sides.
Classified by sides	
Equilateral	All sides are congruent.
Isosceles	At least two sides are congruent.
Scalene	No two sides are congruent.
Classified by angles	
Right	One angle is a right angle.
Acute	All angles are smaller than a right angle.
Obtuse	One angle is larger than a right angle.
Convex Quadrilaterals	
Convex quadrilaterals	Convex polygons with exactly four sides.
Kite	Two opposing pairs of congruent adjacent sides.
Trapezoid	At least one pair of parallel sides.
Isosceles	A pair of opposite sides is congruent.
Parallelogram	Two pairs of parallel sides.
Rectangle	Parallelogram with a right angle.
Rhombus	Parallelogram with all sides congruent.
Square	Parallelogram with a right angle and all sides congruent.

*Some definitions of trapezoid are exclusive and specify *only one* pair of parallel sides, in which case, parallelograms would not be trapezoids. We use the more inclusive definition of "at least one pair," meaning that parallelograms and rectangles are trapezoids. Some states mandate one definition over another, so consult your local curriculum standards.

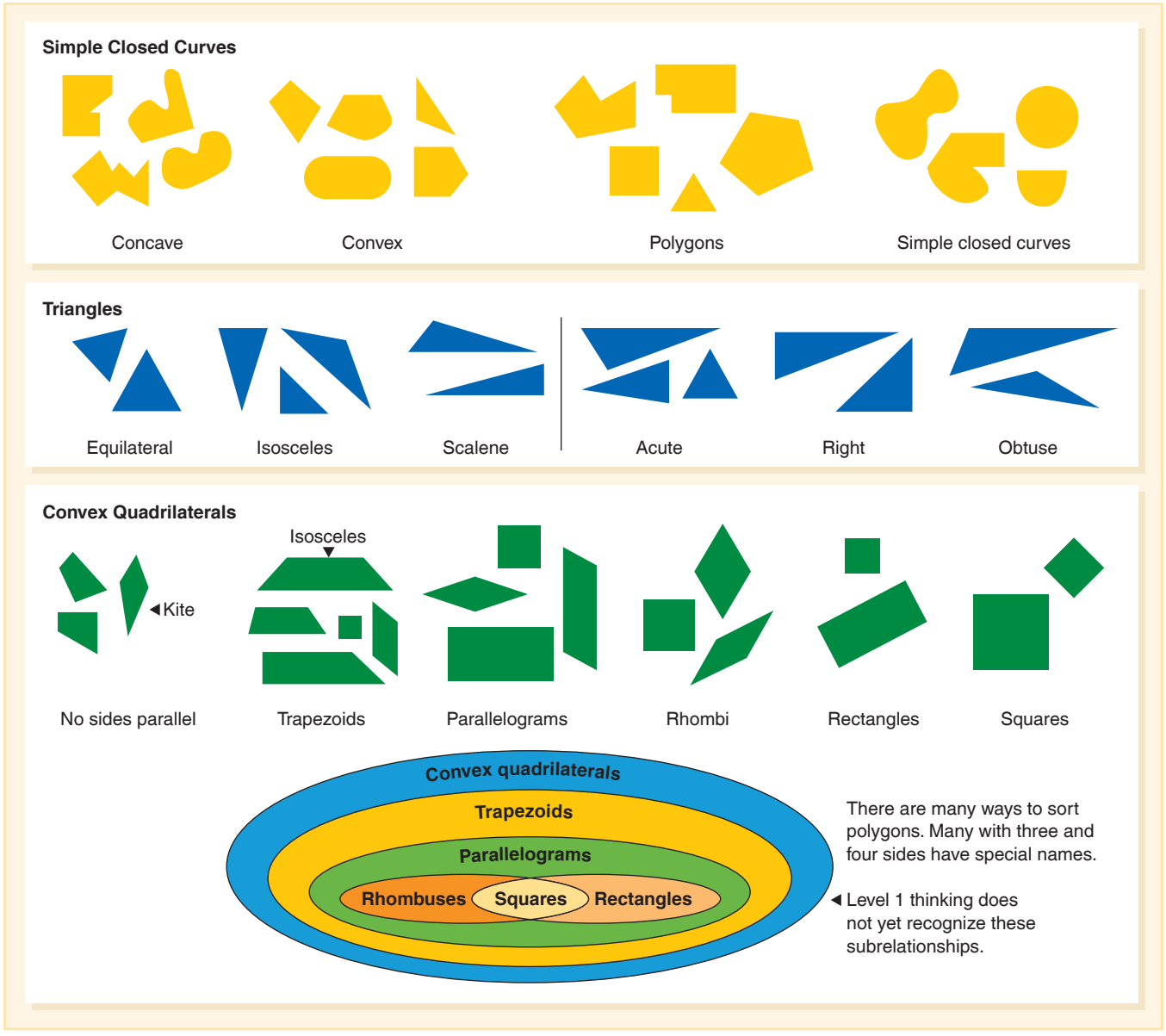


FIGURE 19.10 Classification of two-dimensional shapes

Activity 19.12 Shifting Shapes

CCSS-M: K.G.A.1; K.G.B.4; 1.G.A.1; 1.G.A.2; 2.G.A.1; 3.G.A.1; 4.G.A.2

The *Greedy Triangle* (Burns, 1995) is a delightful book about a triangle that is very busy being a sail on a boat or a roof on a house. Soon he becomes bored and travels to a shape-shifter to get one more side and one more angle. Now a quadrilateral, he fits into four-sided figures in the environment. There are several other shape-shifts with interesting results. Using a meter-long loop of yarn for every pair of students (or three ace bandages tied in a loop for a demonstration), have students follow and discuss events in the book by creating matching shapes with the loop (holding it in the air between their hands). First, they can explore different triangles, and eventually, they can investigate attributes of shapes as they shift from shape to shape. They can also move to level 2 thinking—by providing proof that they have a square, for example.

TABLE 19.5 CATEGORIES OF THREE-DIMENSIONAL SHAPES

Shape	Description
Sorted by Edges and Vertices	
Spheres and egg-like shapes	Shapes with no *edges* and no *vertices* (corners). Shapes with *edges* but no *vertices* (for example, a saucer). Shapes with *vertices* but no *edges* (for example, a football).
Sorted by Faces and Surfaces	
Polyhedron	Shapes made of all faces (a face is a flat surface of a solid). If all surfaces are faces, all the edges will be straight lines. Some combination of faces and rounded surfaces (circular cylinders are examples, but that is not a definition of a cylinder). Shapes with all curved surfaces. Shapes with and without edges and with and without vertices. Faces can be parallel. Parallel faces lie in planes that never intersect.
Cylinders	
Cylinder	Two congruent, parallel faces called *bases*. Lines joining corresponding points on the two bases are always parallel. These parallel lines are called *elements* of the cylinder.
Right cylinder	A cylinder with elements perpendicular to the bases. A cylinder that is not a right cylinder is an oblique cylinder.
Prism	A cylinder with polygons for bases. All prisms are special cases of cylinders.
Rectangular prism	A cylinder with rectangles for bases.
Cube	A square prism with square sides.
Cones	
Cone	A solid with exactly one face and a vertex that is not on the face. Straight lines (elements) can be drawn from any point on the edge of the base to the vertex. The base may be any shape at all. The vertex need not be directly over the base.
Circular cone	Cone with a circular base.
Pyramid	Cone with a polygon for a base. All faces joining the vertex are triangles. Pyramids are named by the shape of the base: *triangular* pyramid, *square* pyramid, *octagonal* pyramid, and so on. All pyramids are special cases of cones.

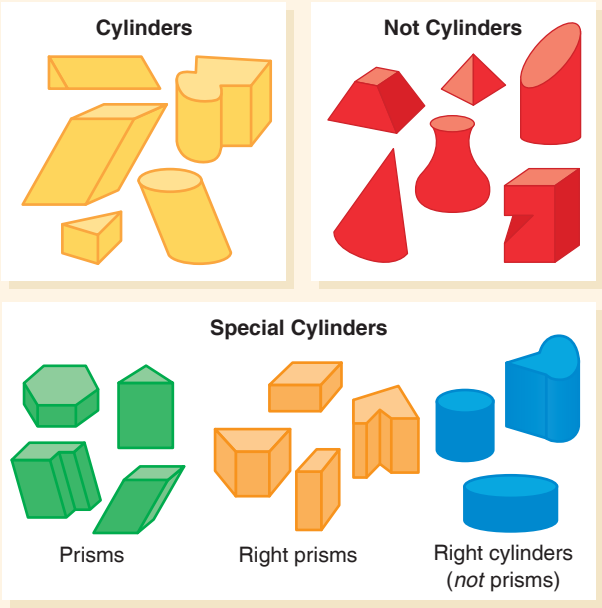
Pearson eText
Video Example 19.3
In this video the late professor Janet Caldwell discusses students' thinking about classes of shapes. Notice how she suggests the use of the language "special cases" to support students' thinking.

Three-Dimensional Shapes. Important shapes and relationships also exist in three dimensions. Table 19.5 describes classifications of solids (see https://www.nctm.org/Classroom-Resources/Illuminations/Interactives/Geometric-Solids/) for virtual images. See also the Categories of Two- and Three-dimensional Shapes activity page for reference. Three-dimensional shapes also have subcategories. Figure 19.11 shows examples of cylinders and prisms. Notice that prisms are defined here as a special category of cylinder—a cylinder with a polygon for a base (Zwillinger, 2011). Figure 19.12 shows a comparable grouping of cones and pyramids. Some textbooks may limit the definition of cylinders to just circular cylinders. Under that definition, the prism is not a special case of a cylinder. As you can see, definitions are conventions, and not all definitions are universally agreed on.

Pearson eText Activity Page: Categories of Two- and Three-Dimensional Shapes

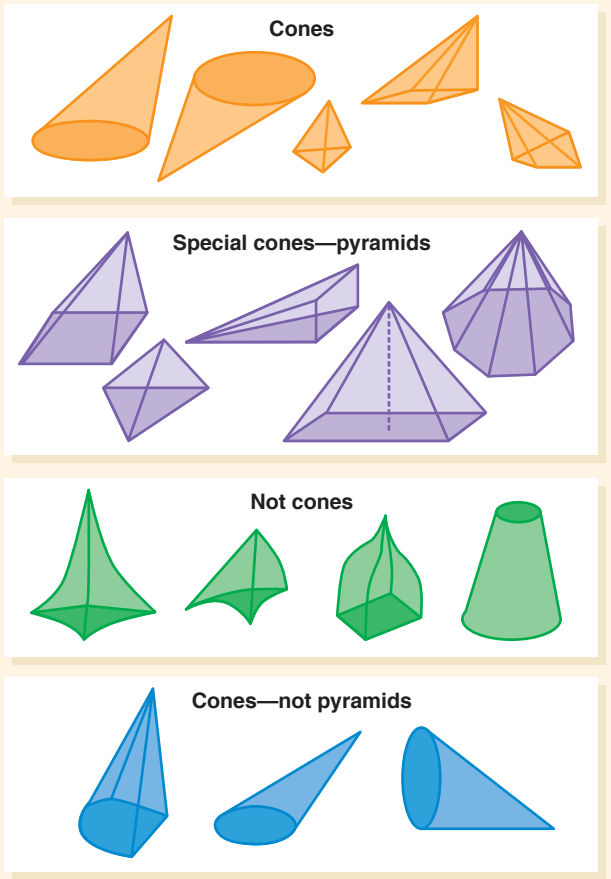
Construction Activities

Building or drawing two- and three-dimensional shapes is an important activity to help students think about properties and defining attributes of shapes. To help young learners explore solids and focus on learning about volume, there is good reason to



Cylinders have two parallel faces, and parallel lines join corresponding points on these faces. If the parallel faces are polygons, the cylinder can be called a prism.

FIGURE 19.11 Cylinders and prisms



Cones and cones with a polygon base (pyramids) all have straight-line elements joining every point of the base with the vertex. (Yes, a pyramid is just a special type of cone.)

FIGURE 19.12 Cones and pyramids

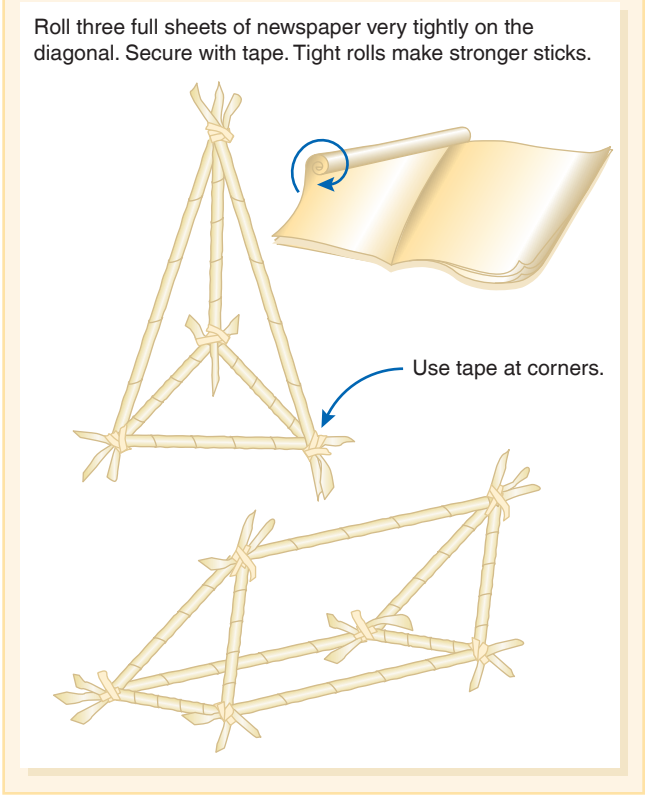

Roll three full sheets of newspaper very tightly on the diagonal. Secure with tape. Tight rolls make stronger sticks.
Use tape at corners.

FIGURE 19.13 Large skeletal structures can be built with tightly rolled newspaper

explore the construction of three-dimensional shapes. Building three-dimensional shapes is more difficult than building two-dimensional shapes but is worthwhile to gain insights about the shapes.

Activity 19.13 Constructing Three-Dimensional Shapes

CCSS-M: K.G.A.3; K.G.B.5; 1.G.A.2; 2.G.A.1; 6.G.A.4

Collect one of the following sets of materials for students to use for construction:

1. Plastic coffee stirrers cut into a variety of lengths with twist ties inserted in the ends to use for attaching, or modeling clay to connect corners
2. Plastic drinking straws cut lengthwise from the top down to the flexible joint—insert the slit ends into the uncut bottom ends of other straws, making strong but flexible joints
3. Rolled newspaper rods connected with masking or duct tape to create skeletons (see Figure 19.13)

With these constructions of concrete representations, discuss the strength and rigidity of triangular components in the structures. Point out that this strength is why triangles are used in bridges, the booms of construction cranes, and in the structural parts of buildings. These models can also be used to explore volume and surface area.

Applying Definitions and Categories

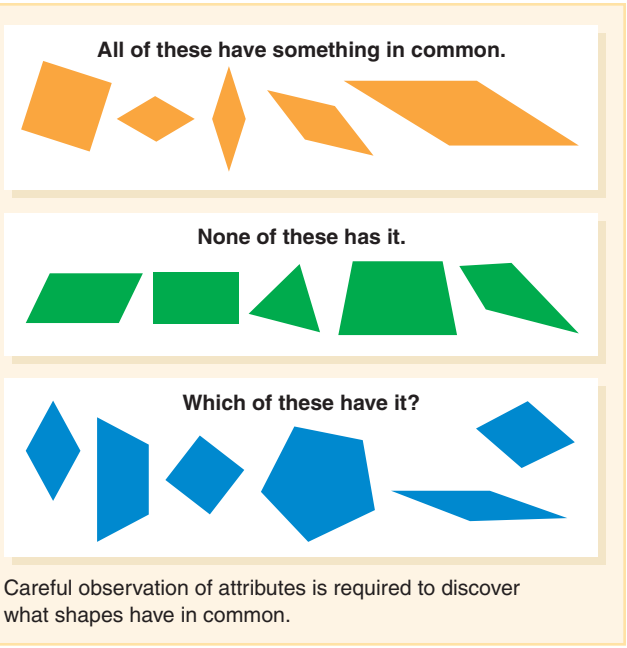
Using definitions and categories helps students focus more deeply on the properties that make the shape what it is (and what it is not). The next activity provides a way to introduce a category of shapes.

Activity 19.14 Mystery Definition

CCSS-M: 1.G.A.1; 2.G.A.1; 3.G.A.2; 4.G.A.2

Give students the Mystery Definition activity page, or project a grade-level-appropriate logic problem (such as the example in Figure 19.14). At the top of your sheet, for your first collection, be certain that you have allowed for all possible correct options. For example, in the first grouping, a square is included in the set of rhombuses. Also choose non-examples to be as close to the positive examples as is necessary to help develop a more precise definition. The third or mixed set should also include non-examples that will challenge students or confuse them. This contrast between examples and non-examples develops definitional reasoning and generalizations (Kobiela et al., 2018). Use the class discussion to hear students' justifications. The use of non-examples is particularly important for students with special needs in math.

Pearson eText Activity Page: Mystery Definition



Careful observation of attributes is required to discover what shapes have in common.

FIGURE 19.14 A mystery definition

The value of the Mystery Definition activity is that students develop ideas and informal definitions based on their own concept development. After their definitions have been discussed, compared, and refined, you can contrast their ideas to the conventional definition for that shape.

Exploring Triangles

Determining classes of triangles is introduced in fourth grade, with the concept of right triangles in fifth grade as students continue a focus on categories and subcategories of shapes. Then in seventh grade, students focus on the properties based on the measures of the sides and angles. For defining classes or categories of triangles, the next activity is a good starting place.

Activity 19.15 Triangle Sort

CCSS-M: 2.G.A.1; 3.G.A.2; 4.G.A.2; 5.G.B.3; 5.G.B.4; 7.G.A.2

Have teams cut out the triangles from the Assorted Triangles activity page, which includes examples of right, acute, obtuse, equilateral, isosceles, and scalene triangles. Ask, "Can you sort the entire collection into three discrete (non-overlapping) groups so that no triangle belongs to two groups?" When this first sort is done and descriptions of the groupings have been written, have students find a second criterion for creating three different groupings. Students with special needs in math may need a suggestion to look only at angle sizes or only at the issue of congruent sides, but delay giving these hints if possible. Once the two groupings have been determined, have students discuss and share appropriate terminology. For multilingual learners, focus on the specialized meaning of these terms (such as contrasting "acute pain" and "acute angle") and root words (*equi-* meaning *equal* and *lateral* meaning *side*). As a follow-up, challenge students to sketch a triangle in each of the nine cells of the Triangle Sort Chart. Extend the activity by repeating the process, using kites and trapezoids or introduce three-dimensional shape definitions.

Pearson eText Activity Page: Assorted Triangles

Pearson eText Activity Page: Triangle Sort Chart

Pause & Reflect

Of the nine cells in the Triangle Sort Chart, two of them are impossible to fill. Can you tell which ones and why? ●

Pearson eText
Video Example 19.4
In this video the late professor Janet Caldwell discusses the importance of considering the properties of shapes in figuring out how to identify a particular shape.

Standards for Mathematical Practice
MP6. Attend to precision

An important aspect of looking at attributes and classes of triangles in intermediate and middle grades is to consider when given conditions result in a unique triangle, more than one triangle, or no triangle—the focus of Activity 19.16.

Activity 19.16 Can You Build It?

CCSS-M: 4.G.A.2; 4.G.A.3; 5.G.B.3; 5.G.B.4; 7.G.A.2

Provide questions about unique shapes and ask students if they can build or draw the shapes according to specifications. Always add the question—"How many ways?"

Can you make:

A. A shape with exactly one right angle and exactly four sides?
B. A shape with exactly one line of symmetry (or two lines of symmetry)?
C. A triangle with one 90-degree angle and two sides, of 5 cm and 8 cm?
D. A triangle with two angles of 45 degrees?
E. A triangle with sides of 4 cm, 8 cm, and 13 cm?
F. A right triangle with legs (once right triangles are introduced; use the term *legs* for the two sides and hypotenuse to name the third side) of 3 units and 4 units?
G. An isosceles triangle with an 80-degree angle and a side of 5 cm?

Angles of a Triangle. Middle-school students explore the relationship of angles within a triangle (adding to 180 degrees) and supplementary, complementary, vertical, and adjacent angles to solve for an unknown angle in a figure. Activity 19.17 explores interior angles.

Activity 19.17 Angle Sum in a Triangle

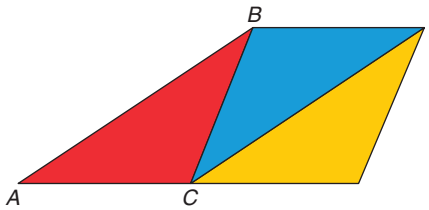
CCSS-M: 5.G.B.3; 7.G.A.2; 8.G.A.5

Distribute three copies of the Three Congruent Triangles activity page to pairs of students and have them label each triangle with angles *A, B*, and *C*, making sure the corresponding angles on the equivalent triangles have the same letters, and then cut out each set of triangles. Ask students to explore what they notice about the angles in one set of congruent triangles. If they do not discover that together they sum to 180 degrees, offer the following steps:

1. Place one triangle on a line and the second directly next to it in the same orientation.
2. Place the third triangle in the space between the triangles as shown in Figure 19.15(a). Ask, "Will this relationship be true for any kind of triangle?"
3. Test other sets of different equivalent triangles.
4. What conjecture can you make about the sum of the angles in a triangle?

Pearson eText Activity Page: Three Congruent Triangles

(a) Three congruent triangles can be arranged to show that the sum of the interior angles will always be a straight angle or 180 degrees.



(b) Draw *CE* parallel to *AB*. Why is angle *BAC* congruent to angle *ECD*? Why is angle *ABC* congruent to angle *BCE*?

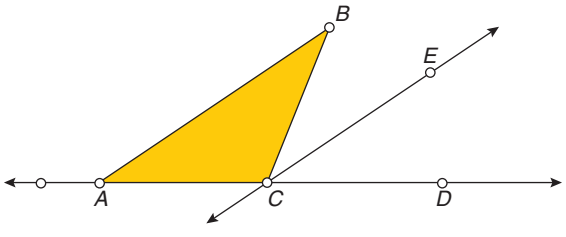


FIGURE 19.15 Deductive, logical reasoning can support proving relationships that appear true from observations

Technology can be an important tool in exploring these relationships, with the ability to drag points as one of the most critical features to developing ideas of what varies and what stays constant (Sinclair et al., 2016; Battista, 2007). For example, have students use a dynamic geometry environment to draw a triangle, measure all the angles, and add them up. As the triangle vertices

are dragged around, the sum of the angles remains at 180 degrees. Students can conjecture that the sum of the interior angles of a triangle is always 180 degrees (they can be led to the same conjecture by tearing off the angles of paper triangles and combining them to create a straight line). To know why this is true requires examining exterior angles as illustrated in Figure 19.15(a). There are lines parallel to each side of the original triangle. By using facts about supplementary, complementary, vertical, and adjacent angles formed by cutting parallel lines with a transversal, it is straightforward to argue that the sum of the angles will always be a straight line. See Figure 19.15(b) to explore a way to think about congruence involving exterior angles.

Exploring Quadrilaterals

Quadrilaterals (polygons with four sides) are a rich source of investigations, as shared in Activities 19.3 and 19.4, where students create property lists with parallelograms, then rhombuses, then rectangles, and finally squares (19.3) and then create minimal defining lists (definitions). Generating definitions and comparing them to the common definition for that shape helps students understand the nesting of shapes (e.g., squares are a subset of rectangles).

As new relationships come up, introduce precise terminology, such as if two diagonals intersect in a square corner, then they are perpendicular. Other terms, such as *parallel*, *congruent*, *bisect*, *midpoint*, can be clarified. Also, introduce symbols such as $\cong$ for *congruent*, or $\|$ for *parallel*. Once students are familiar with the concepts and terminology of right, obtuse, and acute angles, congruence of line segments and angles, and line symmetry, Activity 19.3, "Property Lists for Quadrilaterals," is a good way to bring these ideas together. This language is also reinforced as students explore Activity 19.4, "Minimal Defining Lists." One of the most interesting properties that can be a minimal defining property relates to diagonals of quadrilaterals. For example, a minimal defining list for a rhombus can be: quadrilateral with diagonals that bisect each other and are perpendicular. Activity 19.18 explores the properties of diagonals of various classes of quadrilaterals.

Standards for Mathematical Practice
MP7. Look and make use of structure

Pearson eText
Video Example 19.5
In this video the late professor Janet Caldwell discusses lines, sides, and angles as ways to identify shapes.

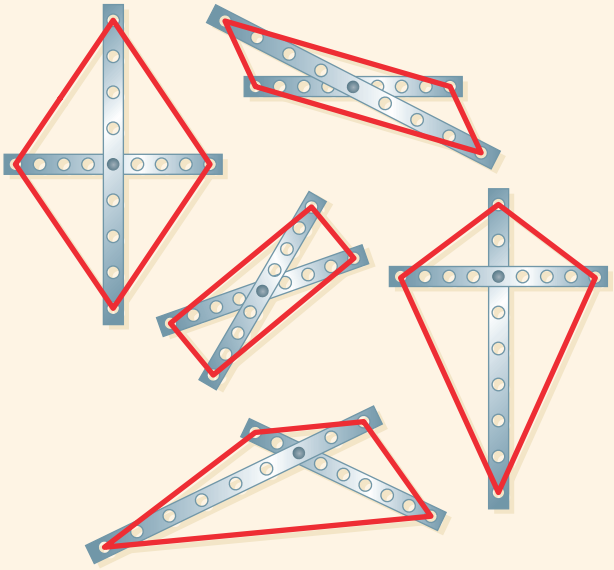
Activity 19.18 Diagonals of Quadrilaterals

CCSS-M: 4.G.A.2; 5.G.B.3; 5.G.B.4; 6.RP.A.1; 7.G.A.2

Give each student three card stock diagonal strips. Punch the nine holes as marked. Use a brass fastener to join two strips. A quadrilateral is formed by joining the four end holes as shown in Figure 19.16. Provide students with the list of possible relationships for angles, lengths, and proportional comparisons of parts (ratios). Ask, "What quadrilaterals can these diagonals form when diagonals are the same length? Perpendicular? Bisect each other? Perpendicular and the same length? Have students use the strips to determine the properties of diagonals that will produce different quadrilaterals. Students can make drawings on 1-centimeter dot paper to test various hypotheses and record their findings on the Properties of Quadrilateral Diagonals activity page. See the Expanded Lesson: Diagonals of Quadrilaterals for a description of this instructional experience.

This activity can also be explored with dynamic geometry environments such as GeoGebra, where points, lines, and geometric figures are easily constructed and objects can be moved about and manipulated in endless ways.

Pearson eText Activity Page: Diagonal Strips
Pearson eText Blackline Master: 1-Centimeter Dot Paper
Pearson eText Activity Page: Properties of Quadrilateral Diagonals
Pearson eText Expanded Lesson: Diagonals of Quadrilaterals



Quadrilaterals can be determined by their diagonals. Consider the length of each, where they cross, and the angles between them. What conditions will produce parallelograms? Rectangles? Rhombuses Challenge: What properties will produce a non-isosceles trapezoid?

FIGURE 19.16 Diagonals of quadrilaterals

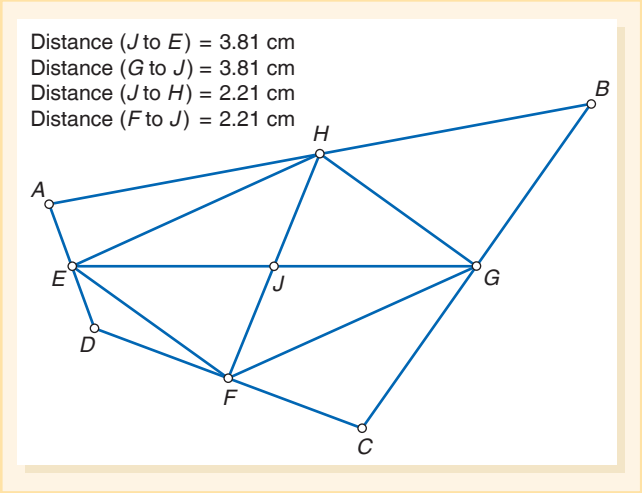


FIGURE 19.17 A construction made with dynamic geometry environments to illustrate what happens to the diagonals in quadrilateral EFGH

Every type of quadrilateral can be uniquely described in terms of its diagonals, using only the conditions of length, proportional comparison of parts, and whether they are perpendicular. Dynamic geometry virtual tools allow students to drag, invert, stretch and shrink shapes to notice what is changing and what is staying the same. For example, in Figure 19.17, the midpoints of quadrilateral *ABCD* have been joined. The diagonals of the resulting quadrilateral (*EFGH*) are also drawn and measured.

Exploring Circles

Circle measurements (circumference and area) are discussed in Chapter 18. Beyond those measurements, many interesting relationships can be observed among measures of different parts of the circle. One of the most astounding and important is the ratio between measures of the circumference and the diameter: pi.

Activity 19.19 Discovering Pi

CCSS-M: 7.G.B.4

Have groups of students carefully measure both the circumference and diameter of a collection of circular items such as jar lids, cans, wastebaskets, and even a quarter. Consider also measuring large circles marked on gym floors and playgrounds with a trundle wheel or rope. To measure circumference, wrap string once around the object and then measure that length of string.

Students gather the circumference and diameter from a variety of circular items and enter the measures in the Circle Table activity page. Have them divide to find the ratio of the circumference to the diameter for each circle. The exact ratio is an irrational number, about 3.14159, represented by the Greek letter π (pi).

Pearson eText Activity Page: Circle Table

What is most important in Activity 19.19 is that students develop a clear understanding of π as the ratio of circumference to diameter in any circle regardless of size. The quantity π is not some mystical number that appears in mathematics formulas; it is a naturally occurring and universal ratio.

Conjectures and Proofs

Remember, if you write a theorem on the board and ask students to prove it, you have already told them that it is true. If, by contrast, a student makes a statement about a geometric situation the class is exploring, it can be written with a question mark as a conjecture—a statement whose truth has not yet been determined. You can ask, "Is it true? Always? Can we prove it? Can we find a counterexample?"

Standards for Mathematical Practice
MP7. Look for and make use of structure

On the path to more formal deduction, students should construct diagrams and consider pertinent definitions (Sinclair, Pimm, & Skelin, 2012). Students should begin to notice evidence of a pattern and then check it with multiple approaches based on mathematical properties and structure.

Exploring Polygons. As students develop an understanding of various geometric properties and attach these properties to categories of shapes, it is essential to encourage conjecture and explore informal deductive arguments to develop logical reasoning. This classifying includes exploring polygons and examining the properties and conditions that make shapes unique or possible. Students can begin to analyze and create simple proofs.

Activity 19.20 uses quadrilaterals, but the activity also applies to other polygons and three-dimensional shapes.

Activity 19.20 True or False?

CCSS-M: 4.G.A.2; 5.G.B.3; 5.G.B.4; 7.G.A.2

Prepare a set of true/false statements of the following forms: "If it is a _____, then it is also a _____." "All _____ are _____." "Some _____ are _____." Or use the True/False Statements activity page. Ask students to determine whether the statements are true or false and present a logical argument to support the decision (see Figure 19.18). Reasoning about four or five true/false statements will make a good lesson. Once this format is understood, let students challenge their classmates by making their own lists of a mixture of true/false statements.

Pearson eText Activity Page: True/False Statements

FORMATIVE ASSESSMENT Notes. The True or False? activity is also a good diagnostic assessment. Notice how the student's work (Figure 19.18) gives insights into ideas, representations, and emerging arguments for the choices made. ■

The Pythagorean Theorem. The *Pythagorean theorem*, commonly explored in eighth grade, is one of the most important mathematical relationships and warrants in-depth conceptual investigation. In geometric terms, this relationship states that if a square is constructed on each side (leg) of a right triangle, the areas of the two smaller squares, when combined, will equal the area of the square on the longest side, the hypotenuse.

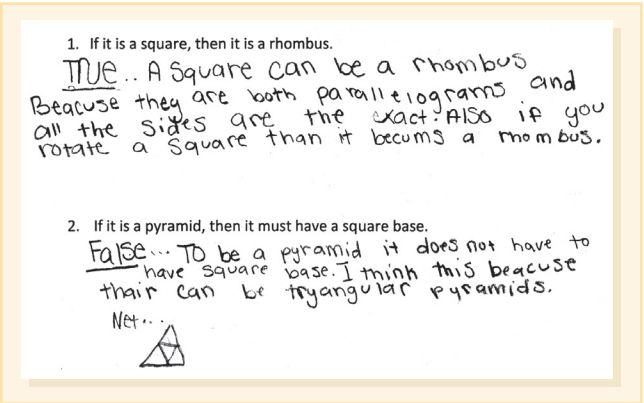
Activity 19.21 The Pythagorean Relationship

CCSS-M: 8.G.B.6; 8.G.B.7

Have students draw a right triangle on 0.5-centimeter grid paper. Assign each student a different triangle by specifying the lengths of the two legs. Students should draw a square on each leg and the hypotenuse and find the areas of all three squares. For the square on the hypotenuse, the area can be found by making each of the sides the diagonal of a rectangle (see Figure 19.19, which shows $4 + 16 = 20$, the area in square units of the square on the hypotenuse). Have students record their own data on the Table of the Areas activity age and then have them collect and record data from two teams with different triangles. Ask students to look for a relationship between the three squares of a particular triangle.

Pearson eText Blackline Master: 0.5-Centimeter Grid Paper

Pearson eText Activity Page: Table of the Areas


1. If it is a square, then it is a rhombus.
TRUE.. A Square can be a rhombus Beacuse they are both parallelograms and all the sides are the exact. Also if you rotate a Square than it becums a rhombus.

2. If it is a pyramid, then it must have a square base.
False... To be a pyramid it does not have to have square base. I think this beacuse thair can be tryangular pyramids.
Net...

FIGURE 19.18 True or false? A fifth grader presents an argument through logical reasoning

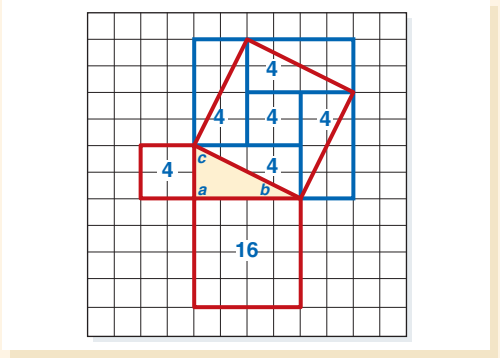
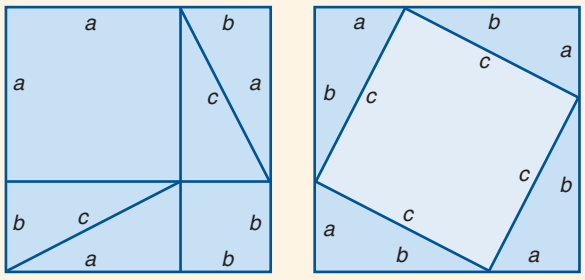


FIGURE 19.19 The Pythagorean relationship



$a^2 + 2ab + b^2 = c^2 + 2ab$

$a^2 + b^2 = c^2$

FIGURE 19.20 The two large squares together are a "proof without words." Can you supply the words?

The two large congruent squares in Figure 19.20 together show a visual proof of the Pythagorean theorem (Nelson, 2001). Notice that both large squares are decomposed into squares and triangles—and that the four triangles in each large square are the same but arranged differently. By adding up the areas of the squares and the triangles in each large square and setting them equal to each other, the Pythagorean relationship can be found by subtracting out the common areas in both squares. An algebraic recording of the thinking process is shown below the drawings. Share with students the applet at Illuminations entitled, "Proof without Words: Pythagorean Theorem," or the video that follows here.

Pause & Reflect

Use the two drawings in Figure 19.20 to create a proof of the Pythagorean relationship. ●

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Pearson eText
Video Example 19.6
In this video consider how water is used to prove the Pythagorean Theorem without words.

https://www.youtube.com/watch?v=CAkMUdeB06o

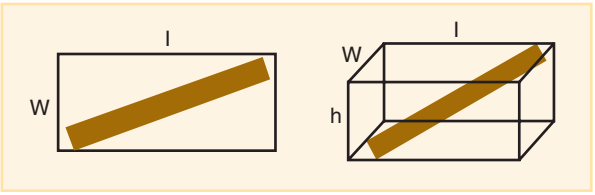
Pearson eText
Video Example 19.7
In this video consider how the Pythagorean Theorem can be demonstrated with a "Proof without Words" by exploring the relationship using paper cut outs of triangles and a square.

https://www.youtube.com/watch?v=gjSAE8_FahM

Activity 19.22 Saber in a Box

CCSS-M: 8.G.B.6; 8.G.B.7

Have students work in pairs. Students are asked to ship a 30-inch lightsaber to a friend. They have two tasks on the Saber in a Box activity page, which asks them to select from three box options. First, they will consider just the two dimensions of the box bottom, and then they will move to a box, where three dimensions offer another way of thinking.



Pearson eText Activity Page: Saber in Box

Eighth-grade students should become familiar with Pythagorean triples. Any set of three whole numbers that satisfy the Pythagorean theorem is called a *Pythagorean triple*. Pythagorean triples occur often in geometry tasks and are often disguised as multiples of commonly recognized triples.

Activity 19.23 Finding Pythagorean Triples

CCSS-M: 8.G.B.6; 8.G.B.7

Begin with the most common Pythagorean triple, 3-4-5. Ask, "Will triangles that are similar to the 3-4-5 triangle also be Pythagorean triples?" Give students a ruler, grid paper, and a calculator (or have them explore with dynamic geometric environments). Ask, "Can you find at least three triples that form triangles similar to the 3-4-5 triangle?" (There are infinitely many, so once students notice a pattern, stop the exploration and discuss strategies for how to recognize the 3-4-5 in disguise.)

Pearson eText Blackline Master: 1-Centimeter Grid Paper

Transformations

Learning Outcome 19.3 **Identify approaches to teaching geometric transformations.**

Transformations are changes in position or size of a shape and are a major focus of eighth grade. The study of line symmetry (introduced in fourth grade) is also included under the study of transformations as reflections. Movements that do not change the size or shape of the object moved are called *rigid motions*. *Translations*, *reflections*, and *rotations* are rigid transformations that result in congruent shapes (Figure 19.21). Conversely, movements that change size and/or shape are called nonrigid motions and include dilations (Figure 19.28).

Here are the definitions of the rigid motions:

Translation: A translation (slide) requires a *direction* and a *distance*. In a translation, every point on the preimage moves in the same direction for the same distance to form the image. On a coordinate grid, for example, you would be moving up 2 and over 3 for each of the points in the figure.

Reflection: A reflection (flip) requires a *line of reflection*. A reflection is a transformation in which an object is flipped across a line of reflection. The line of symmetry can be the x-axis, the y-axis, or any other line. If a shape is reflected over the x-axis, for example, the x-values of the pre-image are the opposite of the x-values of the image, and the y-values in both images are the same.

Rotation: A rotation (turn) requires a *center of rotation* (point) and a *degree of rotation*. The point can be any point on the coordinate axis, although in middle school, rotation around the origin is most common. A figure can be rotated up to 360 degrees.

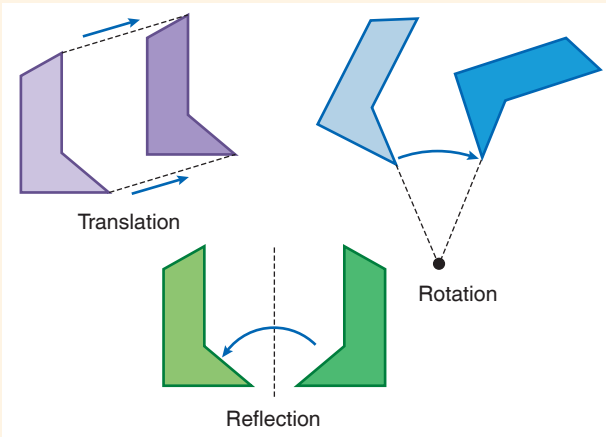


FIGURE 19.21 Rigid transformations: Translation (slide), reflection (flip), and rotation (turn)

The Motion Man activity described next can also be used to introduce students to the terms *slide*, *flip*, and *turn*.

Activity 19.24 Motion Man

CCSS-M: 8.G.A.1a, b, c

Make a copy for each student of the Motion Man activity page and the Mirror Image Motion Man activity page, back to back, so that they match when held to the light, as in Figure 19.22. First, have students demonstrate each of the rigid motions. A slide is simply that—Motion Man moves over a particular distance. Say, "Translate Motion Man 4 inches right and 2 inches down." A reflection requires movement on a line; say, "Draw a vertical or horizontal line and reflect Motion Man over that line." Have them demonstrate a horizontal flip (top goes to bottom) and a vertical flip (left goes to right). A rotation requires a center and a clockwise degree measure. Say, "Rotate Motion Man 90 degrees." For all students, multilingual learners in particular, ensure that these demonstrations include explicit practice with the vocabulary and matching visuals.

Pearson eText Activity Page: Motion Man

Pearson eText Activity Page: Mirror Image Motion Man

Begin with the Motion Man in the left position shown in Figure 19.22 (preimage). Now place a second Motion Man next to the first (image). How did he get there? Will it take one move or more than one move (transformation) to get from the first to the second Motion Man? The task is to decide and test what motion or combination of motions will get the man on the left to match the man on the right. If both men are in the same position, call that a slide. There are often numerous ways to get him to the new position.

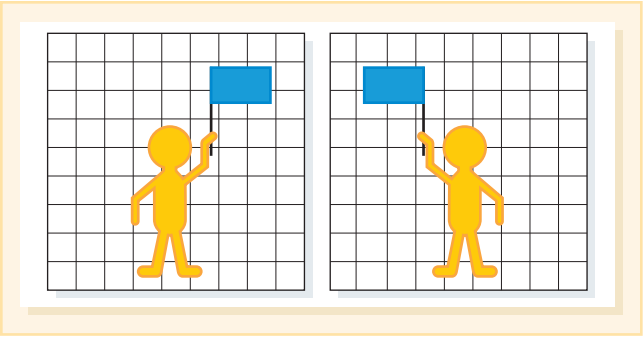


FIGURE 19.22 Motion Man is used to show slides, flips, and turns

Notice that although the center of the rotation will be the center of the figure, and reflections will be flips over vertical or horizontal lines, these restrictions are for initial instruction. In fact, the center of rotation can be anywhere on or off the figure, and lines of reflection can be anywhere. Also see the app https://www.teacherled.com/2015/01/13/symmetry-shapes/. Over time, students can use blank paper with an oblique line of reflection (Teuscher et al., 2021).

Symmetry

The bridge between transformations at the elementary level and the middle school is the area of reflection and symmetry. At first, students may think line symmetry is only vertical and rotation symmetry is only around a central point, but by offering many experiences, they will be challenged to expand their definitions.

Line Symmetry. If a shape can be folded on a line so that the two halves match precisely, then it is said to have *line symmetry* (sometimes called *reflectional* or *mirror symmetry*). Notice that the fold line is actually a line of *reflection*—the portion of the shape on one side of the line is reflected onto the other side. Again, that movement is the connection between line symmetry and transformations.

One way to introduce line symmetry to students is to show examples and non-examples, using an all-of-these/none-of-these approach, as in Activity 19.14. Here's another possibility:

Fold a sheet of paper in half and cut out a shape of your choosing on the side with the fold. When you open the paper, what do you notice?

Another approach is to use mirrors. When you place a mirror on a picture or design so that the mirror is perpendicular to the table, you see a shape with symmetry when you look in the mirror. Explore the "Shape Tool" interactive on the NCTM Illuminations website where students can investigate symmetry with a virtual mirror. See also how art can be integrated with transformations with https://www.mathsisfun.com/geometry/symmetry-artist.html.

Activity 19.25 Pattern Block Mirror Symmetry

CCSS-M: 4.G.A.3

Give students a plain sheet of paper with a straight line drawn through the middle. Ask students to use about six to eight pattern blocks to make a design on only one side of the line that touches the line in some way. Then, students try to make the mirror image of their design on the other side of the line. After the design is complete, have students use a mirror to check their work. They place the mirror on the line and look into it from the side of the original design. With the mirror in place, they should see the same image they see when they raise the mirror. For students with special needs in math, at first make the line of reflection vertical with a left and right side. Then move to a line oriented horizontally or diagonally. Challenge students to make designs with more than one line of symmetry.

The same task can be done with tangram pieces or created on a geoboard. If students try the geoboard, first they stretch a band to represent the line of symmetry. Then they make a design on one side of the line and its mirror image on the other. Students can also investigate quilt patterns to identify how many lines of symmetry are in traditional block patterns (Roscoe & Zephyrs, 2016). Isometric or rectangular dot grids can support creating and analyzing shapes for symmetry, as in the following activity.

Activity 19.26 Dot Grid Line Symmetry

CCSS-M: 4.G.A.3

Give students a piece of either 1-centimeter isometric dot paper or 1-centimeter dot paper. Students should draw a horizontal, vertical, or diagonal line through several dots. Students make a design completely on one side of the drawn line that touches the line, as in the figures shown on the left in Figure 19.23. Have students make the mirror image of their design on the other side of the line or have them exchange their partial design with a peer who must finish it. When finished, they can check by placing a mirror on the line and looking into the mirror to see the same image as when they lift the mirror.

Pearson eText Blackline Master: 1-Centimeter Isometric Dot Paper
Pearson eText Blackline Master: 1-Centimeter Dot Paper

Have students try these two problems:

A shape with one line of symmetry has exactly 6 sides and two 90-degree angles. Can you draw the shape?

A quadrilateral has diagonals that do not form lines of symmetry, but the quadrilateral is symmetrical (one line). Can you draw the quadrilateral?

These exercises combine several key areas of geometry such as line symmetry, properties of shapes, and visualization with reasoning. Try to have students come up with other problems for their classmates to draw.

Rotational Symmetry. A shape has *rotational symmetry* (also referred to as *point symmetry*) if it can be *rotated* about a point and land in a position exactly matching the one in which it began.

A good way to understand rotational symmetry is to trace around a shape with rotational symmetry, such as a square, on a piece of paper. Call this tracing the shape's footprint. The degrees refer to the smallest angle of rotation required before the shape matches itself or fits into its footprint. The parallelogram in Figure 19.24 has *180-degree rotational symmetry*. A square has *90-degree rotational symmetry*.

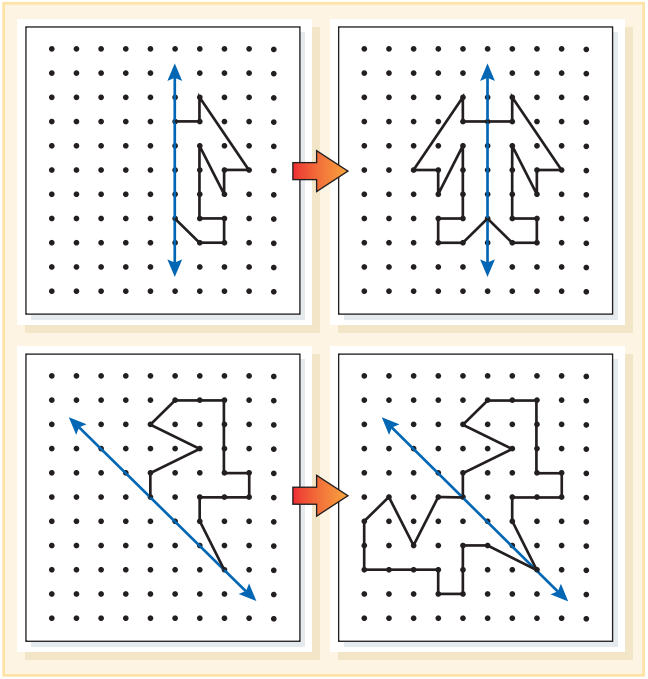


FIGURE 19.23 Exploring symmetry on dot grids

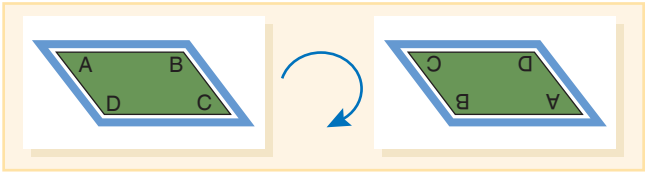


FIGURE 19.24 A parallelogram is rotated 180 degrees and matches the original position

Composition of Transformations

One transformation can be followed by another. For example, a figure can be reflected over a line, and then that figure can be rotated about a point. A combination of two or more transformations is called a *composition*.

Exploration with these ideas can result in interesting generalizations and conjectures (Lo & Cox, 2020). For example, have students experiment with compositions of two or even three transformations, using a simple shape on 1-centimeter dot paper as a step toward using coordinates on the coordinate axis. For example, have students draw an L shape on a dot grid and label it L_1 (Figure 19.25). Reflect it over a line and then rotate the image $\frac{1}{4}$ turn clockwise about a point not on the shape. Call this image L_2, the image of a composition of a reflection followed by a rotation. Notice that if L_1 is rotated $\frac{1}{4}$ turn clockwise about the same point used before (the result of which we will call L_3), there is a relationship between L_2 and L_3. Continue to explore different combinations of transformations, including translations (slides), too.

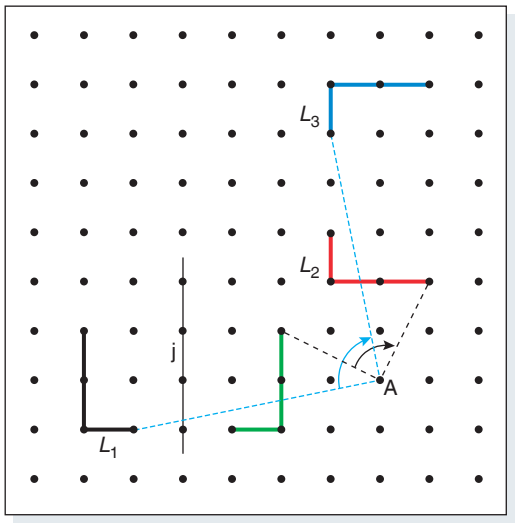


FIGURE 19.25 How are L_2 and L_3 related? Will this approach always work?

Pearson eText Blackline Master: 1-Centimeter Dot Paper

Compositions do not have to involve different types of transformations. For example, a reflection can be followed by another reflection. Take advantage of engaging technology for compositions of transformations such as one at GeoGebra (https://www.geogebra.org/m/fHxE39Uf).

Tessellations. Tessellations are a motivating and artistic application of transformations. A *tessellation* is a tiling of the plane, using one or more shapes in a repeating pattern with no gaps or overlaps (see Figure 19.26). Tessellations are based on a circle—if the angle measures add up to 360 degrees, the shapes will fit together at a vertex with no overlaps or gaps. A *regular tessellation* is made of a single polygon with congruent sides and angles.

Pause & Reflect

Which regular polygons can be used to form regular tessellations? ●

A *semiregular tessellation* is made of two or more different regular polygons. At each vertex of a semiregular tessellation, the same collection of regular polygons comes together in the same order. Students can figure out which polygons are possible at a vertex and design their own semiregular tessellations. Have them use either transformations or combinations of compatible polygons on 1-centimeter grid paper to create tessellations that are artistic and quite complex. Digital versions of pattern blocks allow children to develop more complicated and innovative designs than with actual pattern blocks (Sinclair & Bruce, 2015). NCTM's Illuminations website also has a tool called Tessellation Creator that allows shapes to be combined to make patterns that repeat and cover the plane.

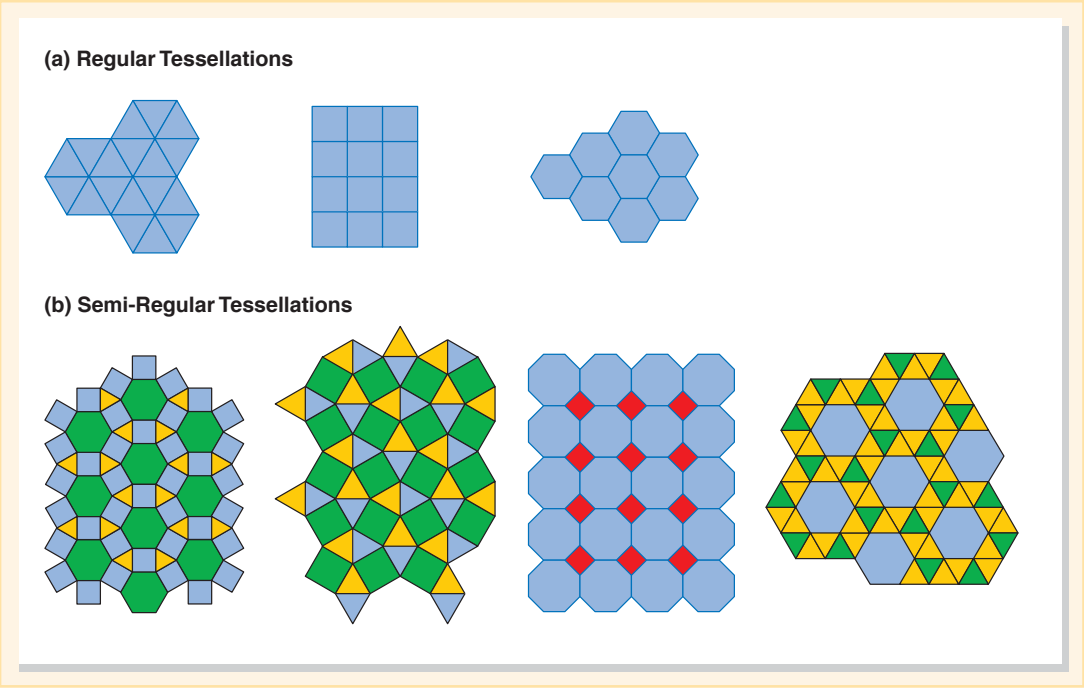


FIGURE 19.26 Tessellations

Pearson eText Blackline Master: 1-Centimeter Grid Paper

The Dutch artist M. C. Escher is famous for his tessellations, in which the tiles are very intricate and are often shaped like animals. Escher took a simple shape such as a triangle, parallelogram, or hexagon and altered the sides, applying transformations (see Figure 19.27). Once a tile has been designed, trace it over and over again.

Pearson eText Blackline Master: 1-Centimeter Dot Paper

Congruence

Congruent shapes are defined in terms of transformations. Two shapes are considered congruent if you can apply rigid transformations to one shape to obtain the other and describe the process.

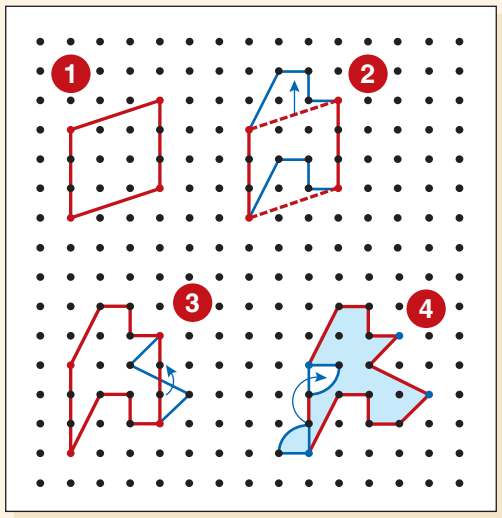
Having explored foundational experiences like Motion Man and transformations on dot paper, students are ready to explore transformations and compositions of transformations on the coordinate axis. A focus on congruence helps connect these two ideas, as in Activity 19.27.

Activity 19.27 Are They Congruent?

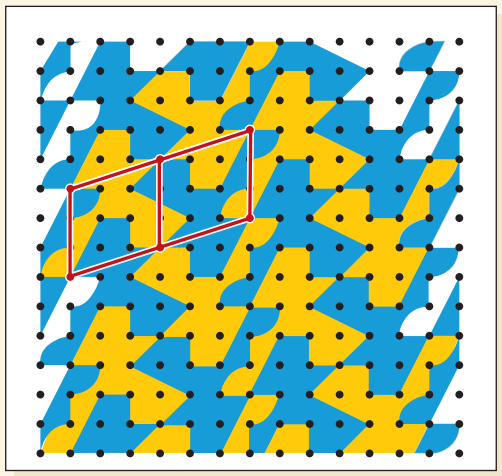
CCSS-M: 8.G.A.1; 8.G.A.2

Draw various triangles on the coordinate grid, some congruent and some not. Ask students to find a match of two congruent triangles and prove they are congruent by stating the transformations that they applied, in order, to get one shape to cover the matching shape precisely.

The coordinate axis is addressed in detail in a later section, "Location," in which transformations will continue to be explored.



1. Start with a simple shape.
2. Draw the same curve on two opposite sides. This tile will stack up in columns.
3. Rotate a curve on the midpoint of one side.
4. Rotate a curve on the midpoint of the other side. Use this tile for tessellation (below).



A column of the resulting tile will now match a like column that is rotated one complete turn. Find these rotated columns in the tessellation shown here.

FIGURE 19.27 Creating an Escher-type tessellation

Similarity

Two figures are *similar* if all of their corresponding angles are congruent and the corresponding sides are proportional. This can be proved if one triangle can be superimposed on the other triangle through a series of transformations—translations, reflections, rotations, or dilations. The connection between proportional reasoning and similarity is very important because similar figures provide a visual representation of proportions, and proportional thinking enhances the understanding of similarity (as in scale drawings).

Pearson eText Video Example 19.8

In this video consider how dynamic geometry environments create tessellations through transformations.

https://www.youtube.com/watch?v=ulnelraidNY

Dilation

A *dilation* is a nonrigid (can change size) transformation that produces similar two-dimensional figures (think of pinching or spreading an image with your fingers on a phone or computer screen). A dilation requires a *scale factor*. Scale factors less than 1 produce smaller figures (pinches), and scale factors greater than 1 (spreading) produce larger figures (see Figure 19.28).

If different groups of students use the same scale factor to dilate the same figure, they will find that the resulting figures are all congruent, even with each group using different dilation points. Dynamic geometry environments make the results of this exercise quite dramatic because they allow for the scale factors to be set at any value.

To understand what is required to have a rigid motion means understanding what types of transformations distort a shape. In the next activity, students see that multiplying a constant times the coordinates is a transformation that is *not* a rigid motion.

Activity 19.28 Polygon Dilations

CCSS-M: 6.G.A.3; 7.G.A.1; 8.G.A.1; 8.G.A.3

Distribute a coordinate grid and ask students to create a four-sided shape in the first quadrant. Next, ask them to record the coordinates for their quadrilateral. With their list, they are to create a new set of coordinates by multiplying each of the original coordinates by 2. Ask, "What do you think the new polygon will look like? How will its perimeter change? Its area?" Then have students multiply each of the original coordinates by $\frac{1}{2}$. Ask the same questions and have students record their predictions. Have students plot the new shapes. Discuss how their predictions compared to what they see. Finally, ask students to draw a line from the origin to a vertex of the largest shape. Repeat for one or two additional vertices and ask for observations. (An example is shown in Figure 19.28.)

Pearson eText Blackline Master: Coordinate Grid–Quadrant 1

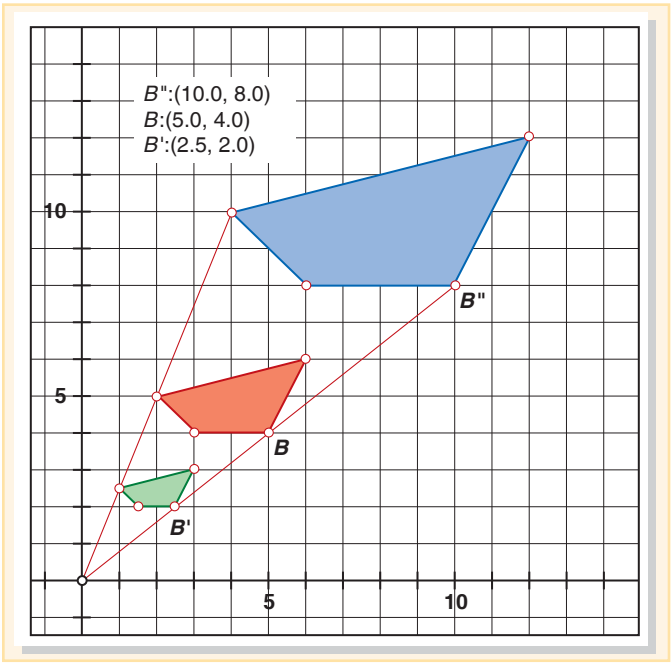


FIGURE 19.28 Dilations of a trapezoid (original in red) when the scale factors are 2.0 and 0.5

Your students may enjoy exploring dilations a bit further, including the connection to scale drawings. If they start with a simple drawing of a face, boat, or some other shape drawn with straight lines connecting vertices, they will create an interesting effect by multiplying just the (1) first coordinates and (2) second coordinates, or (3) use a different factor for each. When only the second coordinate is multiplied, the vertical dimensions alone are dilated, so the figure is proportionately stretched (or shrunk) vertically. Students can explore this process to see how an arithmetic operation can control a figure through slides, flips, turns, and dilations, not just in the plane but also for three-dimensional figures. This process is identical to computer animation techniques.

Location

Learning Outcome 19.4 **Explore ways to engage students in thinking about location.**

In pre-K and kindergarten, students learn about everyday positional descriptions—*above*, *below*, *beside*, *in front of*, *behind*, *and next to*. These place learnings (Sarama & Clements, 2009) are useful for helping students begin to specify locations. Additionally, helping students refine the way they think and reason about direction, distance, and location enhances spatial understandings. Geometry, measurement, and algebra are all supported by using a grid system with numbers or coordinates that can specify location.

After early development of terms for how objects are located with respect to other objects ("the ball is under the table"), location activities involve analysis of paths from point to point as on a map, and the eventual use of coordinate systems. The first quadrant of the coordinate plane is introduced in grade 5, with all quadrants included in grade 6, scale drawing and constructions added in grade 7, and the coordinate axis in grade 8 for graphing lines, performing transformations, and exploring distance.

The next activity can serve as a readiness task for learning about coordinates as students refine ways to describe location.

Activity 19.29 Hidden Positions

CCSS-M: 5.G.A.1

Give each student a Hidden Positions game board. Two students sit with a screen separating their desktop space so that neither student can see the other's grid (see Figure 19.29). Each student has four different pattern blocks. The first player places a block on four sections of the grid. They then use words to tell the other player where to put blocks on her grid to match his own. When all four pieces are positioned, check the two grids to see that they are alike. Then players switch roles. Model the game once by taking the part of the first student. Use words such as *top row, middle row, left, right, above, below, next to, column,* and *beside*. For students with special needs in math, consider starting with just one shape and then move to two, and so on. For a challenge, extend the grids up to 6×6. As the grid size increases, notice how the need for a system of labeling positions increases.

Pearson eText Activity Page: Hidden Positions Game Board

The next activity explores the notion of different paths on a grid.

Activity 19.30 Paths

CCSS-M: 5.G.A.1

On a sheet of 2-centimeter grid paper, mark two points, A and B, as shown in Figure 19.30. Using a projection display or floor tiles, demonstrate how to describe a path from A to B. For the points in the figure, one path is "up 5 and right 6." Another path might be "right 2, up 3, right 4, up 2." Count the length of each path. As long as you always move toward the target point (in this case either right or up), the paths will always be the same length. Here they are 11 units long. Students draw three paths on their papers from A to B using different-colored crayons. For each path, they write directions that describe their paths. Ask, "What is the greatest number of turns that you can make in your path?" "What is the smallest number?" "Where would A and B have to be to have a path with no turns?" For a challenge, add a coordinate system on the grid and have students describe their paths, using coordinates: For example: (1, 2), (3, 2), (3, 5), (7, 5), (7, 7).

Pearson eText Blackline Master: 2-Centimeter Grid Paper

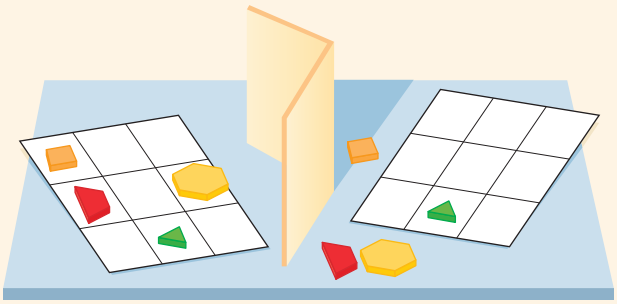


FIGURE 19.29 The Hidden Positions game

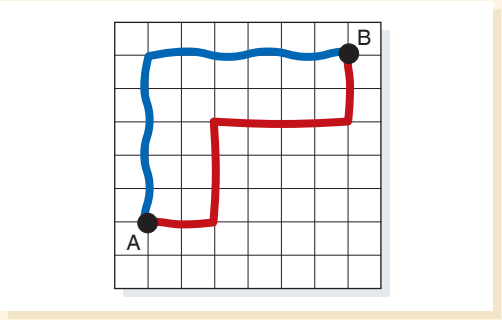


FIGURE 19.30 Different paths from A to B on a grid

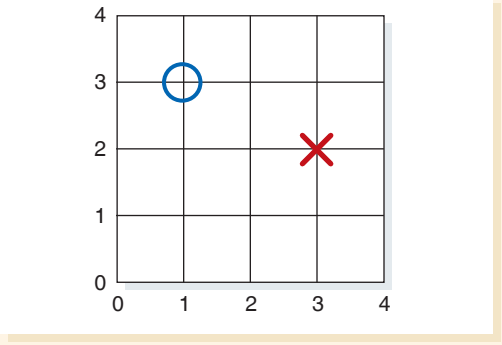


FIGURE 19.31 A simple coordinate grid

Coordinate Plane

To describe a location, students need a reference point (origin) and information about distance and direction (Goldenberg & Clements, 2014). Use a coordinate grid like the one shown in Figure 19.31 and explain how to use two numbers (start with positive whole numbers) to designate an intersection point on the grid. In the figure, the X is at (3, 2) and the O is at (1, 3). Students can use the grid to play "Three in a Row," putting the marks on the intersections, not in the spaces. In Hidden Positions (Activity 19.29), objects were placed within the space and not on the grid.

So, if you've used it as an introductory activity, when transitioning to this coordinate grid, emphasize the difference between the two representations. Initially, use words along with the numbers: 3 right and 0 up (don't avoid 0). Select points on the grid and have students decide what two numbers name that point. If your point is at (2, 4) and students incorrectly say "four, two," then simply show where the point is that they named. Then try the next activity.

Activity 19.31 Step Right Up

CCSS-M: 5.G.A.1

Create a coordinate grid on the floor, using painter's tape, or on the school playground with chalk or paint (or a coordinate grid that is projected). Give each student a small whiteboard for recording. Select a student and secretly give them a set of coordinates. The student moves to that location. Other students write the coordinates for that location and display their answers on the whiteboard. They should also find that same location on their personal coordinate grid and add the correct location and coordinates. If you repeatedly call this activity "Step Right Up," that can act as an initial mnemonic for students with special needs in math, because they will remember to step right first, and then up. But make sure you tell the students that there are other directions they will be learning in the future, and a "right step" or "up" will not always be the "right way to go!"

Pearson eText Blackline Master: Coordinate Grid—Quadrant I

Students can also examine transformations on a coordinate plane—commonly a standard in grade 8—as developed in the following activities.

Activity 19.32 Coordinate Slides

CCSS-M: 6.G.A.3; 8.G.A.1; 8.G.A.3

Ask students to plot and connect five or six points on 1-centimeter grid paper to form a small shape (see Figure 19.32). Begin with all coordinates in Quadrant I with x and y values between 5 and 12, and then begin to explore various slides, such as these:

- Add 6 to each of the x-coordinates of their shape, leaving the y-coordinates the same. (e.g., for the point [5, 10], a new point [11, 10] is plotted, as shown in Figure 19.32).
- Add 9 to each y-coordinate of the original coordinates (see Figure 19.32).
- "How can you change coordinates to slide your figure **up** and **to the right**? Propose an idea and test your conjecture."
- "How can you change the coordinates to slide your figure **down** and to **the left**?"

After exploring these slides, ask students, "What does adding (or subtracting) a number from the first coordinates cause? What if the number is added or subtracted from the second coordinates? From both coordinates?" Have students draw lines connecting corresponding points in the original figure with one of those where both coordinates were changed. What do they notice? (The lines are parallel and the same length.)

Pearson eText Blackline Master: 1-Centimeter Grid Paper

In Coordinate Slides, the figure did not twist, turn, flip over, or change size or shape. The shape slid along a path that matched the lines between the corresponding points—a *translation*. *Reflections* can be explored on a coordinate grid just as successfully. Begin with using the x- or y-axis as the line of reflection, as in the following activity.

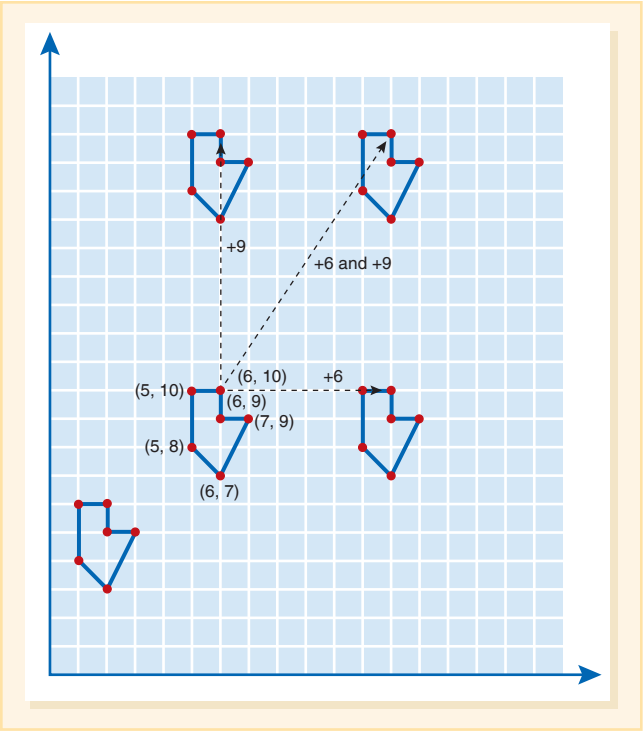


FIGURE 19.32 By adding or subtracting from the original coordinates, new shapes are found that are translations

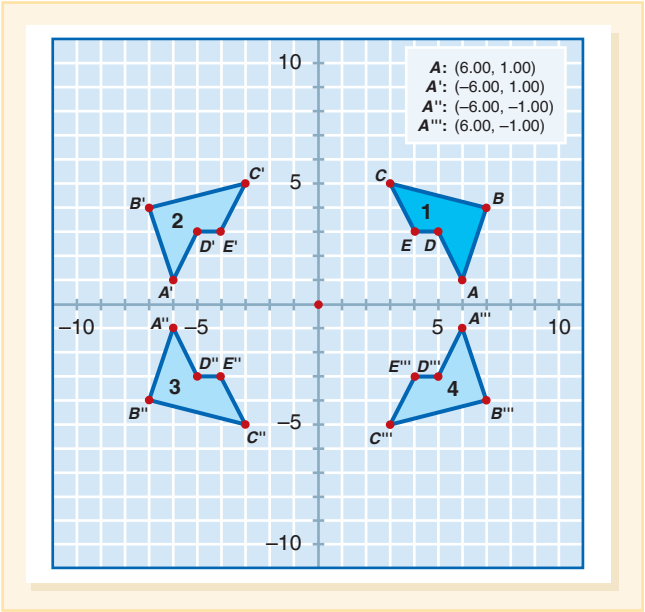


FIGURE 19.33 Exploring reflections on a coordinate grid

Activity 19.33 Coordinate Reflections

CCSS-M: 8.G.A.1; 8.G.A.3

Ask students to draw a five-sided shape in the first quadrant on a coordinate grid. Label the figure *ABCDE* and call it Figure 1 (see Figure 19.33). Use the *y*-axis as a line of symmetry and draw the reflection of the shape in the second quadrant. Label the reflected points $A'B'C'D'E$ and call it Figure 2. Now use the *x*-axis as the line of reflection and create Figure 3 (in Quadrant III) and the *y*-axis again to make Figure 4 (in Quadrant IV). Label the points of these figures with double and triple primes (A'' and A''' and so on). Write in the coordinates for each vertex of all four figures.

- How is Figure 3 related to Figure 4? How else could you have created Figure 3 from the original figure (Figure 1)? How else could you have created Figure 4?
- How are the coordinates of Figure 1 related to its image in Figure 2? What can you say about the coordinates of Figure 4 as compared to Figure 1?
- Make a conjecture about the coordinates of a shape reflected on the *y*-axis and a different conjecture about the coordinates of a shape reflected on the *x*-axis.
- Draw lines from the vertices of Figure 1 to the corresponding vertices of Figure 2. What can you say about these lines? How is the *y*-axis related to each of these lines?

Pearson eText Blackline Master: Coordinate Grid—4 Quadrants

While exploring the transformation activities, students might be challenged with questions that deepen their understanding of transformations, such as the following:

- How should the coordinates be changed to cause a reflection if the line of reflection is not the *y*-axis but is parallel to it?
- Can you discover a single rule for coordinates that would cause a reflection across one of the axes, followed by a rotation of a quarter turn? Is that rule the same for the reverse order—a quarter turn followed by a reflection?

- If two successive slides are made with coordinates and you know what numbers were added or subtracted, what number should be added or subtracted to get the final figure in only one move?
- What do you think will happen if different factors are used for different coordinates in a dilation?

Measuring Distance on the Coordinate Plane

Measuring in the coordinate plane begins in grade 6 with students measuring vertical and horizontal lines and moves in grade 8 to student exploration of distance—eventually leading to the distance formula. In the following activity, students use the coordinate grid and the Pythagorean relationship to develop a formula for the distance between points.

Activity 19.34 Finding Distance Using the Pythagorean Theorem

CCSS-M: 6.G.A.3; 8.G.B.8

Ask students to draw a line between two points in the first quadrant that are not on the same horizontal or vertical line (use the coordinate grid or dynamic geometry environments). What is an estimate of the length of this line? Give students time to share estimates as well as strategies to find the actual length, and then ask students to draw a right triangle, using the line as the hypotenuse. (The vertex at the right angle will share one coordinate with each end point.) Students then apply the Pythagorean theorem to find the distance. Ask students to do two to four more examples, asking them to look for patterns across their examples.

Next, have them look through all of their calculations and see how the coordinates of the two end points were used. Challenge students to use the same type of calculations to get the distance between two new points without drawing any pictures.

Pearson eText Blackline Master: Coordinate Grid—Quadrant I

Students do not need to construct proofs independently at this time but should be able to follow the rationale if shown proofs. By using the Pythagorean theorem to find the length of one line (or the distance between the end points), you provide the opportunity for students to make an important connection between two big mathematical ideas.

Visualization

Learning Outcome 19.5 Illustrate approaches that develop students' mental imagery and visualization skills.

Spatial visualization might be called "geometry done with the mind's eye." It involves being able to create mental images of shapes and then turn them around mentally, thinking about how they look from different viewpoints and predicting the results of various transformations. It includes the mental coordination of two and three dimensions— for example, by determining the net (a two-dimensional drawing) for a three-dimensional shape. Any activity that requires students to think about, manipulate, or transform a shape mentally or to represent a shape as it is seen visually will contribute to the development of students' visualization skills.

Two-Dimensional Imagery

At first, students are thinking about shapes in terms of the way they look (level 0), so visualization activities will challenge students to think about two-dimensional shapes in different orientations.

Activity 19.35 Can You Remember?

CCSS-M: K.G.B.4; 1.G.A.2; 2.G.A.1

Display one of these simple sketches of figures (see Figure 19.34) for about 5 seconds. Then have students attempt to reproduce it on their own. Show the same figure again for a few more seconds and allow students to modify their drawings. Repeat with additional figures. For students with special needs in math share a copy of the figure at their location for 5 seconds.

Pearson eText Activity Page: Sketches of Figures

Have a class discussion during which students are asked to describe how they thought about the figure or give examples of attributes of the figure that helped them remember what they saw. As students learn to describe verbally what they see, their visual memory will improve. Another option, helpful to students with special needs in math, is to match the displayed figure to an image from a set of figures that look alike.

Activity 19.36 employs students' visualization skills as they look for different ways to make shapes with five tiles. This task engages students in mentally flipping and turning shapes in their minds and finding ways to decide whether they have found all possible options.

Activity 19.36 Pentominoes

CCSS-M: 1.G.A.2; 2.G.A.1; 3.G.A.1

A pentomino is a shape formed by joining five squares as if cut from a square grid. Each square must have at least one side in common with another. Provide students with five square tiles and a sheet of 1-centimeter grid paper for recording. Challenge them to see how many pentomino shapes they can find. Shapes that are flips or turns of other shapes are not considered different. Do not tell students how many pentomino shapes there are. Good discussions will come from deciding whether some shapes are really different and if all shapes have been found. Look at the Pentominoes activity page to see all the possible options, or refer to https://mathigon.org/polypad#pentominoes.

Pearson eText Blackline Master: 1-Centimeter Grid Paper

Pearson eText Activity Page: Pentominoes

Once students have decided that there are just 12 pentominoes (see Figure 19.35), challenge them to try to fit all 12 pieces into a 6×10 or 5×12 rectangle. Another task is to examine each of the 12 pentominoes and decide which of these nets will fold up to make an open box. For those pentominoes that are open-box makers, which square is the bottom? The use of nets can be also linked to volume activities (Prummer, Amador, & Wallin, 2016).

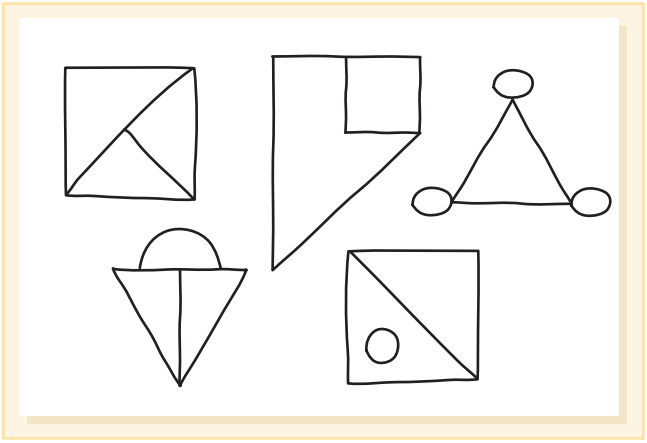


FIGURE 19.34 Examples of quick images to use in the Can You Remember? activity

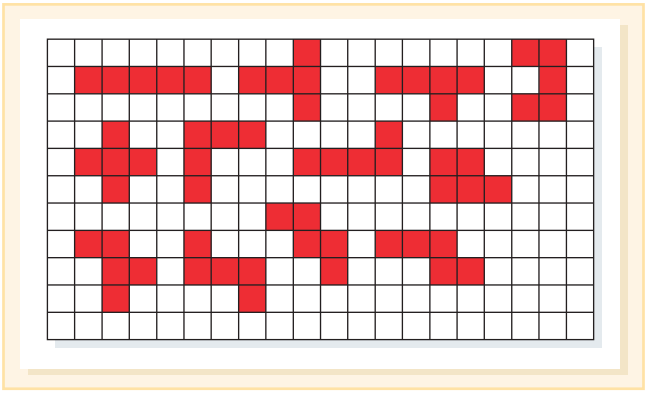


FIGURE 19.35 There are 12 pentomino shapes

Students can explore the number of shapes that can be made from six equilateral triangles or from four 45-degree right triangles (halves of squares). With the right triangles, sides that touch must be the same length. How many of each of these -ominoes do you think there are? What about using 6 squares for hexominoes (there are 35). You can also use five cubes to create unique, three-dimensional structures (there are 29) (Copley, 2017).

Activity 19.37 Geometry Necklaces

CCSS-M: 3.G.A.1; 4.G.A.1; 5.G

Make necklaces from Geometry Necklace cards with two holes punched at the top and a loop of yarn long enough to go over all students' heads easily. Each card should have a two- or three-dimensional shape or any geometric figures appropriate for your students. Hang the necklace on each student's back so they cannot see it. Students should not try to look at their necklace or ask others to tell them what they are wearing. Then students walk around the room and ask only yes/no questions to help them collect information to determine their geometric figure or term. Students may only ask one question per person, keeping track of questions and answers on a piece of paper. As they think about the different responses, they must mentally visualize precisely what is being described. When students believe they have precisely identified what is on their necklace, they should check with the teacher.

Pearson eText Activity Page: Geometry Necklace Cards

Three-Dimensional Imagery

Another aspect of visualization for young students is to be able to think about three-dimensional shapes in terms of their two-dimensional representations—focusing on faces.

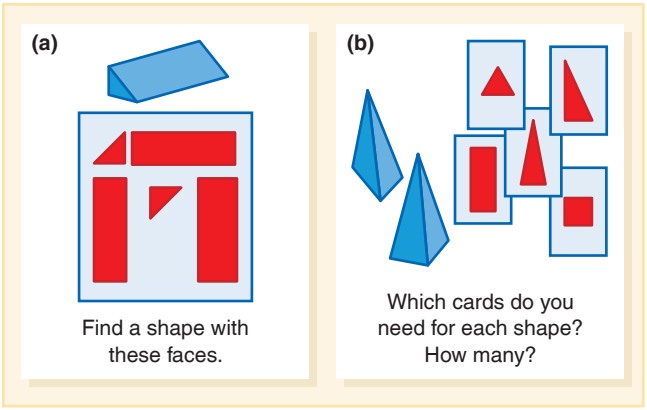


FIGURE 19.36 Matching face cards with solid shapes

Activity 19.38 Face Matching

CCSS-M: 1.G.A.2; 2.G.A.1; 3.G.A.1; 4.G.A.2

Provide students with a Find a Shape activity page and sets of cardstock Face-Matching cards (see Figure 19.36). There are two options: Given a Face-Matching card, find the corresponding solid or, given a solid, find the matching card. For another variation, stack all of the Face-Matching cards for one solid face-down. Turn the cards up one at a time as clues to identifying the solid. Use the Face-Matching Card Questions activity page and collect student responses.

Pearson eText Activity Page: Find a Shape
Pearson eText Activity Page: Face-Matching Cards
Pearson eText Activity Page: Face-Matching Card Questions

Students can use three-dimensional drawing programs such as SketchUp or Tinkercad (https://www.tinkercad.com) to draw shapes and actually develop solids through the use of three-dimensional printers (Cochran et al., 2016).

One of the main goals of the visualization strand is to be able to build three-dimensional figures from two-dimensional images as developed in the next activity.

Activity 19.39 Building Views

CCSS-M: 6.G.A.4; 7.G.A.1

For this activity, students will need 1-centimeter grid paper for drawing a building plan, and 1-inch blocks for constructing a building with *orthogonal* views.

- Version 1: Students begin with a building made of blocks and draw the left, right, front, and back views (these drawings are called elevations) on grid paper (see bottom of Figure 19.37).
- Version 2: Also in Figure 19.37, the building plan shows a top view of the building with the number of blocks in each position. Give students the Building Views activity page and ask them to build the corresponding building out of blocks.
- Version 3: Using the routine of quick images shared in other topics, create block images that are shown for a short period of time that students must copy (Matney, Porcella, & Gladieux, 2020).

Pearson eText Activity Page: Building Plans
Pearson eText Blackline Master: 1-Centimeter Grid Paper
Pearson eText Blackline Master: 2-Centimeter Isometric Grid Paper
Pearson eText Blackline Master: 1-Centimeter Isometric Dot Paper

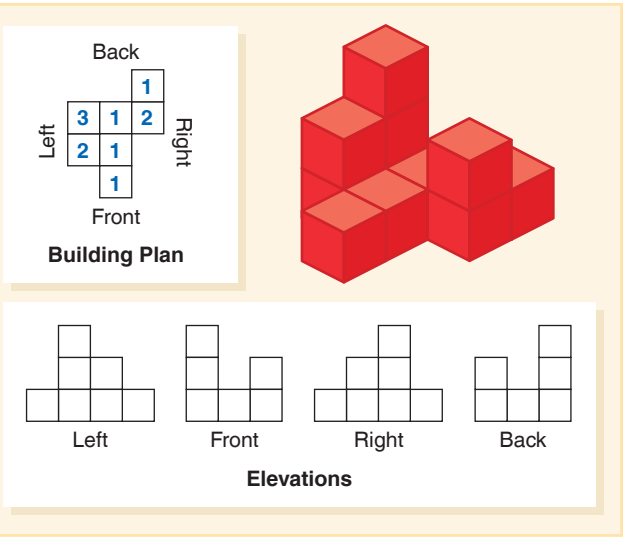


FIGURE 19.37 Building Views tasks 1 and 2

Isometric grids allow an *axonometric* drawing, in which the scale is preserved in all dimensions (height, depth, and width). The next activity provides a glimpse at this form of visualization.

Activity 19.40 Three-Dimensional Drawings

CCSS-M: 6.G.A.4; 7.G.A.1

- Version 1: Pairs of students begin with the Three-Dimensional Drawings activity page or other isometric drawings of a building created by the teacher on isometric grid paper or dot paper. The assumption is that there are no hidden blocks. From the drawing (see top of Figure 19.38), the students build the actual building with blocks. To record the final result, they draw a building plan (top view) indicating the number of blocks in each position. For students with special needs in math or others who need support, have them use the Building Mat activity page labeled with the words "front," "back," "left," and "right" on the four edges to highlight the locations of the different elevation views.
- Version 2: Give pairs one of the two pages in the Elevations activity page. They should use these four diagrams of the different views of a building to create a block structure (see bottom of Figure 19.38). To record the final result, they draw a building plan (top view) indicating the number of blocks in each position.
- Version 3: For a challenge, students can make their own three-dimensional isometric drawings to match three-dimensional block buildings they create (use 2-centimeter isometric dot paper or 1-centimeter isometric dot paper or the following NCTM technology tool.)

Pearson eText Activity Page: Three-Dimensional Drawings
Pearson eText Activity Page: Building Mat
Pearson eText Activity Page: Elevations
Pearson eText Blackline Master: 2-Centimeter Isometric Grid Paper
Pearson eText Blackline Master: 1-Centimeter Isometric Dot Paper.

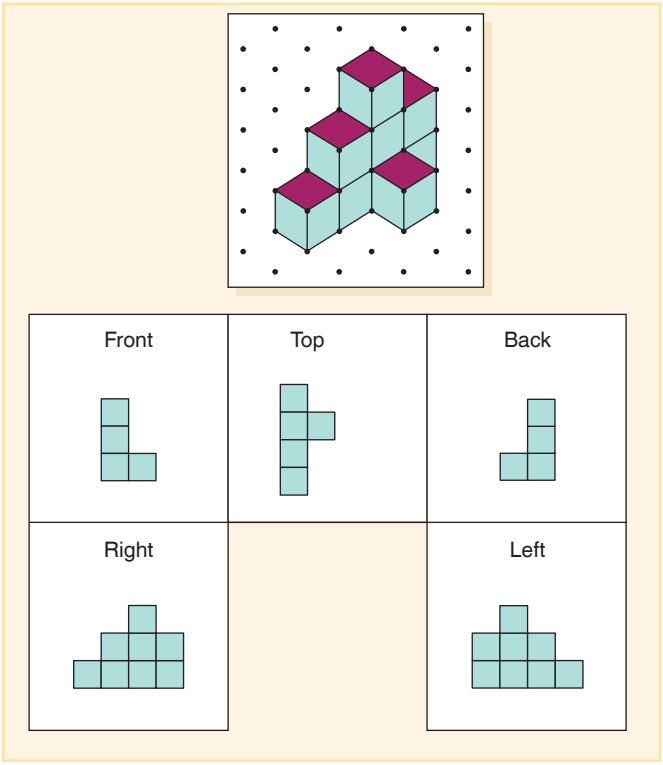


FIGURE 19.38 Isometric dot drawing with elevations and building plan

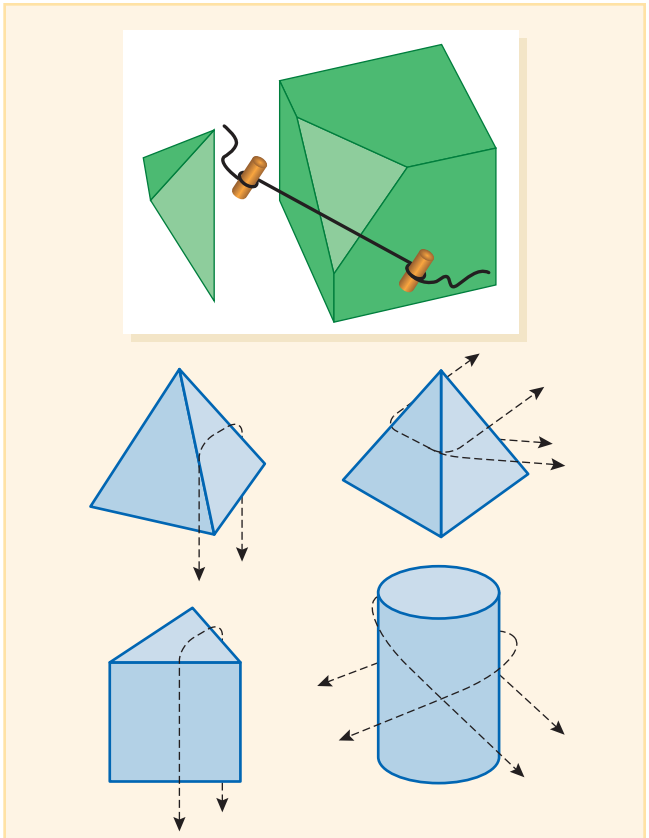


FIGURE 19.39 Predict the shape of the slice face and then cut the clay model with potter's wire

***TECHNOLOGY* Note.** An amazing computer tool for drawing two-dimensional views of block buildings is the Isometric Drawing Tool, available at the NCTM Illuminations website. Using mouse clicks, students can draw either whole cubes or faces, or just lines. The drawings, however, are actually buildings and can be viewed as three-dimensional objects that when rotated can be seen from any viewpoint. Prepared investigations lead students through the tool's features. ■

Activity 19.41 Slicing Solids

CCSS-M: 7.G.A.3

When a solid is sliced into two parts, a two-dimensional shape is formed on the slice faces. Provide groups of students with clay solids and potter's wire or a plastic knife (see Figure 19.39). Pose the situation, "What shaped face will the remaining surface of a cube be if you slice off a corner?" Ask students, "Can you slice the solid you have to create a trapezoid face? A square face?" What will be the shape from each cut in Figure 19.39?

Pearson eText

Video Example 19.9

In this video consider what happens when you slice a solid and the relationships between three dimensional figures and two dimensional shapes.

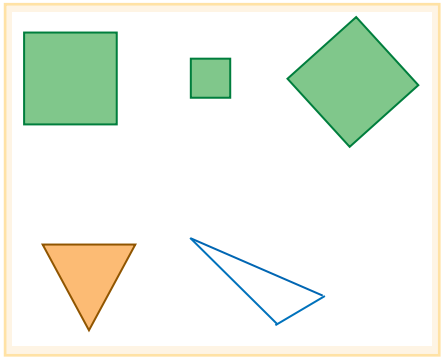
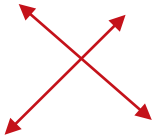
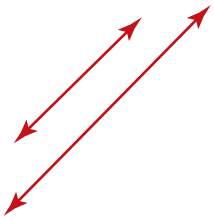
https://www.youtube.com/watch?v=hlD_j3AtxGs

Another engaging task is to partially fill a translucent or clear plastic solid with water. The surface of the water simulates a slice and models the face of the solid as if it were cut at that location. By tilting the shape in different ways, every possible slice can be observed. Prior to testing with water, students might use a list of possible triangles and quadrilaterals to predict for a given solid which shapes can be made and which are impossible. Have them offer reasons for their conjectures.

Explore the following Table 19.6 to see students' common challenges across the four major geometry strands.

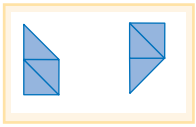
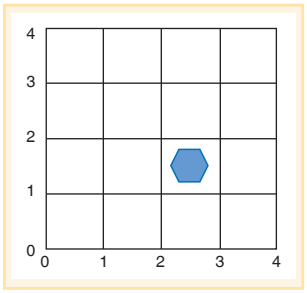
Common Challenges in Geometry

TABLE 19.6 COMMON CHALLENGES IN GEOMETRY AND HOW TO HELP

Common Challenge	What It Looks Like	How to Help
1. Incorrect identification of a shape due to a focus on non-defining characteristics such as orientation, size, or color.	Students call the third shape a diamond (not a mathematical shape) because it is tilted and no longer looks like a square. And they may just say the last shape is pointy but is not a triangle. 	• Expose students to a variety of examples of a given shape and ask them what the shapes have in common. • Use examples and non-examples of shapes to focus on the defining attributes. • Carefully select posters, children's literature, and examples to avoid using/displaying inaccurate or imprecise examples of shapes.
2. A focus on the two-dimensional face can confuse the identification of three-dimensional shapes.	Students call a cube a square, a right rectangular prism a rectangle, or a sphere a circle.	• Have students build three-dimensional shapes by first constructing and naming the two-dimensional faces and then naming the three-dimensional shape. • Avoid letting students say a box (rectangular prism) is a rectangle. Highlight that the face on one side of the box is a rectangle. • Put a three-dimensional shape into a bag, box, or sock and ask students to describe what they feel. If the student names a face, show the two-dimensional model and ask whether that is what is in the bag.
3. Unsure of definitions that involve inclusion relations of quadrilaterals: Is a rectangle a square? A rectangle a parallelogram? A square a parallelogram?	When students are asked whether this shape is a rectangle, they say "no." 	• Remind students that they could be in more than one school club or team. Shapes can be in more than one category in the same way. • Focus on Activities 19.2 and 19.3, in which students can compare the attributes and definitions of shapes. • Have students create the nested image in Figure 19.10 as a poster.
4. Perpendicular lines must be horizontal and vertical.	Students will say these lines are *not* perpendicular. 	• This is an artifact of having perpendicular lines shown only as a vertical and a horizontal line. Show perpendicular lines in a variety of orientations.
5. Parallel lines must be horizontal.	Students will say these lines are *not* parallel. 	• This is an artifact of having experience only with the iconic image of parallel lines as two horizontal lines. Show parallel lines in a variety of orientations.

(*Continued*)

TABLE 19.6 ***(CONTINUED)***

Common Challenge	What It Looks Like	How to Help
6. Line of symmetry must be vertical or horizontal, or they must be adjacent to the side of the image (Ronau, Meyer, & Crites, 2015).	For example, given a square, students will say there are only two lines of symmetry. In this case, they miss seeing the diagonal lines of symmetry.	• Have students use many shapes and initially tell them how many lines to look for. Then fade that support so they have to seek out the correct number of lines of symmetry in the figure.
7. Identifying a shape in different orientation as congruent shape.	When finding all the ways to rearrange three right triangles, the student thinks the following designs are different because of their orientation. 	• Provide experiences, using and describing *slides, turns*, and *flips*, such as when putting together puzzles, making designs with pattern blocks, and using interactive applets. • Provide several copies of the shape being composed. When one design is found, have them tape it together. When another design is found, ask them to turn and flip their existing designs to see if it is new.
8. Visualize a three-dimensional shape from a two-dimensional image (net).	Given a collection of nets on a grid, students are unable to fold the shape mentally to predict which can make a shape.	• Try Activity 19.36, in which students suggest which nets can be folded into a box with no top.
9. Incorrectly describes shapes they cannot see (struggles with creating a mental image) or is unable to predict what a shape looks like after it has been flipped, turned, or slid.	Student is unable to predict what Motion Man, for example, would look like if it were flipped over the indicated line segment.	• Engage students in activities where they can check their predictions by manipulating concrete or semi-concrete representations. • Have children work puzzles (see Activities 19.8 and 19.9).
10. Incorrectly identifies the space as opposed to the intersection of the grid for location.	When asked to place a block on the location indicated by 3 right and 2 up, the student places the block in the space, not on the intersection of the grid. 	• Use two pipe cleaners or straws to help the student align with the horizontal and vertical lines that match the coordinates. Where the two pipe cleaners intersect shows the location point. • Use a simple local map laid out on a grid to find a location. Explain that because we travel along the streets, we will indicate our location along the grid lines, not in the square.
11. Reverses the ordered pair.	Students suggest that a point is located at (y, x) rather than at (x, y).	• Students need to know that there is a convention to the labeling of a point using ordered pairs. They need to practice the consistent pattern of the x value first then the y value (see Activity 19.31).

RESOURCES FOR CHAPTER 19

WRITING TO LEARN

Assess your understanding and application of chapter content.

1. Describe in your own words the first three van Hiele levels of geometric thought (levels 0, 1, and 2). How would activities aimed at levels 0, 1, and 2 differ?
2. How are preK-1 students' learning of positional ideas such as below or beside useful in their learning of coordinate geometry?
3. Describe how transformations are used to support one of the following ideas: congruence, similarity, or symmetry.
4. Find one of the suggested technology supports and explain how it can be used. What are the advantages of using technology instead of concrete materials or drawings?
5. What are ways to assess your students' spatial sense?

LITERATURE CONNECTIONS

There are numerous children's books about geometry. Here we include a teacher resource of a collection of popular stories linked to geometry concepts.

Teacher Resource: Children's Literature Books on Geometry

Here is just one book highlighted from the list:

I Spy Shapes in Art

Micklethwait (2007)

Museum Shapes *Metropolitan Museum of Art (2005)*

Using paintings from such renowned artists as Matisse and Warhol, challenge students to look for particular shapes through an I-spy approach. This activity can be extended to postcards of artwork from local museums or act as a catalyst for budding mathematical artists in the classroom.

RECOMMENDED READINGS

Articles

Edwards, M. T., & Harper, S. R. (2010). Paint bucket polygons. *Teaching Children Mathematics*, *16*(7), 420–428.

To advance understanding of polygons, examples and non-examples are used to help refine definitions. Using the paint bucket feature found in many graphics programs or the Microsoft Word fill color feature, students can define properties of polygons.

Koester, B. A. (2003). Prisms and pyramids: Constructing three-dimensional models to build understanding. *Teaching Children Mathematics*, *9*(8), 436–442.

Explorations with third to fifth graders include using straws and pipe cleaners to build models. The activities involve classification and definitions of shapes and seeking patterns in the number of faces, vertices, and edges.

Books

Copley, N. (2017). *Putting Essential Understanding of Geometry into Practice in Grades PreK–2*. NCTM.

Crites, T., Dougherty, B. J., Slovin, H., & Karp, K. (2018). *Putting Essential Understanding of Geometry into Practice in Grades 6–8*. NCTM.

These two books link the most important ideas in geometry for pre-K-2 or grades 6–8 with the most effective teaching strategies. Topics that are challenging for students are tackled, with approaches for assessment and next instructional steps provided.

CHAPTER

20 Developing Concepts of Data and Statistics

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you will be able to:

- **20.1** Explain differences between statistics and mathematics, including what is meant by "doing statistics."
- **20.2** Engage students in designing meaningful statistical questions.
- **20.3** Describe techniques for collecting data and quality sources for finding data.
- **20.4** Identify meaningful ways for students to classify data.
- **20.5** Determine developmentally appropriate ways for students to analyze data.
- **20.6** Illustrate and explain the meaning of measures of center and variability.
- **20.7** Compose questions that help students interpret data.

Our lives are heavily influenced by data. That means that teaching data is essential to our students' well-being. As described in the 2020 report titled *Pre-K–12 Guidelines for Assessment and Instruction in Statistics Education II: A Framework for Statistics and Data Science Education Report (GAISE II)* (Bargagliotti et al., 2020), never have data and statistical literacy been more important. Individuals must synthesize information about global issues, such as the worldwide pandemic and a changing planet with extreme weather conditions, and important social issues such as Black Lives Matter and immigration. Statistical literacy is critical to understanding the world around us, essential for social responsibility, and vital for developing the ability to question information presented in the media. The following are the big ideas that help students become statistically literate.

BIG IDEAS

- Statistics is a different field of study from mathematics; although mathematics is used in statistics, statistics is concerned with analysis of data and the resulting practical implications.
- Statistical problem solving involves a four-step process: formulating questions, collecting data, analyzing data, and interpreting results.
- Different types of graphs or representations provide different information about the data and, hence, the population from which the data were taken. The choice reflects what the author wants noticed and affects how well the data are understood.

- Measures that describe data with numbers are called *statistics*.
- The shape of data provides a big picture of the data rather than a collection of numbers. Graphs and statistics can provide a sense of the shape of the data, including how spread out or how clustered they are.

What Does It Mean to Do Statistics?

Learning Outcome 20.1 **Explain differences between statistics and mathematics, including what is meant by "doing statistics."**

The purpose of learning about statistics is to be able to answer statistical investigative questions such as, 'What is the most common lunch option at our elementary school?' *Doing statistics* means that the context is a central focus of the investigation.

Is It Statistics or Is It Mathematics?

Statistics and mathematics are different fields; however, statistical questions are often asked in assessments with questions that are mathematical in nature rather than statistical. The harm in this is that students are not focusing on statistical reasoning, as shown by the following excellent exemplars from Scheaffer (2006).

Read the following questions and label each as "doing mathematics" or "doing statistics."

1. **The average weight of 50 prize-winning tomatoes is 2.36 pounds. What is the combined weight, in pounds, of these 50 tomatoes? (NAEP sample question)**
   a. **0.0472**
   b. **11.8**
   c. **52.36**
   d. **59**
   e. **118**
2. **Joe had three test scores of 78, 76, and 74, whereas Mary had scores of 72, 82, and 74. How did Joe's average (mean) compare to Mary's average (mean) score? (TIMSS eighth-grade released item)**
   a. **Joe's was one point higher.**
   b. **Joe's was one point lower.**
   c. **Both averages were the same.**
   d. **Joe's was 2 points higher.**
   e. **Joe's was 2 points lower.**
3. **Table 20.1 gives the times each girl has recorded for seven trials of the 100-meter dash this year. Only one girl may compete in the upcoming track meet. Which girl would you select for the meet and why?**

TABLE 20.1 **RACE TIMES IN SECONDS FOR THREE RUNNERS**

Runner	1	2	3	4	5	6	7
Suzie	15.2	14.8	15.0	14.7	14.3	14.5	14.5
Tanisha	15.8	15.7	15.4	15.0	14.8	14.6	14.5
Dara	15.6	15.5	14.8	15.1	14.5	14.7	14.5

Which of these three questions involves statistical reasoning? Only question 3. The first requires working backward through a formula—mathematical thinking, not statistical thinking. The second problem requires computation. Only #3 is statistical in nature because the situation requires analysis—graphs or averages might be used to determine a solution.

Notice that the context is irrelevant in the first two problems and central to the third question. In statistics, the context is essential to analyzing and interpreting the data (Bargagliotti et al., 2020;

Lovett & Lee, 2016; Scheaffer, 2006). Statistics anticipates and analyzes *variability*: how results, responses, or data related to a statistical question will vary.

Recently, researchers and organizations have begun to advocate for the inclusion of meaningful social and political issues in school mathematics curriculum (Wager & Stinson, 2012; Weiland, 2017). Sociopolitical issues provide opportunities to grapple with and make sense of issues meaningful to their lives outside of school, using statistics. Large-scale sociopolitical issues, such as climate change, elections, immigration, racism, poverty, food shortages and waste, and environmental pollution, could be addressed through age-appropriate statistical investigations so that students are not only aware of these issues but know how to investigate them logically (while also seeing that mathematics is useful). Sample investigations include:

How might you group people (shapes) into districts to have the minority group win the election (gerrymandering)? (Safi, Bush, & Desai, 2018)
What does the distribution of wealth look like in the United States? More concretely, if 10 students each represent 10% of the U.S. population, how will they share 10 chairs (representing money) to reflect the distribution of wealth in the U.S. (Gutstein & Peterson, 2013)?

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

***TECHNOLOGY* Note.** The *New York Times* offers many math lessons that engage students in analyzing graphs and other data to explore social issues. New content is added regularly so that the issues and data are current (https://www.nytimes.com/spotlight/learning-lessons-math/). NCTM Classroom Resources (https://www.nctm.org/classroomresources/) also offers free lessons on various topics such as pandemics. ■

The Process of Doing Statistics

Just as learning addition involves much more than the procedure to combine quantities, doing statistics is much more than being able to create a graph or carry out a computational procedure for finding the mean.

To learn and do statistics meaningfully, students should be involved in the full process, from asking and defining questions to interpreting results. This process may sound sophisticated, but with interesting, age-appropriate questions and data analysis, students in kindergarten can meaningfully participate in this process (Hourigan & Leavy, 2015/2016). This chapter is organized around this process, which is presented in Figure 20.1.

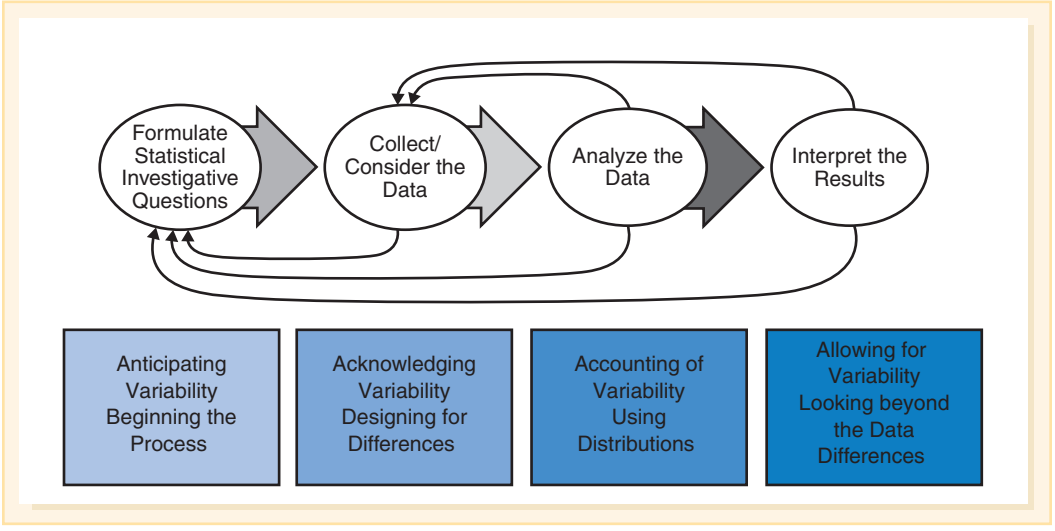

Formulate Statistical Investigative Questions
Collect/ Consider the Data
Analyze the Data
Interpret the Results
Anticipating Variability Beginning the Process
Acknowledging Variability Designing for Differences
Accounting of Variability Using Distributions
Allowing for Variability Looking beyond the Data Differences

FIGURE 20.1 Statistical problem-solving process

Source: Bargagliotti et al. (2020) *PreK-12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II).* American Statistical Association. All rights reserved.

All students, K–8 (and beyond), should have regular opportunities to engage in the statistical process. In the early grades, students learn how to ask a statistical investigative question, consider how to gather data, categorize and display data in various graphical forms (e.g., picture graph), and make statements about their findings. In the upper elementary grades, students continue to engage in the process, using more sophisticated data collection and data analysis methods, including frequency tables, bar graphs, dot plots (line plots), and picture graphs. In middle school, new data representations are learned (e.g., histograms, box plots, scatter plots, and stem-and-leaf plots), and students explore statistics (measures of center and measures of variability) (NGA Center & CCSSO, 2010). GAISE II provides a framework, identifying priorities within each step at Level 1 (elementary), Level 2 (middle), and Level 3 (high). Although grade-bands are named, no matter what grade level, students must engage in Level A processes to be ready to learn at Level B (and eventually Level C in high school) (Bargagliotti et al., 2020).

Pearson eText Teacher Resource: GAISE II Level 1 and Level 2 Learning Goals

Pearson eText
Video Example 20.1
In this video author Jennifer Bay-Williams summarizes important ideas of statistics.

Anticipating Student Statistical Thinking

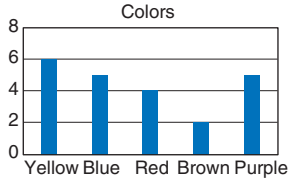
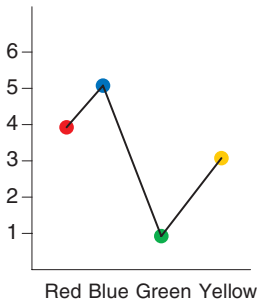
Statistical reasoning is different from mathematical reasoning and therefore can be challenging. If students have not had the opportunity to engage in the Level A statistical problem-solving process, they are more likely to struggle with the ideas at level B. It is important to listen to students and help them understand the statistical problem-solving process. Common challenges are briefly described in Table 20.2, organized across the statistical problem-solving process, as a preview to the sections that follow.

TABLE 20.2 COMMON CHALLENGES RELATED TO DATA AND STATISTICS

Common Challenge	What It Looks Like	How to Help
1. Posing a question that is statistical in nature	Student poses questions that are not statistical, such as • How old is my dog? • How much time did I spend on homework last night?	• Give examples and non-examples of statistical questions and ask students to identify the differences by sorting them (see examples on p. 553 and 557). • Help the student modify their nonstatistical question so that it becomes a statistical question (e.g., "What are the ages of our pets?" or "What is the typical amount of time our class members spent on homework?"). • Emphasize that statistical questions rely on the context, can only be answered by collecting data, and involve some sense of variability (i.e., if there is only one answer, it is not a statistical question).
2. Determining categories for grouping data	After collecting data (e.g., favorite book), a student lists each classmate's name along with the name of the favorite book, rather than sorting into categories.	• Use categorization activities (see Activities 20.4–20.6 and those listed in Table 20.3 that are based on social studies and science content). • Provide categories or brainstorm possible categories.
3. Sorting objects based on more than one rule	A student is asked to sort attribute materials, using the rule small triangles, but sorts based only on the attribute of triangle (ignoring the size attribute).	• Initially sort materials that have visible attributes (like color). • Scaffold first by only sorting by one attribute. Increase the number of attributes as the child demonstrates successful sorts. • Have student state the sorting rules for each shape that is sorted, emphasizing both attributes, for example, saying, "This shape is small *and* this shape is a triangle."

(*Continued*)

TABLE 20.2 ***(CONTINUED)***

Common Challenge	What It Looks Like	How to Help
4. Looking at the data set within a graph as a whole (rather than at single data points)	When asked to describe something about the class graph that shows the ways their classmates travel to school, the student explains, "That's my sticky note there! I ride the bus."	• Explicitly ask questions about the shape of the data. • Draw a line or curve (e.g., connecting the tops of bar graphs) to illustrate the general shape of the data. • Compare the shapes of two different graphs (e.g., two bar graphs, or two line plots).
5. Bar graph creation: Dealing with 0	Student creates a bar graph from a survey, but leaves out the category that had zero responses. Third Grade Favorite Colors Yellow 6 Blue 5 Green 0 Red 4 Brown 2 Purple 5 	• Have students count the survey items and count the bars. • Include categories for which you anticipate there will be 0 responses so they encounter 0 more often and interpret the meaning of 0 respondents. • Discuss why there still needs to be a bar for the category.
6. Pictographs: Understanding the scale	Student sees a pictograph of favorite animals and interprets the 5 dolphins as 5 students selecting dolphins (rather than referring to the scale and seeing that each dolphin picture represents 3 students).	• Explicitly ask students why the number of pictures is not the same as the number of students surveyed. • Skip-count for each animal category. • Ask students to discuss different results.
7. Histograms: Understanding the meaning of the bars (as describing the relative frequency of data within the interval) and the meaning of the *x*- and *y*-axes	Student reads the bar as an actual data value rather than as the frequency for the interval; student uses the *y*-axis to determine the center of the data; student thinks the height of bar captures the variability in the data.	• Compare histograms to bar graphs. • Compare histogram to the related numerical data. • Spend adequate time on analysis and interpretation (rather than devoting most of the instructional time to graph construction).
8. Box plots: Understanding that each quartile has 25 percent of the data points	A student thinks a smaller box means fewer data points.	• Compare box plots to histograms. • Discuss which sections have the most data points (none) and which have the most variability.
9. Selecting appropriate graph for the data	Student uses a line to connect categorical data. 	• In a visible place in your classroom, post a list of options for graphing. • Before students graph, ask, "Which graphs are options for these data" (and why)? "Which graphs are not options" (and why)? • After students graph and are sharing their displays, ask students to critique the display in terms of whether it is appropriate and whether it is a good way to communicate the data collected.
10. Determining the mean from a frequency table or dot plot (line plot)	Student cannot figure out how many of each data point there is, or does not know how many data points in all.	• Ask students to write an expression for each row and column in the table or chart. For example, if there are 6 tallies for students with a 7-inch hand span, they record 6(7) next to the tally. • Ask students to revisit their answers and see if they are a reasonable measure of center.

Formulating Statistical Investigative Questions

Learning Outcome 20.2 **Engage students in designing meaningful statistical questions.**

At the heart of statistics is asking and answering questions about our world. Data collection should be for a purpose, to answer a question, not for the sake of learning how to construct a graph. Students, even young students, should have opportunities to generate their own questions, decide on appropriate data to help answer these questions, and determine methods of collecting the data (Arnold & Franklin, 2021; Bargagliotti et al., 2020). Generating questions in which they are interested is also a way to be culturally responsive in your teaching!

To begin, students need to be able to determine what types of questions lead to statistical investigations. These are questions that include variability and for which data can be gathered. Providing examples and non-examples can help students, in particular students with special needs in math, focus on the elements of an appropriate statistical question. For example, consider which of the following questions can be answered using statistics and which cannot.

1. How much money do I have in my pocket?
2. What is the typical number of coins a person carries in their pocket?
3. What cereal is most healthy?
4. What reasons do people use to select gum (e.g., taste, cost, bubble-making quality, long-lasting, good breath)?
5. How long do different kinds of gum keep their flavor?
6. Which store has the best prices?
7. Where will you buy shampoo?

Pause & Reflect

Which of the previous questions are statistical and which ones are not statistical in nature? ●

Questions 1, 3, and 7 are *not* statistical in nature. Question 3 could be adapted to a statistical question with more specific criteria on what is meant by healthy. Similarly, question 7 is very broad and will need to be focused to collect the data needed to answer it. Facilitating discussions with students about examples and non-examples as well as questions they develop improves their ability to generate appropriate statistical questions.

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

When students formulate the questions, the data they gather become more meaningful. How they organize the data and the techniques for analyzing them have a purpose. Often, questions will come naturally during the course of discussion or from questions arising in other content areas. The next two sections suggest many ideas.

Classroom Questions

Students want to learn about each other, their families and pets, measures such as arm span or time to get to school, their likes and dislikes, and so on. To begin, consider questions that can be answered by each class member contributing one piece of data. Here are a few ideas:

- *Favorites:* TV shows, games, movies, ice cream, video games, sports teams, music (when there are lots of possibilities, start by restricting the number of choices).
- *Numbers:* Number of pets or siblings, hours watching TV or hours of sleep, bedtime, time spent on the hand-held device.
- *Measures:* Height, arm span, area of foot, long-jump distance, shadow length, seconds to run around the track, minutes spent traveling to school.
- *Choice:* Ask your students, "What do we want to know about ourselves?" This is a way to connect to their personal interests and cultural backgrounds, and they learn how to formulate statistical investigative questions.

As noted earlier, a distinguishing feature of statistics is that the context is front and center. Therefore, it is particularly important that the context be culturally meaningful. Culturally meaningful contexts create a supportive classroom environment (McGlone, 2008). Such connections include exploring favorite family food or games or can include an exploration of family customs—the key to having such questions lead to a discussion of results that helps others in the class appreciate and respect the unique lives of their classmates.

Activity 20.1 Who Is in Our Village?

CCSS-M: 2.MD.D.10; 3.MD.B.3; 6.SP.A.1; 7.SP.B.3

Read one of the picture books, such as, *If America Were a Village: A Book about the People of the United States* (Smith, 2009) or *If the World Were a Village: A Book about the World's People* (Smith, 2011), both of which explore such topics as wealth and language, providing statistics in the form of a village of 100 people. After reading, ask, "What questions might we ask about our village (our classroom or community)?" Engage students in a conversation about questions, including the ones in the book, as well as other questions about themselves. With a list in hand, either select one to explore as a class, or have small groups select the question they want to investigate.

As an additional or extended opportunity, challenge students to revise one of the pages (because the data are more than 10 years old). Then they can compare the two villages. Try the article by Riskowski, Olbricht, & Wilson, 2010, for a project exploring concepts of statistics using 100 students.

Questions beyond Self and Classmates

The statistical questions in the previous section are designed for students to contribute data about themselves. As in Activity 20.1, these questions can grow into questions that look beyond the classroom, such as, "How might our data compare to another class?" For example, do other second-grade students like the same video games that we like? Comparison questions are a good way to help students focus on the data they have collected and the variability within that data and between data sets (Russell, 2006). As students get older, they can begin to think about various populations and differences between them. For example, how are sixth-graders similar to or different from eighth graders?

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Exploring the local community through formulating statistical investigative questions is an excellent way to engage in culturally responsive mathematics teaching (and integrate social studies and science). The following activity can be done early in the year and then revisited regularly to explore new statistical questions and answers.

Activity 20.2 What Can We Learn about Our Community?

CCSS-M: 1.MD.C.4; 2.MD.D.10; 6.SP.A.1; 6.SP.B.4; 6.SP.B.5

This activity plays out over several days. First, ask students to turn in a note card with three statistical questions they would like to investigate during the year. This could be assigned as homework, with students' families helping to brainstorm ideas. Collect these ideas. When you have time to start the investigation, take one question from the set. As a class, refine the question to one that can be answered using statistics. Questions students might submit include:

- How many kinds of restaurants or stores are in our community (fast food restaurants versus sit-down restaurants; Italian, Mexican, or American; convenience stores, grocery stores, clothing stores, department stores)?
- How many responses are made by local firefighters each month? How many types of responses are made by local firefighters each month (fire, medical, hazardous, public service)? Data can usually be found on websites of local institutions.
- How many state and local government officials do voters elect?

Discuss ways to gather the data. Set up a plan and deadline for gathering the data. When students bring in the data, encourage them to select and use data displays (e.g., circle graph, pie chart, line plot, etc.). Invite students to share. Return to the question and ask, "What does our data tell us about _____?" Consider how the different data displays communicate the answer to this question. See Expanded Lesson: Using Data to Answer a Question for a full lesson of this activity.

Pearson eText Expanded Lesson: Using Data to Answer a Question

To help students identify questions, look through local newspapers (online or in print). Additionally, the newspaper itself can lead to brainstorming different statistical questions. For example,

- What is being advertised (and lots of other questions about advertising!)? Are there different ads in different sections (i.e., sports, next to the comics, near the obituaries)?
- How many sports stories are about different types of sports (e.g., basketball, football, swimming, etc.)? How many about male teams or female teams?
- How many pages is the local paper each day of the week? How many stories about crime? How many about positive events in the community?
- What types of stories are on the front page across several weeks?

Science is about inquiry and is full of measurements and data, and therefore provides excellent opportunities for interdisciplinary learning experiences. For example, consider this short list of ideas:

- How many plastic bottles are placed in the school's recycling bins over a given week?
- How many days does it take different types of beans, squash, and pea seeds to germinate when kept in moist paper towels?
- Which brand of bubble gum will give you the largest bubble?
- Does a plant grow faster when watered with water, soda, or milk?

Before going on a field trip, students can develop questions for data they can gather on the trip (Mokros & Wright, 2009). For example, when visiting the zoo, students might record data (e.g., keeping tallies in a chart) related to questions such as:

- How many animals at the zoo are bigger than me?
- How many animals have no legs, two legs, four legs, and more than four legs?
- Are there more animals with fur or without fur?
- From what continents are the animals at the zoo?

To include statistical investigations that are culturally responsive (in other words, connect meaningfully to your students), ask them what issues or topics they wish to investigate. Sarfo-Mensah (2020), for example, asked her seventh graders, mostly Black and Latino/a/x students, to select a topic, and they selected police brutality (see Edutopia story for more details on connecting student interests to standards and learning outcomes https://www.edutopia.org/article/bringing-culturally-responsive-lens-math-class).

Data Collection

Learning Outcome 20.3 **Describe techniques for collecting data and quality sources for finding data.**

How to collect data effectively is an important (and sometimes skipped) part of the discussion as students learn statistics. Figure 20.2 shows how a kindergarten class decided to organize their data collection on favorite ice cream flavors (Cook, 2008). Students may gather data by hand-raising, interviewing their classmates, collecting tallies, or using a ballot.

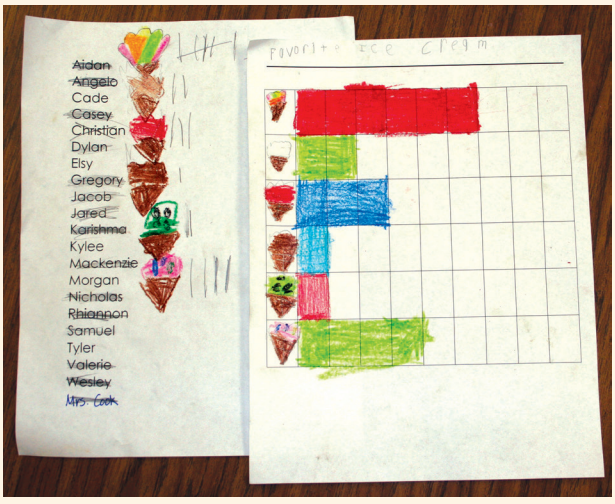


FIGURE 20.2 Kindergartners collect data on favorite ice cream flavors

Source: C. D. Cook (2008). "I Scream, You Scream: Data Analysis with Kindergartners." *Teaching Children Mathematics, 14*(9), p. 539. Reprinted with permission. Copyright © 2008 by the National Council of Teachers of Mathematics. All rights reserved.

Pearson eText
Video Example 20.2
In this video, what do you notice about how the teacher helps students focus on the big idea of how to collect data?

The following (true) scenario describes data collection in a first-grade classroom. As you read, reflect on how this data collection is different from collecting data about favorite ice cream or jelly bean flavors:

A teacher asks her students to gather data to find out how many of their classmates are 6. The students agree on the question, "Are you 6?" The teacher instructs students to gather data from each of their classmates and gives them a signal to begin. Eighteen eager students stand up and begin asking each other "Are you 6?" tallying yes or no, then walking up to another classmate. After about 5 minutes of exchanges, the teacher asked everyone to sit down to look at the data they had collected and count how many students were 6. Each student had different data to report.

The students did not pay attention to whom they had surveyed, asking some students more than once and other students not at all. This provided an excellent entry into a discussion about how to collect data.

Beyond surveys, data can be collected through experiments or observations. For example, set up a bird feeder outside the classroom window and collect data at different times during the day to count the number or type of birds. Students can also conduct observational data collection events on field trips and in evening or weekend activities with their families.

As illustrated in these examples, different types of data can be collected. Data are often classified as categorical (data grouped by categories such as ice cream flavors) and numerical (data that are counts or measures of things, like number of children who are 6), but more and more data come in the form of pictures, sounds, videos, or words (Bargagliotti et al., 2020).

Sampling

When asking a question about a small population, like your class, data can be gathered on everyone. But statistics generally does not involve gathering data from the whole population and instead samples the population. Recall from Figure 20.1 that data collection *acknowledges variability*. This means that the sample you select must *represent* your whole population. For example, a poll on favorite TV shows will produce different answers from a survey of seventh-grade students than from a group of teachers or third-grade students. It may also vary culturally or by sex or gender. Answers also may vary based on the day the question is asked or whether a particular show has been recently discussed.

To help students determine whether they have identified a representative sample, ask, "What is the population for your question?" or "Who or what is the subject of your question?" Then ask students to consider how they will gather data that will include representatives across that population. For example, if they are hoping to learn the movie choice for a seventh-grade movie night at your school, they need to poll girls and boys across the seventh-grade teams within the middle school. Provide opportunities for students to justify whether the data are representative and to critique the explanation of others as they describe what a representative sample is.

Standards for Mathematical Practice
MP6. Attend to precision

Even when it may appear that a sample is representative, it may not be. Unintentional biases can occur, and we cannot always know what subsets might exist within a population. Therefore, *random sampling* is used. It increases the validity of the results and therefore gives more confidence in being able to make inferences. Seventh-grade students who begin informal work with random sampling start to consider the importance of representative samples for drawing inferences. Activity 20.3 can help students develop an awareness of the importance of sampling.

Activity 20.3 How Do We Compare?

CCSS-M: 6.SP.A.1; 6.SP.A.2; 7.SP.A.1; 7.SP.B.3

Using *Book of Lists: Fun Facts, Weird Trivia, and Amazing Lists on Nearly Everything You Need to Know!* by Buckley and Stremme (2006), or a similar book or online resource, find a list that includes sampling a group of people. Use the How Do We Compare? activity page. Read the question to the students and gather data, using the class as the sample. Ask students, "Do you think our class will be a representative sample of the population targeted in this question?" Ask them to offer why the class might be or might not be a representative sample. Gather data from the class. Compare the class data to the list in the book. Create a circle graph to compare the two data sets. After displaying the two sets, ask students what technique they think the authors might have used to sample the population.

Pearson eText Activity Page: How Do We Compare?

Although this activity involved people, "population" in statistics is used broadly to mean "group or subject of study." The population could be a species of a plant, an insect, or a type of car. The previously mentioned activity can be replicated with other data. For example, you can look up favorite car colors online and then see if cars that drive by the school form a representative sample of all cars.

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Using Existing Data Sources

Data do not have to be collected by survey; existing data abound in various places, such as the following sources of print and Web data.

Print Resources. Newspapers, almanacs, sports record books, maps, and various government publications are possible sources of data that may be used to answer student questions. Children's literature is another excellent and engaging resource. Young students can tally words in a repeating verse like "Hickory, Dickory, Dock" (Niezgoda & Moyer-Packenham, 2005). Similarly, books like *Goodnight Moon* (Brown, 1947) or *Green Eggs and Ham* (Seuss, 1960) have many repeated words or phrases. Nonfiction literature can be another source of data and has the added bonus of learning about the real world. For example, *A Negro League Scrapbook* (Weatherform, 2005) has very interesting statistics about the league's players. Ask your librarian for ideas!

Web Resources. More and more data sets are available online. Here are a few excellent choices:

1. *Census at School survey* (www.amstat.org/censusatschool) is an international classroom project that engages fourth- through twelfth-grade students in statistical problem solving, using their own real data (students respond to a 13-item survey).
2. *Tuva* (tuvalabs.com) has access to more than 350 authentic databases from primary sources, complete with lessons for grades 3–12. You can also input your own data sets and use Tuva tools to explore, visualize, and analyze your data.
3. *Common Online Data Analysis Platform* (CODAP) (https://codap.concord.org/) provides an easy-to-use data analysis tools for middle school students. CODAP offers over 50 data sets and data games (games that generate data sets).

Students may be interested in facts about another country as a result of a social studies unit or a country in the news. Olympic records in various events over the years or data related to environmental issues are other examples of topics around which student questions may be formulated. For these and hundreds of other questions, data can be found online. These websites provide a lot of interesting data.

World focus:

- Google Public Data Explorer is a direct source of many interesting large data sets. (https://www.google.com/publicdata/directory)
- The Central Intelligence Agency (CIA) *World Fact Book* provides demographic information for every nation in the world: population, age distributions, death and birth rates, and information on the economy, government, transportation, and geography. (https://www.cia.gov/the-world-factbook/)
- The Official World and Olympic Records website provides information about the games and events. (https://www.worldathletics.org/records/by-category/world-records)

U.S. focus:

- Data.gov provides access to all data sets available from the U.S. government on one site.
- NASA's Earth Data provides global observations of the Earth's surface, revised every 2–3 days at 30-meter spatial resolution. The data here are a great connection to earth science! (https://earthdata.nasa.gov/)
- The U.S. Census Bureau has copious statistical information by state, county, or voting district. (https://www.census.gov/)
- USDA Economic Research Service Food Consumption offers wonderful data sets on the availability and eating of hundreds of foods. Annual per capita estimates often go back to 1909. (https://www.ers.usda.gov/)

Data Analysis: Classification

Learning Outcome 20.4 **Identify meaningful ways for students to classify data.**

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Data analysis begins with organizing the data in a meaningful way, using techniques such as sorting or graphing, with the resulting visual format providing a snapshot of the data. The data are then analyzed with the goal in mind of discovering how the information answers the questions that started the statistical inquiry and provided the purpose for collecting the data.

Classification involves making decisions about how to categorize things. This is not easy for young students, who may just list the data (Clements & Sarama, 2020). For example, when asked to represent class data about how they arrived at school (bus, car, walk, logging in), young students may just list each classmates' response, rather than group them into categories. Students need opportunities to discuss ways to categorize their data that aligns with their question. For example, students who posed the question, "What types of animals live on farms?" might group the animals by those that work, provide food, or are pets; by size or color; and so on. Each of these groupings is based on a different *attribute* of the animals. Classifying by attributes is typically a kindergarten topic.

Pearson eText
Video Example 20.3
In this video author Karen Karp discusses representing and interpreting data through pictographs and bar graphs.

Attribute Materials

Attribute materials can be any set of objects that lend themselves to being sorted and classified in different ways—for example, seashells, leaves, pencils, or students' shoes. The *attributes* are the ways that the materials can be sorted. For example, pencils have length, erasers, tip sharpness, and colored sides. Each attribute has a number of *values:* for example, a pencil may be tall or short (length), new or worn down (eraser), sharpened or not (tip), and yellow, blue, or red colored sides. Commercially available attribute blocks are designed for sorting, with each

block having four attributes: color (red, yellow, blue), shape (circle, triangle, rectangle, square, hexagon), size (big, little), and thickness (thick, thin).

Classifying shapes with students can be done by sitting in a circle where all students can see, with the objects to be sorted in the middle. Grouping circles or hula hoops can be used to categorize groups. For example, if sorting leaves, students might pick "with stems and without stems," placing the ones with stems in the grouping circle and those without stems outside the circle or in another circle. Then students are ready to explore overlapping categories, the focus of Activity 20.4

Activity 20.4 What about "Both"

CCSS-M: K.MD.3; 1.MD.C.4

Give students two grouping circles (or two pieces of string to be formed in a circle) and Woozle cards (see Figure 20.3), attribute blocks, or other sets of objects with varying attributes. To begin, ask students using attribute blocks to put all the *red* pieces inside one circle and all *triangles* inside the other circle. Let the students grapple with what to do with the *red triangles*. After some time exploring, bring students back to share what is in their circles. Ask, "What did you do with the red triangles?" When the notion of overlapping the strings or circles to create an area common to both loops is clear, more challenging activities can be explored, for example, three attributes. As shown in Figure 20.4, the labels need not be restricted to single attributes. If a piece does not fit in any region, it is placed outside all of the loops. To support multilingual learners and students with special needs in math, provide cards with attribute labels to place in the circles to show the rule (the cards can be translated into the students' native language).

Pearson eText Activity Page: Woozle Cards

Notice that the categories in Figure 20.4 include "not" as well as a combination "red or yellow," Another option could be "and" (e.g., blue and happy). This use of *not*, *and*, and *or* significantly widens students' classification schemes. An engaging and challenging activity is to infer how things have been classified when the loops are not labeled. The following activities require students to make and test conjectures about how things are being classified.

Standards for Mathematical Practice
MP6. Attend to precision

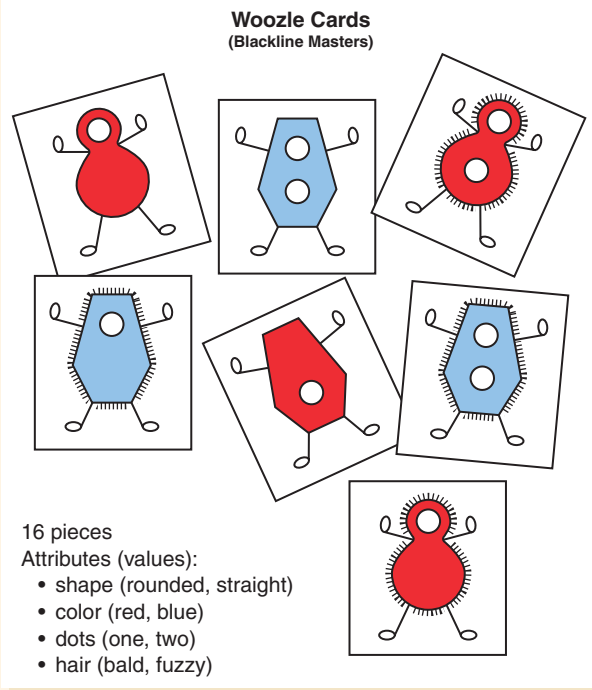


FIGURE 20.3 Sample set of Woozle cards and list of attributes

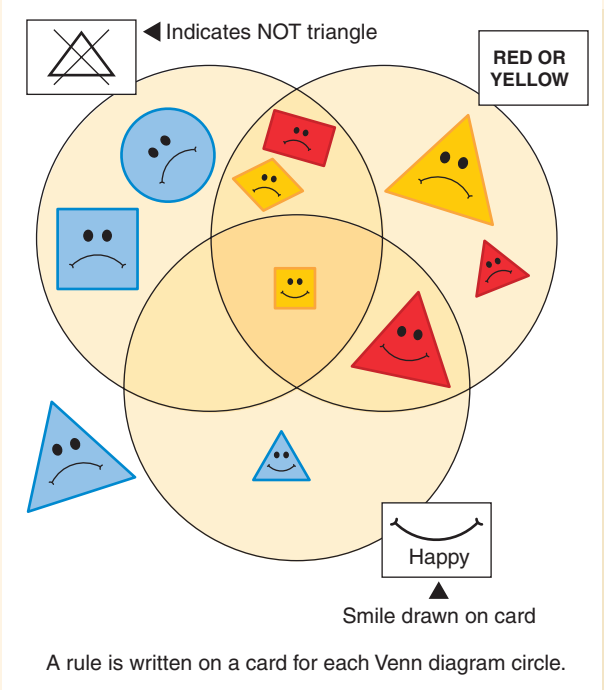


FIGURE 20.4 A Venn diagram activity with attribute pieces

Activity 20.5 I.D.the Attribute

CCSS-M: K.MD.3; 1.MD.C.4

This activity can be done with attribute blocks but is particularly engaging to use with mystery clothing attributes such as "wearing blue jeans" or "stripes on clothing." Ask for a volunteer. After they stand, move the student to the left or right according to this secret rule. Tell students that when they think they have identified the attribute, they are *not* to tell, but they can be the Sorter. Continue the activity for a while so that others in the class will have an opportunity to determine the attributes. Repeat this activity regularly. You can use any materials that can be sorted, such as students' shoes, shells, or buttons.

Activity 20.6 Hidden Labels

CCSS-M: K.MD.3; 1.MD.C.4

Place grouping circles or loops of string (one, two, or three) in the form of a Venn diagram. Use attribute blocks, Woozle cards (see Figure 20.3), or illustrations of geometric figures. Use attribute label cards or write labels on notecards for each loop of string. Place the labels *facedown* so you know the labels, but students do not. Ask students to select a block (or Woozle or geometry card) for you to place based on the hidden rules. As you sort, have students try to determine what the labels are for each of the loops. When they think they know, they raise their hand, but keep quiet. They get to become the teacher (sorter), telling where the next block goes. Wait to turn the cards face up until most students have figured out the rule. For multilingual learners and students with special needs in math, provide a list of the labels with pictures and/or translations for each as a reference.

Pearson eText Activity Page: Woozle Cards

Pearson eText Activity Page: Illustrations of Geometric Figures

Pearson eText Activity Page: Attribute Label Cards

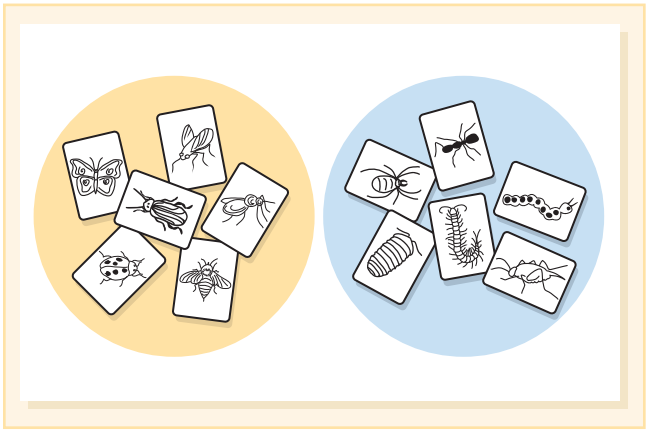


FIGURE 20.5 Can you ID the attribute that was used to sort these bugs?

ID the Attribute and Hidden Labels can be repeated with real-world materials connected to other content areas and to students' experiences. For example, if you were doing a unit on wildlife in the backyard, you could use pictures of creatures (see Figure 20.5). To connect reading and data, students can sort and analyze data within a book (e.g., types of events or settings) or across books (e.g., kinds of books). Students can bring in things they can recycle, and these objects can be sorted by type (English, 2013). Table 20.3 lists additional ideas that can be used for classification activities related to social studies and science. Using ideas related to content and the real world helps students understand that data analysis is useful as well as engaging!

Data Analysis: Graphical Representations

Learning Outcome 20.5 **Determine developmentally appropriate ways for students to analyze data.**

Graphs summarize data. Representing and interpreting data typically begins in grade 1 and is included at each grade thereafter, with increasingly complex representations and analysis expected. How data are organized should be directly related to the question that caused you to collect the data in the first place. Elementary students might create a bar graph to illustrate their favorite type of fruit, or a line plot (dot plot) to tell how many pockets each person has.

TABLE 20.3 TOPICS IN SOCIAL STUDIES AND SCIENCE THAT CAN BE CLASSIFIED

Social Studies	Science
• Country or state of origin of classmates' ancestors • Past, present, and future events • Goods (such as bread, milk, apples, pants, socks, shoes) and services (such as mowing yards, repairing cars) • Continents and oceans • U.S. presidents according to the state where they were born • Availability of items (scarce, average, abundant) • Jobs in the local community • Climates in the United States	• Sensory descriptors for items (hard/soft, rough/smooth, cold/warm, loud/quiet, sweet/sour/bitter/salty, high/low, and bright/dull) • Relative size or weight of a collection of objects (big/little, large/small, heavy/light, wide/thin, long/short) • Speed (fast/slow) of different animals (ants, turtles, snakes, cheetahs) or types of transportation (car, bicycle, airplane, walking) • Materials or objects that float/sink or dissolve/don't dissolve in water • Weather observations (sunny, cloudy, raining, snowing) • Characteristics of plants (edible/nonedible, flowering/nonflowering, evergreen/deciduous) • Characteristics of animals (methods of movement; wild/tame, water homes/land homes, hibernate/do not hibernate, migrate/do not migrate, camouflage/no camouflage)

Middle school students might have questions about music, such as, "How many songs do students in our class listen to in one day?" For data collection, you might decide to have students keep track of the songs on a school day (Tuesday, for example) and come to class on Wednesday with their own totals. Each student records the number on a sticky note and places it on the board. In middle school, a dot plot might be used to illustrate the spread and shape of the data or a histogram to capture how many students fall within a range of number of songs listened to (e.g., between 0 and 10, 11 and 20, etc.) or a box plot to highlight the center of the data as well as the range. The point is that each of these displays gives a different snapshot of the data and provides different insights into the question posed.

Students should be involved in deciding how they want to represent their data, but they will need to be introduced to what the options are and when each display can and cannot be used. Creating graphs requires care and precision, including determining appropriate scales and labels so that an audience can see at a glance the summary of the data gathered on a particular question. Instruction should not focus solely on the details of graph construction. The focus must be on what the graph tells the people who see it as it relates to the question posed.

Standards for Mathematical Practice
MP6. Attend to precision

***TECHNOLOGY* Note.** Technology makes graph creation so much easier, freeing up time to focus on analyzing and interpreting the data. Computer apps and programs (e.g., Microsoft Excel or Google sheets) provide a variety of graphical displays. CODAP is an excellent option (https://codap.concord.org/). Canva's Graph Maker (https://www.canva.com/graphs) makes many kinds of graphs. For younger students or students who benefit from a step-by-step process, Create a Graph (NCES Kids' Zone) is a good option for various graphs (e.g., bar graph, pie charts, line graph, etc.). ■

Graphs for Categorical Data

Bar graphs and circle graphs are ways to summarize categorical data.

Bar Graphs. Bar graphs include object graphs (real objects [including humans] used in the graph), picture graphs, and bar graphs. These graphs display categorical data.

An ***object graph*** uses the actual objects being graphed. Examples include types of shoes, favorite crayon color, energy bar wrappers, and books. Each item can be placed in a square or on a floor tile so that comparisons and counts can be made easily. Notice that an object graph is a small step from sorting. If real objects are sorted into groups, those groups can be lined up for comparison—an object graph!

Picture graphs (also called *pictographs*) move up a level of abstraction by using a drawing or picture of some sort that represents what is being graphed. The picture can represent one piece of data, or it can represent a designated quantity (scaled). Students learn to interpret the scale for the pictograph.

Students can make their own drawings, but this can often become time-consuming and tedious and take away from the mathematics. There are various ways to make the creation of picture graphs easier to create and thereby keep the focus on the meaning of the graph rather

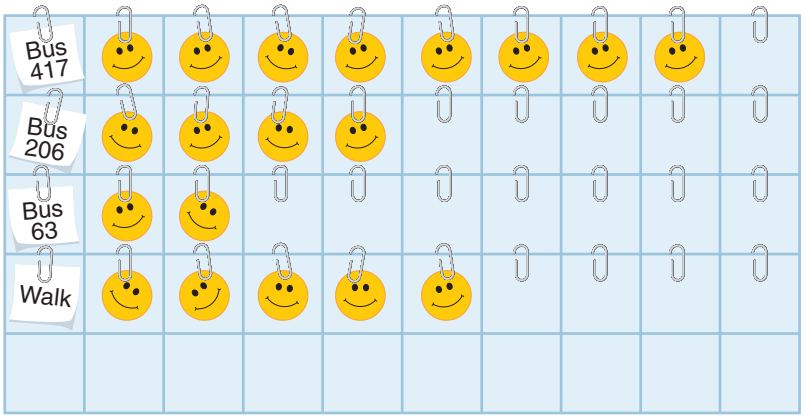
Pearson eText
Video Example 20.4
In this video consider how preschoolers are able to create object graphs. How could they transition to picture graphs?
https://www.youtube.com/watch?v=4SIY1-idRqY&list=PLRY-hkA40X06NRPAADyqsG-FWoNwfU3Sq&index=4

Pearson eText
Video Example 20.5
Consider what strategies the teacher uses to help students understand bar graphs.

than on the creation of it. For example, you can use stickers, objects cut out with a die-cut, or clip art (copied repeatedly on a page that can then be cut out, or students create the graphs electronically).

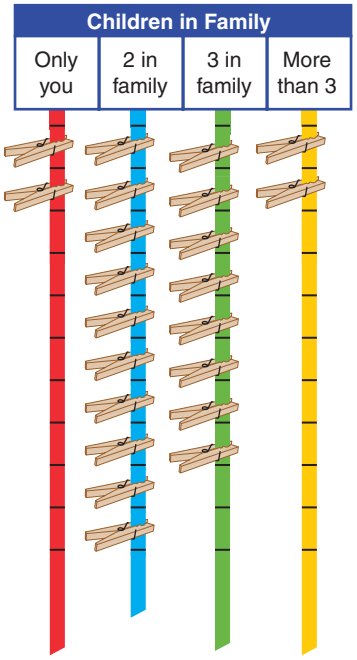
After object and picture graphs, ***bar graphs*** are among the first ways used to group and present data with students in pre-K–2. To help transition from picture graphs to bar graphs, have students use something to represent the pieces of data or the things being counted. Figure 20.6 illustrates a few techniques that can be used to make a graph quickly with the whole class. These examples can be posted in the classroom, changing out the question and the data each week.

Initially, bar graphs should be made with each bar consisting of countable parts so that the individual pieces of data do not seem to disappear. The sticky notes in the next activity, "Picture Graphs to Bar Graphs," serve this purpose.



Clip paper pictures or symbols on a chart that has a paper clip prepared in each square.

A graph mat can be made on a sheet of plastic about 8 to 10 feet long and used on the floor. Make 5 or 6 columns with 12 to 15 squares in each column. Students place real objects in the columns to show the number of each.



Hang ribbons with marks at even intervals for placing clothespins, and have students clip on pinch-style clothespins.

FIGURE 20.6 Some ideas for quick graphs that can be used again and again

Activity 20.7 Picture Graphs to Bar Graphs

CCSS-M: 2.MD.D.4; 3.MD.B.3

With students, generate a question that lends itself to preset categories (e.g., favorite pet). Give each student a sticky note and ask them to draw a picture of their favorite choice (e.g., cat), place their sticky notes in a row on the whiteboard (or wall) so that there is a little space (equal) between the sticky notes. Invite students to make observations about their picture graph. Discuss with students how they might make a *bar graph* to illustrate their data. Move the sticky notes so there is no space between them to look like a bar. Discuss ways to know how long the bar is and add a scale next to the bars to show the height of each bar. This activity can be done with a scaled situation, too. For example, they might decide to use one picture of a cat to represent three cats. In the bar graph, the scale will skip-count by threes, which can be physically illustrated by stacking three sticky notes on top of each other. Ask students to figure out what to do when there are not three (use thirds). Have students record their scale (e.g., 1 cat picture = 3 real cats).

Once the graph has been constructed, be sure to interpret the data (step 4 of the statistical problem-solving process and addressed later in this chapter).

Circle Graphs and Pie Charts. Circle graphs and pie charts mean the same thing; the term *circle graph* may be more common in curriculum, and the term *pie chart* is used with graphing tools such as electronic spreadsheets. Pie charts, like bar graphs, typically display categorical data. They also typically show percentages and, as such, are inaccurately perceived as too advanced for young students. However, circle graphs can be set up to indicate the number of data points from a total set, without calculating percentages and can be interpreted visually. Also, an understanding of percentages is not required when using computer software to create the graph.

Pie charts and circle graphs are commonly found in newspapers and websites. Pie charts make it more difficult for comparisons *within* a data set (angle measures are harder to compare than lengths of bars) but easier to compare *across* data sets, in particular when the data sets are a different size. Figure 20.7 shows a pie chart for classroom data and schoolwide data illustrating the percentages of students with different numbers of siblings.

One fun and easy option for constructing pie charts is the human circle graph. Students can first form lines for their response to a question, and then the lines rotate to make a circle. Students can each hold a color block to show their response, and string can be held by a person in the center to show the pie pieces (see Figure 20.8). See Expanded Lesson: Bar Graphs to Circle Graphs for a full lesson.

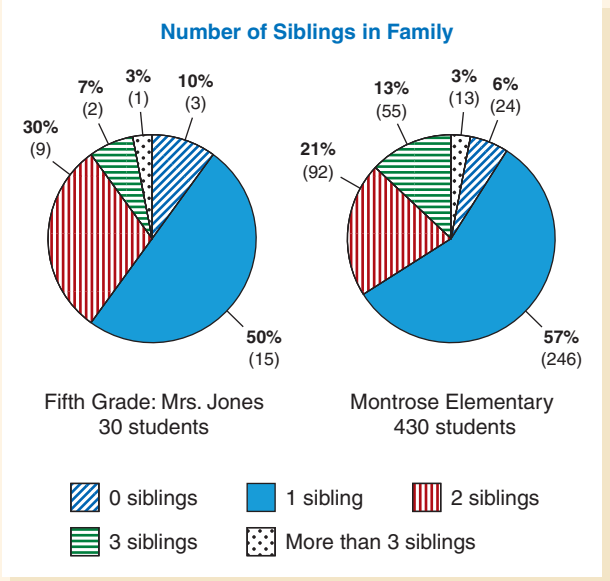


FIGURE 20.7 Pie charts are useful in comparing two data sets

Pearson eText Expanded Lesson: Bar Graphs to Circle Graphs

A second option is to use color tiles, converting from rows to forming a circle. This can then be sketched on paper. You can estimate percentages by using the rational number wheel or percent necklace as described in Chapter 16.

The concrete circle-making process helps make sense of the mathematics of a circle graph. The numbers in each category are added to form the total or whole. The percentage of one section is computed by dividing each of the parts by the whole (and multiplying by 100). It is an interesting proportional problem for students to convert between percentages and degrees, because one is out of 100 units and the other is out of 360 units. It is helpful to start with common benchmarks like 50 percent, 25 percent, and 10 percent before moving to more difficult values.

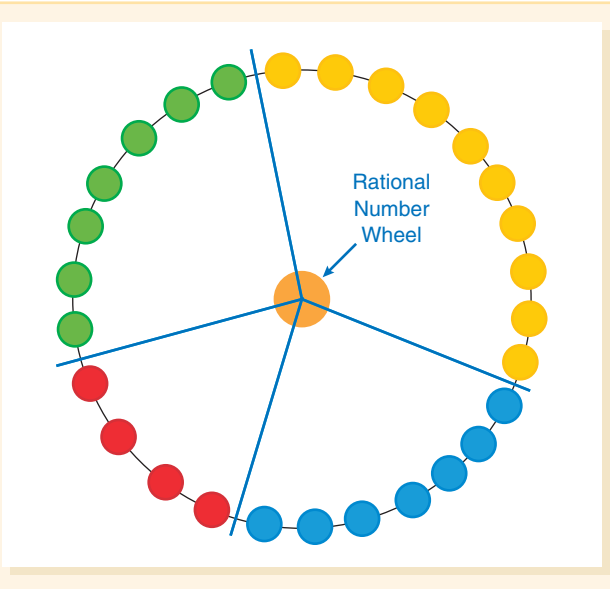


FIGURE 20.8 A human circle graph: Students are arranged in a circle, with string stretched from the center to show the pie pieces

FORMATIVE ASSESSMENT Notes. Students can write in a journal about their graphs, explaining what the graph tells and why they selected that type of graph to illustrate the data. As you evaluate students' responses, it is important not to focus undue attention on the skills of constructing a graph but, instead, to focus on whether they chose an appropriate representation and have provided a good rationale for its selection that connects back to their question (step 1). ■

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

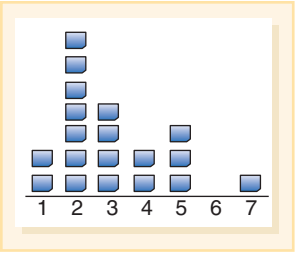


FIGURE 20.9 Dot plot (line plot) displaying number of pockets each student has

Graphs for Numerical Data

Bar graphs or picture graphs are useful for illustrating categorical data. When data are grouped along a continuous scale, they should be ordered along a number line. Examples of such information include temperatures that occur over time, height or weight over age, and percentages of test takers' scoring in different intervals along the scale of possible scores.

Line or Dot Plots. *Dot plots* are counts of things along a numerical scale on a number line. Although CCSS uses the term *line plot* in elementary school (and therefore curricula do as well), statisticians use the term *dot plot*, which is therefore what we use in this section. To make a dot plot, a number line is drawn and a mark (e.g., an X, sticky note, or dot) is made above the corresponding value on the line for every corresponding data point, as illustrated in Figure 20.9 for the question, "How many pockets does a person have?"

Although this looks like a bar graph, the fact that the categories are now numbers may make it more challenging for students to interpret (Russell, 2006). For example, does the stack above the 5 mean that five people have three pockets or three people have five pockets? Activity 20.8 uses the students in the classroom to help make sense of dot plots.

Activity 20.8 Stand by Me

CCSS-M: 1.MD.C.4; 2.MD.D.10; 3.MD.B.4

Use painters' tape to create a line on the floor of the classroom or gymnasium. Label it with numbers 0, 5, 10 . . . 30 (or whatever scale is appropriate for your students). Have students write on a sticky note how many minutes it takes them to travel to school, rounded to the nearest 5 minutes. By groups, ask students to form a row in front of their matching time on the number line. Encourage peers to confirm they are standing in the correct place. Then, ask students to leave their sticky note where they were standing and sit down. Re-create the dot plot on the interactive whiteboard and have students create one on their own grid paper (see Figure 20.10). Ask students what they notice about the data. Multilingual learners and students with special needs in math benefit from sentence frames like, "I notice that _____ students take _____ minutes" or "One interesting thing I noticed is _____..."

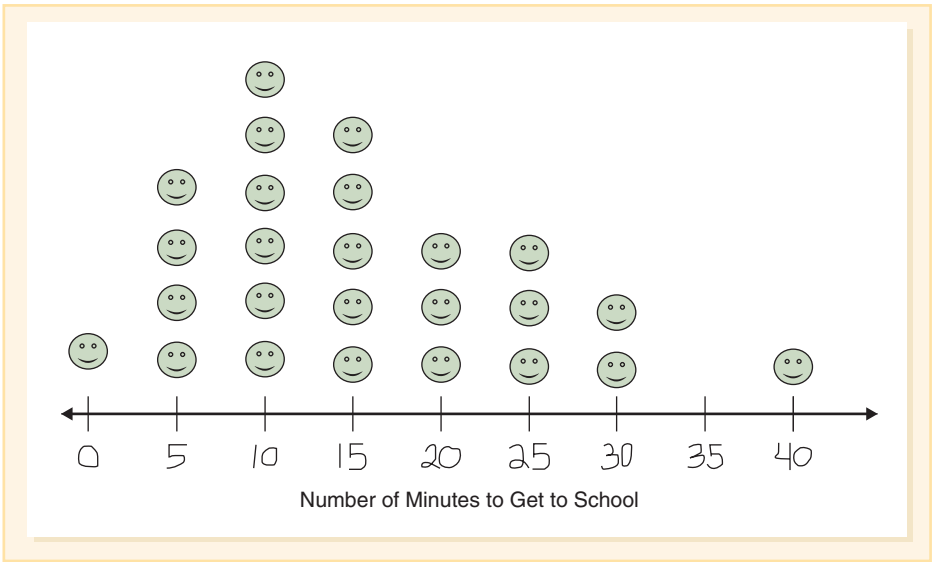


FIGURE 20.10 Class dot plot for Activity 20.8

Dot plots are excellent for measurements, and the scale can include fractional values. Students might measure their foot length, their cubit (length of forearm from elbow to fingertips), height of the stack of books they happen to have brought to class, and so on. Data can also be gathered from plants growing, time passing, or weather, as in the next activity.

Activity 20.9 Storm Plotter

CCSS-M: 3.MD.B.4; 4.MD.B.4; 5.MD.B.2; 6.SP.B.4; 7.SP.B.3

Create a class dot plot (line plot) and chart the amount of rain (or snow) that falls with each storm (install a rain gauge or access the information online). Place the data from each storm on the number line. This can be done all year and color-coded by month. As more data are gathered, you can ask, "What do we notice about rainfall in our area?" If you don't want to take months gathering data, instead look up the rain or snowfall in various cities and towns after a storm has passed through and use that to create a storm plot. Seventh-grade students can explore and compare two locations.

Dot plots work well for comparisons. For example, if data are gathered from two groups, separate dot plots can be created, providing a great visual to see the difference in the shape of the data sets.

Stem-and-Leaf Plots. Stem-and-leaf plots (also called *stem plots*) display numerical data as a list, grouped within ranges of data. By way of example, consider the National League baseball teams' total wins for the 2019 season:

East Division Teams	Number of Wins	Central Division Teams	Number of Wins	West Division Teams	Number of Wins
Atlanta Braves	97	St. Louis Cardinals	91	Los Angeles Dodgers	106
Washington Nationals	93	Milwaukee Brewers	89	Arizona Diamondbacks	85
NY Mets	86	Chicago Cubs	84	San Francisco Giants	77
Philadelphia Phillies	81	Cincinnati Reds	75	Colorado Rockies	71
Miami Marlins	57	Pittsburgh Pirates	69	San Diego Padres	70

Figure 20.11 shows how the graph is constructed. If the data are to be grouped by tens, list the tens digits to form the stem. Next, write the ones digits next to the appropriate tens digit, ordered from smallest to largest, forming the leaves. The result shows the shape of data, indicating where the data clusters and where the outliers are. Furthermore, every data point can be retrieved from the graph. Stem-and-leaf plots are not limited to two-digit data. For example, if the data ranged from 600 to 1300, the stem could be the numerals from 6 to 13, representing hundreds, with the leaves having two-digit numbers.

Histograms. To this point, the types of graphs show each data point, making them easier to interpret. Histograms (and box plots) are more difficult to interpret because they do not illustrate each data point (Groth & Bargagliotti, 2012), but the other options do not make sense with large data sets, which is the case with many real data sets. In this case, a histogram is an excellent choice. A histogram displays numerical data in consecutive, equal intervals. Histograms differ from bar graphs because (1) they are for representing numerical data (not categorical), and (2) the order of the bars matters (Metz, 2010).

Figure 20.12 shows two histograms describing the number of text messages sent in one month for (a) 100 teenage girls who have a cell phone and (b) 100 teenage boys who have a cell phone (Bargagliotti et al., 2020). The number of data elements falling into that particular interval determines the height or length of each bar. Histograms effectively show the distribution of data values, especially the shape of the distribution and any outliers.

Challenges to interpreting histograms are well-documented and point to the need to spend adequate time making sense of these very useful graphs (Cooper & Shore, 2008; Kaplan, Gabrosek, Curtiss, & Malone, 2014; Meletiou-Mavrotheris & Lee, 2010). Challenges include:

1. Understanding the distinction between bar graphs and histograms.
2. Finding the center of the distribution (students focus on the heights of the bars (*y*-axis) instead of on the distribution (*x*-axis).
3. Interpreting a flatter histogram to mean there is less variability in the data.

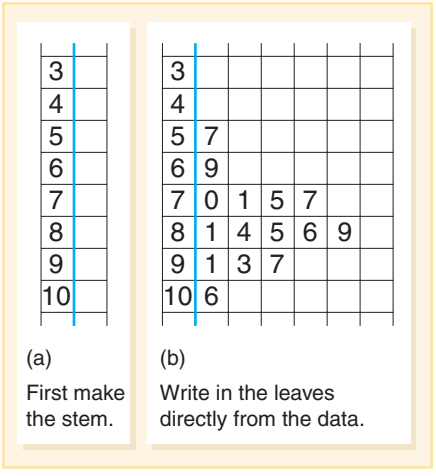


FIGURE 20.11 A stem-and-leaf plot of baseball wins

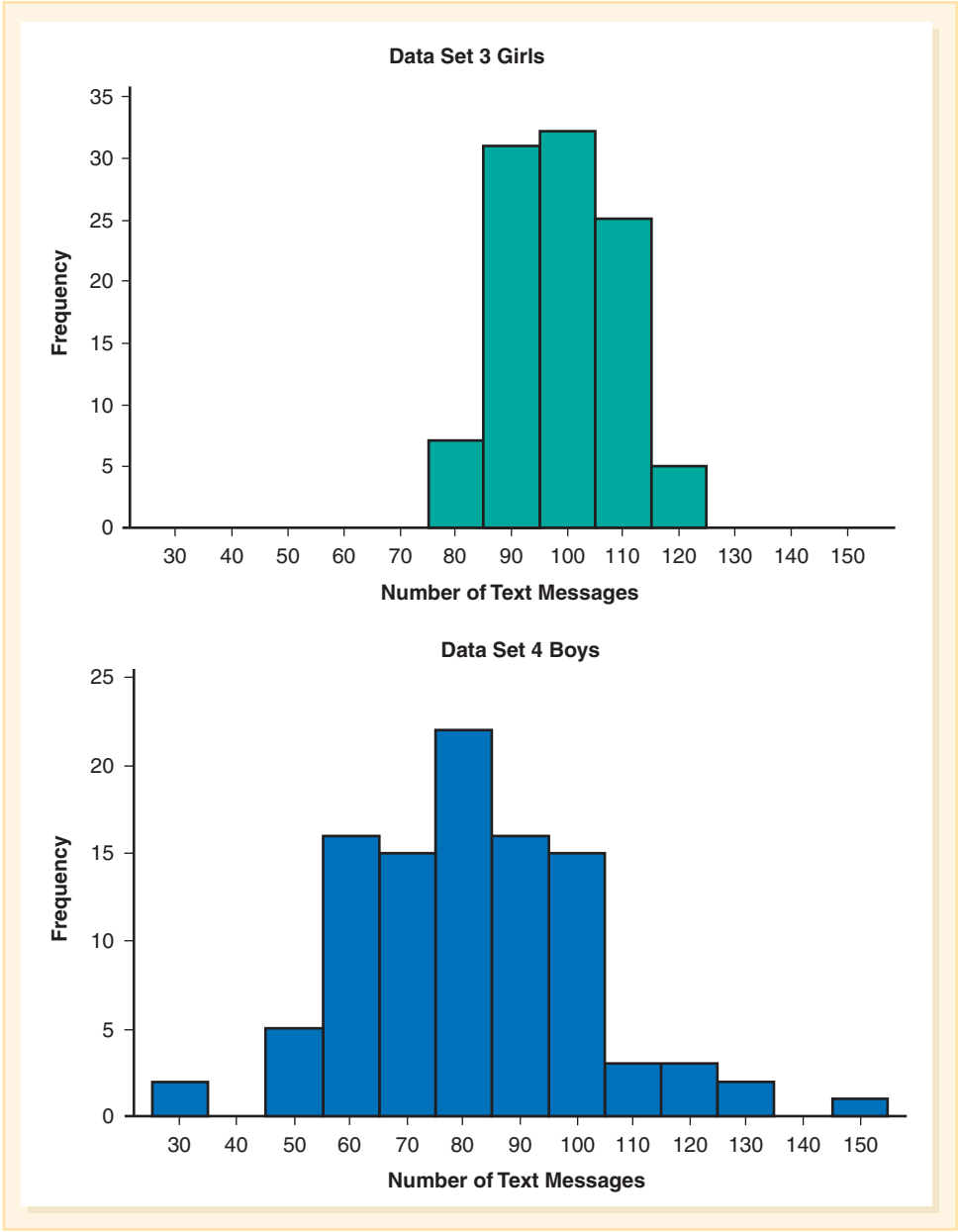


FIGURE 20.12 Histograms displaying data on teenage text messaging

There is also a lot of decision making with constructing histograms: What is the appropriate interval to use for the bar width? What is a good scale to use for the length of the bars? Histograms can and should be created with computer software or calculators to increase precision and so that sufficient attention can be devoted to making sense of the histogram and analyzing and interpreting the data.

***TECHNOLOGY* Note.** Desmos (Desmos.com) has several histogram investigations that connect to social and cultural contexts. Beyond histograms, Desmos has quite a few statistics explorations for middle school students. ■

Box Plots. A box plot (also known as a *box and whisker plot*) is a graph based on partitioning the data into four quarters, with the same number of data values in each quarter (Bargagliotti et al., 2020). Five numbers are needed to construct the graph: (1) minimum, (2) first quartile

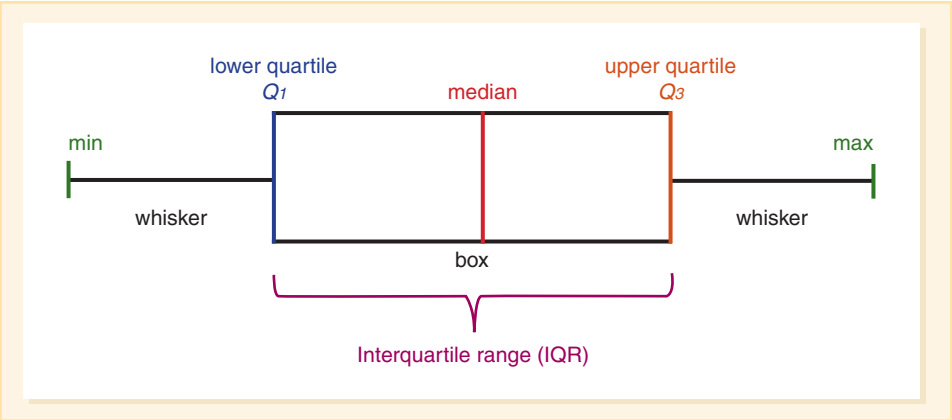


FIGURE 20.13 Parts of a box plot

(marking the end of one-fourth of the data values), (3) second quartile (marking the end at the middle of the data values), (4) third quartile (marking the end of three-fourths of the data values), and (5) maximum or marking the whole. Figure 20.13 illustrates these components.

To make a box plot, put the data points in order. Next, find the median. To find the two quartiles, ignore the median itself and find the medians of the upper and lower halves of the data. Mark the two extremes, the two quartiles, and the median above an appropriate number line. Draw the boxes and the lines (whiskers).

An effective way to introduce box plots is to create a dot plot first, which shows individual data values ordered from least to greatest, and then use those displays to create the box plots. For example, use the data from Storm Plotter (Activity 20.9). Activity 20.10 is another engaging way to explore box plots.

Activity 20.10 Which Paper Airplane Design Is Best?

CCSS-M: 6.SP.B.4; 7.SP.B.3

To begin, you can give students a few airplane-folding options or have students investigate and find their own design. In partners, have students build their airplane until they are satisfied with the design. Then it is time to collect data on the airplane in flight. Have students fly their plane 20 times, each time noting how far it went. (This works well on tile surfaces where the tiles are square feet, or you need to stretch tape measures or mark the floor with painters' tape in intervals so they can get approximate measures quickly.) After they have 20 data values, ask students to create a dot plot. From the dot plot, have students create their box plot. After completion, students can write observations about their own box plots, interpret other students' box plots, and discuss which airplane design is best.

Students do not need to create the box plot themselves; they can create them efficiently by using software (see Technology Note). For a related activity with paper helicopters, see Makar, Doerr, & delMas (2020).

***TECHNOLOGY* Note.** Numerous online sites create graphs. Check out CODAP (https://codap.concord.org/) and Infogram (https://infogram.com). Both offer many types of graphs, and you can import your own data. ■

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Because box plots have so much information and engage proportional thinking, students may be challenged to interpret box plots (Bakker, Biehler, & Konold, 2004). Interpreting box plots is difficult, because each section of the box plot contains approximately the same number of points (approximately 25% of the data). That means the *shorter sections* of a box plot represent the greatest densities of values, whereas the *longer sections* represent the greatest spread.

Pause & Reflect

What might be some interesting numerical contexts for which you can engage your students in gathering data and creating dot plots, histograms, and box plots? ●

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Bivariate Data

In eighth grade, the focus of statistics is on analyzing *bivariate data* (NGA Center & CCSSO, 2010). The phrase *bivariate data* may be new to the eighth-grade curriculum, but the concept is not. Stated simply, *bivariate* means that two things are varying together (e.g., number of people attending, the number of hot dogs sold). Concepts and activities related to covariation are addressed in Chapter 13 (Algebraic Thinking) and in Chapter 17 (Ratios and Proportional Reasoning). In statistics, the focus of bivariate data is on the distribution and related patterns in the covariation. Graphs and tables are effective in displaying these distributions so that patterns or trends are more observable (Kader & Jacobbe, 2013).

Bivariate Categorical Data. As the name implies, these situations involve two variables that are both categorical, such as favorite superpower and grade level. Bivariate categorical data are often presented in a two-way table, as illustrated in Figure 20.14(a). These data can also be analyzed and reported in terms of relative frequency, as illustrated in Figure 20.14(b). Notice that data can be analyzed by looking across one row (e.g., sixth grade students) or down one column (e.g., Super Strength), as well as by analysis of overall total.

a. Surveyed sixth-grade and eighth-grade students report on the superpower they would choose

	Super Strength	Ability to Fly	Invisibility	Razor Vision	Total
Sixth-grade students	15	36	17	6	74
Eighth-grade students	25	17	28	11	81
Total	40	53	45	19	155

b. Relative frequency of sixth-grade and eighth-grade students' superpower selections

	Super Strength	Ability to Fly	Invisibility	Razor Vision	Total
Sixth-grade students	10%	23%	11%	4%	48%
Eighth-grade students	16%	11%	18%	7%	52%
Total	26%	34%	29%	11%	100%

FIGURE 20.14 Example of bivariate categorical data, with raw data and relative frequency data

Line Graphs. A coordinate axis allows for the plotting of bivariate numerical data. When data are continuous, a line can connect the data points and illustrate the trend in the data. Data can be gathered at specific points in time (e.g., every 2 hours), and these points can be plotted. A straight line can be drawn to connect these points, because time is continuous and data points do exist between the plotted points. (The temperature moves through all the in-between degrees at each interval.) See the example in Figure 20.15 for a line graph on temperature change.

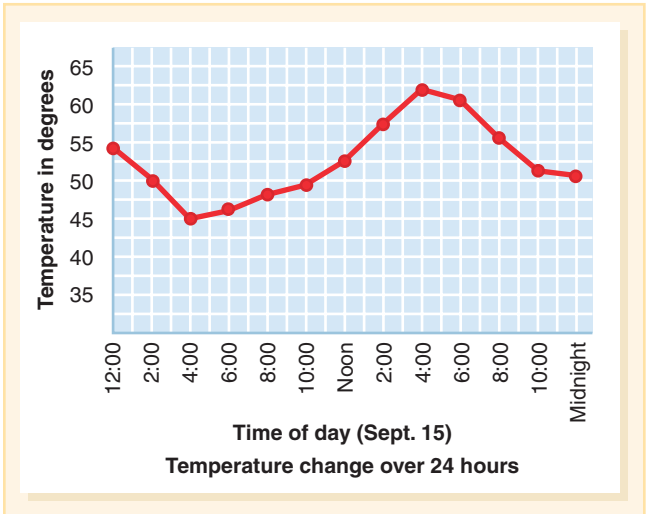


FIGURE 20.15 Line graph of one day's temperatures

Scatter Plots. Bivariate numerical data are often plotted on a scatter plot, a graph of points on a coordinate grid with each axis representing one of the two numerical variables. Each pair of numbers from the data, when plotted, produces a visual image of the data as well as a hint concerning possible relationships. For elementary school–aged students, the focus should be on interpreting the data (Bargagliotti et al., 2020). For example, Figure 20.16(a) shows a scatter plot of weight versus height and 20.16(b) a scatter plot of weight versus name length. Begin by asking about a slice of the graph, for example, what students notice about students with a height of 66 inches, or with a last name with eight letters. Then ask, "Is there a relationship between _____and _____?"

The first scatter plot indicates a positive relationship as it moves up from left to right in a visually observable way, although variation is shown by the points that are not exactly close to the imagined line. There is no relationship between name length

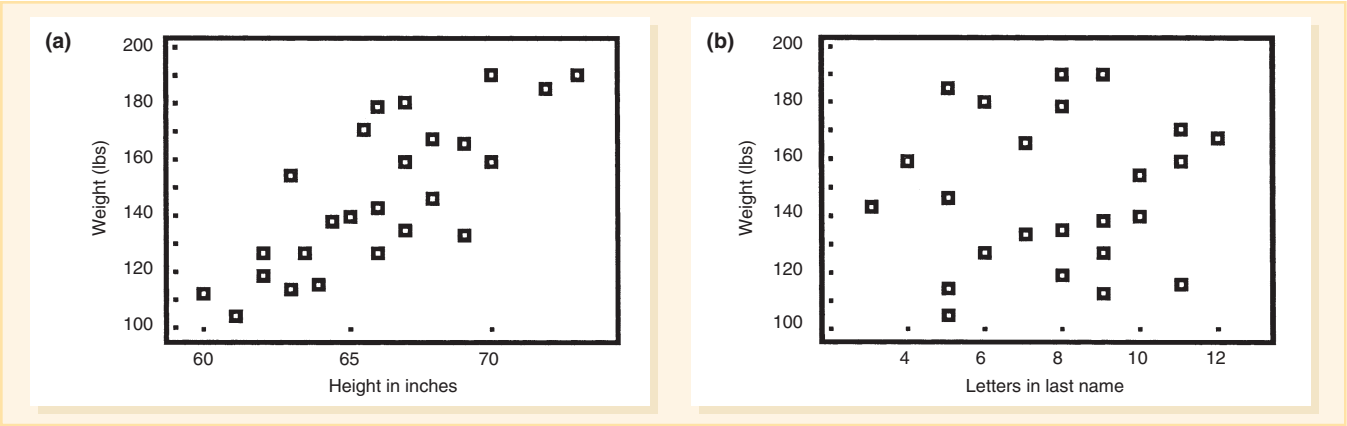


FIGURE 20.16 Scatter plots show potential relationships or lack of relationships

and weight: the points are everywhere with no unified pattern. Encourage students to plot many data sets and look for relationships in the scatter plots, including data sets that suggest linear relationships (like plot [a]) and those that indicate no apparent relationship between the variables (like plot [b]), or a nonlinear relationship (such as parabolic). Activity 20.11 provides engaging ways to explore the relationship between bivariate numerical data.

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

Activity 20.11 Is There a Relationship?

CCSS-M: 8.SP.A.1

Prepare cards with different questions on them about bivariate situations or use Is There a Relationship? cards. For example, is there a relationship between:

- Distance a toy car rolls and its weight?
- Distance a toy car rolls and tilt of a ramp?
- Distance a toy car rolls and how high it starts on a ramp (tilt held steady)?
- Foot length and height?
- Shoulder width and height?
- Months of age and height?
- Head circumference and wrist circumference?
- Minutes watching TV and minutes doing homework?

Distribute cards and a coordinate grid to groups of four students. Ask students to (1) predict whether they think there is a relationship and (2) determine how they will gather data. Have students gather data and create a scatter plot on paper that they will display to the class. Multilingual learners may be more familiar with metric measures and will also benefit from seeing what is being measured through gestures or demonstration. Each group reports on their findings and explains whether they think there is a relationship. Other groups listen and determine whether they agree that there is a relationship, and what that relationship is. As an extended experience, have students generate their own questions where they wonder, "Is there a relationship between x and y?"

Pearson eText Activity Page: Is There a Relationship? Cards
Pearson eText Blackline Master: Coordinate Grid—Quadrant I

Best-Fit Lines. If a scatter plot indicates a relationship, the relationship can be described by using the ratio between the two measures. You can find the ratio between the variables by drawing a straight line through the data points that best represent the pattern or shape of all of the dots, or a *line of best fit*. There are statistical techniques to do this, but to begin, students can estimate what looks like the best line.

Standards for Mathematical Practice
MP4. Model with mathematics

The line you select defines the observed relationship (the ratio or slope), and it can be used to predict other values not in the data set. The more closely the dots in the scatter plot cluster around the line you select, the greater the confidence you would have in the predictive

Activity 20.12 Spaghetti Lines

CCSS-M: 8.SP.A.1; 8.SP.A.2

Engage students in gathering bivariate data (see list in Activity 20.11). For example, ask students to measure foot length and height and record the data in a class table. Give students a copy of a coordinate grid (Quadrant I). Ask students to plot the coordinates on their own graph and then place a piece of uncooked spaghetti until they feel it is the best fit for the data and tape it down. Ask students to write an equation to represent their line. Invite students to compare and discuss the different equations. For example, ask students to use their equation or line to tell how tall Bigfoot might be (e.g., if his foot is 25 inches long).

Pearson eText Blackline Master: Coordinate Grid—Quadrant I

value of the line. Certainly, you could try to draw a straight line somewhere in the name length–weight graph, but you would not have much confidence in its predictive capability because the dots are quite dispersed from the line.

Encourage students to use a mathematical reason for why a line might be best. A good-fitting line is one where the distances from all of the dots to the line are minimal. This general notion of least distance to the line for all points is the basis for an algorithm that will always produce a unique line for a given set of points, which is introduced in high school.

***TECHNOLOGY* Note.** In many places, graphing calculators are being replaced by free software that is more versatile, provides better visuals, and is accessible to all students. Desmos (desmos.com) calculators (e.g., graphing, scientific, and 4-function) is becoming the standard not only in classrooms but also on the state-mandated exams (38 U.S. states and territories as of Fall 2021). There are also smart phone app versions of the traditional graphing calculator interface (and Desmos has an app version of its online calculators). ■

Data Analysis: Center, Variability, and Shape

Learning Outcome 20.6 Illustrate and explain the meaning of measures of center and variability.

When students gather quantitative data, analysis should attend to three key features that describe the distribution of the variables:

- Shape: symmetric or asymmetric and number of modes
- Center: *mean* as a balance point, and *median* as the middle-ordered value
- Variability: interquartile *range* and *mean absolute deviation* (MAD) (Bargagliotti et al., 2020)

Each of these is explored in this section.

Shape of Data

The *shape of data* includes describing how data are spread out or grouped. Graphs, such as dot plots, illustrate the distributions of numerical data (Kader & Jacobbe, 2013). Figure 20.17 shows four dot plots (line plots), each showing a different shape of the data. Although the dot plot in Figure 20.17(a) is not exactly symmetrical, meaning the left side of the graph doesn't look just like the right side, it is considered symmetric because of the general shape.

Just as dot plots provide information about the distribution of numerical data, bar and circle graphs show the distribution of categorical data. The circle graph focuses more on the relative values of this clustering, whereas the bar graph adds a dimension of quantity with an obvious scale. The choice of which graph to use and how many categories to use in these graphs will result in different pictures of the shape of the data.

Part of understanding the shape of data is being aware of how spread out or clustered the data are. In the early grades, this can be discussed informally by looking at almost any graph. For numerical data, there are statistics that tell us how data are spread or dispersed. The simplest of these is the range. In high school, students will learn about the standard deviation

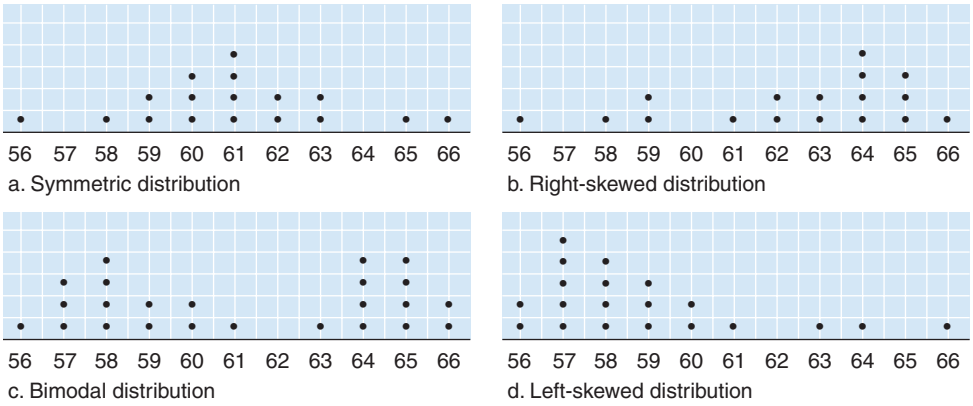

56 57 58 59 60 61 62 63 64 65 66
a. Symmetric distribution
56 57 58 59 60 61 62 63 64 65 66
b. Right-skewed distribution
56 57 58 59 60 61 62 63 64 65 66
c. Bimodal distribution
56 57 58 59 60 61 62 63 64 65 66
d. Left-skewed distribution

FIGURE 20.17 Dot plots showing different distributions (shape) of data

statistic, which is also a measure of spread. At the middle school level, a simple graphical technique called the *box plot* is designed to give us visual information about the spread of data.

A measurement that describes the shape of the data is the mode, the most common or most frequent outcome for a variable. The mode is a useful summary statistic for categorical data (i.e., the category with the most data points). If two categories are tied for the most, both categories are the mode, and the distribution is called bi-modal (Bargagliotti et al., 2020).

Measures of Center

The term *average* is heard quite frequently in everyday usage. Sometimes it refers to an exact arithmetic average, as in "the average daily rainfall." Sometimes it is used quite loosely, as in "She is about average height." In either situation, an average is a single number or measure that describes a set of numbers.

An excellent way to introduce measures of center is to provide a data set with a relevant context and ask students to make decisions that will informally involve ideas of measures of center. Figure 20.18 illustrates such a task implemented with fourth graders and certainly is appropriate across grades 4–8 (Hodges, Johnson, & Roy, 2017). As the authors explain, they did not use statistics terminology, but simply engaged students in reasoning about the situation.

The two most common measures of center are median and mean.

The *median* is the middle of an ordered set of data. Half of all values lie at or above the median and half at or below. The median is easier to understand and compute and is not affected, as the mean is, by one or two extremely large or extremely small values outside the range of the rest of the data (logically called outliers).

Where Do You Want to Work?

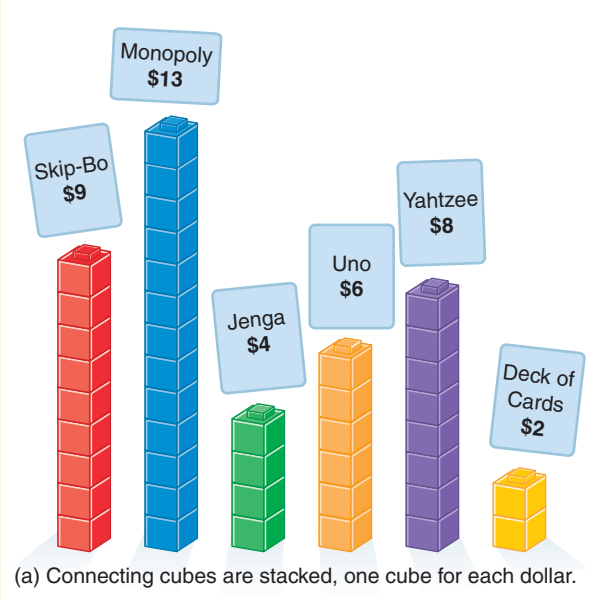
The hourly pay for the employee of three business is included in the table below. If hourly pay was your only consideration, which company would you rather work for?

Acme Business Supplies		Sam'm Materials		Brown's Better Business	
Hourly pay	Peapole	Hourly pay	Peapole	Hourly pay	Peapole
$10	////////	$10	////	$10	//////
$15	////	$15	////	$15	/////
$20	///	$20	//////	$20	//
$40	/	$40	/////	$40	///////
$60	/	$60	/	$60	
$100	///	$100		$100	

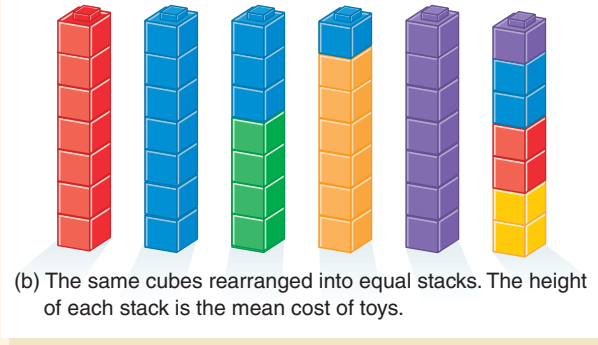
In the space below, explain why your choice is the best.

FIGURE 20.18 Task to use students' intuitive thinking to introduce measures of center (slash marks are counts of people)

Source: T. E. Hodges, M. Johnson, & G. J. Roy (2017). Bridging informal and formal reasoning. *Teaching Children Mathematics, 23*(6), 368–375.



(a) Connecting cubes are stacked, one cube for each dollar.



(b) The same cubes rearranged into equal stacks. The height of each stack is the mean cost of toys.

FIGURE 20.19 Understanding the mean as a leveling of the data

Ask an adult what the mean is, and they are likely to tell you something like, "The mean is when you add up all the numbers in the set and divide the sum by the number of numbers in the set." This is not what the mean is—this is how you calculate the mean. The *mean* can be explained as an equal share and a central balance point.

Understanding the Mean

Equal shares is the first way to develop understanding of the mean (Level A), and then a balance-point interpretation (Level B).

Mean as an Equal Share. Suppose that you hear that the mean hours to study each day in a week is 5. Although study time actually varied, this means that the total amounts of time were combined and the hours were redistributed to have equal time each day. That is called the equal-share approach, which can be illustrated with cubes, as described in the next activity.

Activity 20.13 Mean Cost of Games

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Post a copy of the Games Costs activity page. Have students make a bar graph of the data, using connecting cubes (one cube per dollar). Choose a situation with 5 or 6 values. For example, Figure 20.19(a) shows cube stacks for the price of each game. The task for students is to use the stacks of cubes (bars) to determine what the price would be if all of the games were the same price. Encourage students to use various techniques to rearrange the cubes to level the prices, or make the price the same for each item (see Figure 20.19(b)). Be sure that multilingual learners understand the meaning of leveling the stacks of cubes.

Pearson eText Activity Page: Games Costs

Explain to students that the size of the leveled bars (fair shares) is the mean of the data—the amount that each item would cost if all items cost the same amount but the total of the prices remained fixed. Follow Mean Cost of Games with the next activity to help students develop an algorithm for finding the mean.

Activity 20.14 The Mean Foot

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Pose the following question: "What is the mean length of our feet in centimeters?" Have each student cut a strip of paper (cash register paper works great) that matches the length of their foot. Students record their names and the length of their feet in centimeters on the strips. Begin with groups of four. In each group, have the students tape their foot strips end to end. The task for each group is to come up with a method of finding the mean without using any of the lengths written on the strips. They can only use the combined paper strip. Each group will share their method with the class. From this work, they will devise a method for determining the mean for the whole class. For multilingual students, build background for *foot* (which is not being used as a measurement unit, but as an object) and *mean* (which means something different in mathematics).

Pause & Reflect

Before reading on, what is a method that the students could use in The Mean Foot? ●

To distribute the centimeters for each student's foot equally among the members of the group, they can fold the strip into equal parts so that there are as many sections as students in the group (e.g., fold into fourths for a group of 4). Then they can measure the length of any one part.

How can you find the mean for the whole class? Suppose there are 24 students in the class. Using the strips already taped together, make one very long strip for the whole class. It is not reasonable to fold this long strip into 24 equal sections. But if you wanted to know how long the resulting strip would be, how could that be done? The total length of the strip is the sum of the lengths of the 24 individual foot strips. To find the length of one section if the strip were folded into 24 parts, simply divide by 24. In fact, students can mark off "mean feet" along the strip. There should be close to 24 equal-length feet. This process concretely and dramatically illustrates the algorithm for finding the mean.

Mean as a Balance Point. Statisticians think about the mean as a point on a number line where the data on either side of the point are balanced. To help think about the mean in this way, it is useful to think about the data placed on a dot plot. Then, students attend to (1) how many data values are on either side of the mean *and* (2) the distances of the data values from the mean.

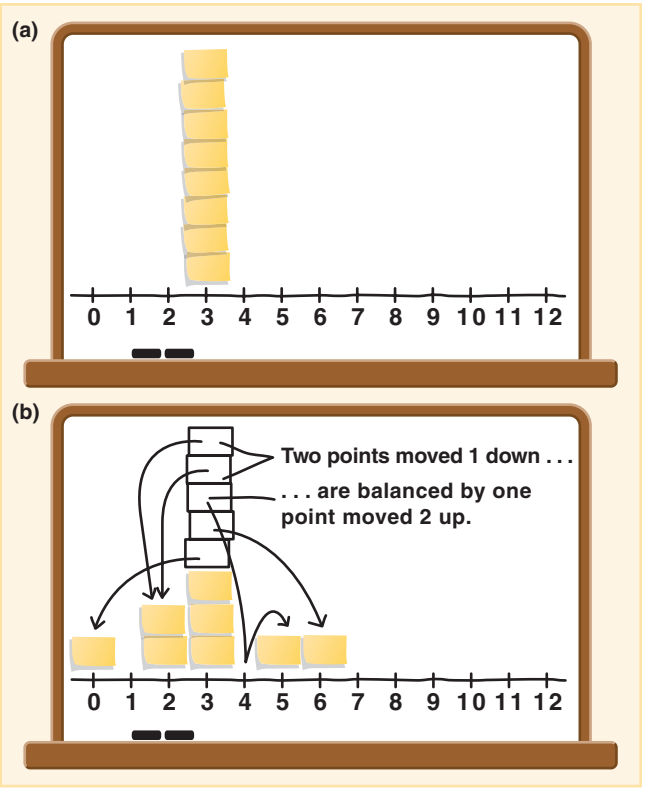


FIGURE 20.20 Starting with all data points the same, sticky notes can be moved to different distributions and still maintain the balance point

To illustrate, draw a number line on the board and arrange 8 sticky notes above the number 3, as shown in Figure 20.20(a). Each sticky note represents one family. The notes are positioned on the line to indicate how many pets the family owns. Stacked like this indicates that all families have the same number of pets. The mean is 3 pets. But different families are likely to have different numbers of pets. We could think of 8 families with a range of numbers of pets. Some may have 0 pets, and some may have 10 pets or even more. Ask students, "How could you change the number of pets for these 8 families so that the mean remains at 3?" Students will suggest moving the sticky notes in opposite directions, probably in pairs. This will result in a symmetrical arrangement. But what if one of the families has 8 pets, a move of 5 spaces from the 3? This might be balanced by moving two families to the left, one 3 spaces to the 0, and one 2 spaces to the 1 (to balance the 5 from the other side). Figure 20.20(b) shows one way the families could be rearranged to maintain a mean of 3. To help students reason about the balancing of the mean, ask students, "Can you find at least two other distributions of the families, each having a mean of 3?"

Going the other direction, students can move sticky notes from actual data points on a dot plot to figure out the mean, as described in Activity 20.15.

Activity 20.15 Balance Point Sticky Notes

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Use the Balance Point activity page, or just have students draw a number line from 0 to 13 with about 2 inches between the numbers. Use six small sticky notes to represent the prices of six games, as shown in Figure 20.21. Have them place a light pencil mark on the line where they think the mean might be. Ask students to determine the mean by moving the sticky notes in toward the center. That is, the students are to find out what price (point on the number line) balances out the six prices. For each move of a sticky note one space to the left, a different sticky must be moved one space to the right. Eventually, all sticky notes should be stacked above the same number, the balance point or mean.

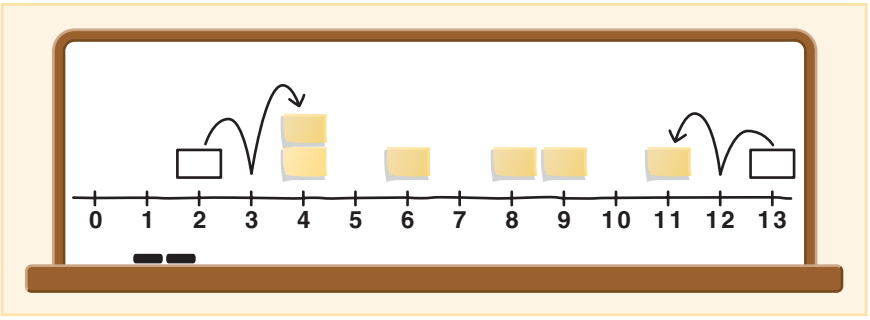


FIGURE 20.21 Sticky notes can be relocated to maintain balance point

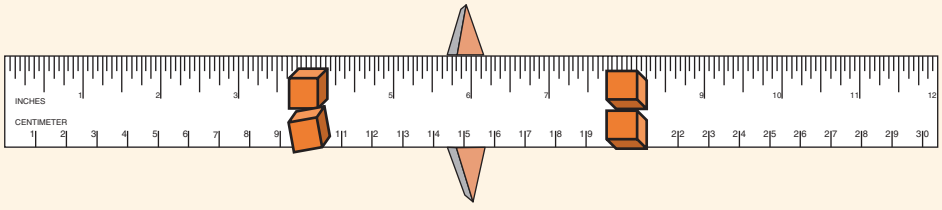
The balance approach to finding the mean clearly illustrates that different data distributions can have the same mean. Especially for small sets of data, the mean is significantly affected by extreme values. For example, suppose another game with a price of $20 is added to the six we have been using in the examples.

A very concrete way to illustrate balance is on a number balance that is adapted to have values between 0 and 20 (see Figure 20.22) (Peters, Bennett, Young, & Watkins, 2016). Notice that as one plastic piece (data point) is placed on the right, one or more other data points must be placed on the left. Using this model, students instantly see if there is a balance because the balance otherwise physically tilts to one side. If you don't have a number balance, a ruler and cubes can be used in a similar way.

Activity 20.16 Balancing Cubes on a Ruler

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Give students a ruler, a block such as a pattern block piece, and cubes, such as connecting cubes. Have students balance the ruler on the pattern block. Notice that the 6-inch mark of the ruler is at the center. Explain that students are going to be creating data sets with a mean of 6 (O'Dell, 2012). Ask students to place 4 cubes on the ruler so that there is a balance point (mean) of 6. Students might, for example, place a cube on 4 (2 away from the mean of 6) and then one on the 8 to balance it:



This photo by Unknown Author is licensed under CC BY-NC.

You can increase the challenge by having students do the following: Use only one data point on each side, use exactly 5 cubes, add one cube that keeps the balance, move two cubes to maintain the balance, and place cubes with a wide distribution or with a narrow distribution.

Pearson eText Activity Page: Balance Point

Pause & Reflect

Stop and try this exercise yourself. What do you notice about how you can move the values? ●

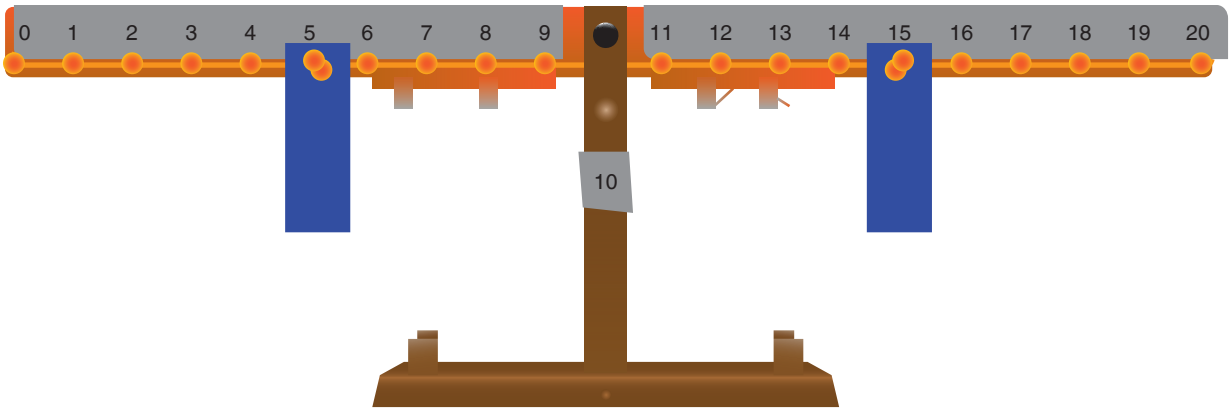


FIGURE 20.22 Adapt a number balance to illustrate the balance interpretation of the mean

Choosing a Measure of Center

The median and the mean can be quite different for a set of data, especially when the data set is skewed left, skewed right, or has outliers (Groth, Kent, & Hitch, 2015/2016). Therefore, it is important to consider the context and data to decide how best to represent the center. For example, in reporting home prices, the median is a better measure—the median is *resistant* to outliers (very expensive homes lead to a higher mean). Activities 20.17–20.19 provide ideas that help students compare and select the best measures of center, beginning with revisiting the game costs.

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Activity 20.17 Average Cost of More Games

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Distribute copies of the Average Game Costs activity page and post the Games Costs activity page. Ask students to predict how the mean and median will change if a new game, costing $20, is part of the game set. What if the $2 game is removed? Then ask students to figure out the price of a game that was added that increased the mean to $9. For a full lesson, see Expanded Lesson: Playing with Measures of Center.

Pearson eText Activity Page: Average Game Costs

Pearson eText Activity Page: Games Costs

Pearson eText Expanded Lesson: Playing with Measures of Center

Activity 20.18 Which Measure of Center Makes Sense?

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Prepare possible questions to investigate, like the ones listed here:

- How many pencils does a sixth grader have?
- What is the cost of used cars in our area?
- What is the average weight of the pumpkin/apples/books in our class?
- What is the average height of a cereal box?
- What is the average monthly cost of a mobile phone?

The students' task is to decide which measure of center makes the most sense and be able to justify their decision. Also discuss the spread and overall shape of the data gathered (e.g., does the height of the cereal box crowd the center? Is there a lot of variation in the phone bills?).

Having to make decisions by using statistics is an authentic reason to choose a measure of center, the focus of the nex activity.

Activity 20.19 You Be the Judge

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Use the You Be the Judge activity page. The gymnastics coach can only send one person to the all-star state competition. She wants to select the student with the best average and most consistency this season. The following table gives overall scores for the eight most recent competitions. Whom should she pick?

Meet	Jenna	Miah	Leah
1	9	9	5
2	3	9	6
3	10	7	7
4	9	8	6
5	7	7	9
6	5	9	8
7	10	9	10
8	9	8	10

Ask students questions such as the following: Which measure of center seems to be the fairest way to judge who goes to the competition? What variability do you notice for each person? Which person would you pick and why?

Pearson eText Activity Page: You Be the Judge

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Although the mean (or the median) provides a representative value of the data, this measure of center is not sufficient for fully describing a distribution (because data distributions with the same value for the mean can be considerably different). This understanding provides an entry point to discuss variability, which is discussed later in this section, with students.

FORMATIVE ASSESSMENT Notes. Consider using a diagnostic interview to assess whether students can determine the best measure of center to use in a given situation, such as the average height of students in the class. You can begin with general questions such as these: What does the mean represent? What does the median represent? What is the difference between the mean and the median? What is each useful for? Then move to more analytical questions: Which should we use for this set of data? Might we use a different measure of center in another class? When you've found the average height of the students in our class, is it possible that no one is that height? Why? ■

***TECHNOLOGY* Note.** Data can be found online and used for comparison with class data. For example, CNN recently reported that the average youth owns seven pairs of jeans. Mooney and Bair (2011) used this data to compare to data gathered by their class, which varied from the reported average, and discussed what might cause the deviation. Also, students can efficiently investigate how the mean and median are affected by each piece of data, using the Desmos calculator and built-in mean and median functions. (See the how-to video at https://youtu.be/upFprrTmCK4.) ■

Variability

Variability is a focus in all four steps of the statistical problem-solving process, as illustrated in Figure 20.1. Increased attention to variability is needed in the study of statistics (Bargagliotti et al., 2020; Kader & Jacobbe, 2013). As noted in the discussion of measures of center, data sets can have the same mean but have very different distributions. Comparing such data sets can help students see the need also to describe the variability in the data in addition to the measures of center.

Instruction on variability logically grows in sophistication (Bargagliotti et al., 2020). In the elementary grades, focus is on variability within a group—for example, the varying lengths of students' names, varying family sizes, and so on. When students create a bar graph of class data and compare the data collected, they are discussing the variability within a group. In middle school, variability within a group continues, but often groups of data are compared. Students might compare the variability of fifth-grade students' favorite music choices with eighth-grade students' music choices.

Variability can be analyzed by looking at data in a table—for example, looking at the frequency of occurrences of categorical data (Kader & Jacobbe, 2013). Figure 20.23 shows the frequency and relative frequency of students' favorite Saturday activity.

Activity	Frequency	Relative Frequency
Play a sport	7	$\frac{7}{28} = .25 = 25\%$
Go to the movies	3	$\frac{3}{28} = .107 = 11\%$
Read	3	$\frac{3}{28} = .107 = 11\%$
Play outside	6	$\frac{6}{28} = .214 = 21\%$
Hang out with friends	4	$\frac{4}{28} = .143 = 14\%$
Play with electronics	5	$\frac{5}{28} = .178 = 18\%$
TOTAL	28	100%

FIGURE 20.23 Frequency and relative frequency describe the variability in the data

Comparing different data sets or playing a game repeatedly and comparing outcomes provides the opportunity for students to analyze the spread of data and think about variation in data (Franklin & Mewborn, 2008; Kader & Mamer, 2008).

One way to help students understand variability is to ask questions about variability in the discussion of data. Friel, O'Conner, and Mamer (2006), using the context of heart rates, suggest the following questions as examples of how to get students to focus on data and variability:

- If the average heart rate for 9- to 11-year-olds is 88 beats per minute, does this mean every student this age has a heart rate of 88 beats per minute? (Note: the real range is actually quite large—from 60 to 110 beats per minute.)
- If we found the heart rate for everyone in our class (of 30), what might the distribution of data look like?
- If another class (of 30) was measured, would their distribution look like the one for our class? What if they just came in from recess?
- Would the distribution of data from 200 students look like the data from our class?

Range. Range is a *measure* of variability. Range of a data set is the difference between the highest and lowest data points. The *interquartile range* of the data is connected to the box plot described earlier. It is the difference between the median values of the lower and upper quartiles (Q3–Q1) and the range of the middle 50 percent of the data. Let's look at an example.

The following data set is the number of hours seventh-grade students spent playing sports or playing outside over the weekend (data has already been placed in order).

0 0 0 1 3 4 4 4 5 5 5 5 6 6 7 8 8 9 10 10

Find the interquartile range. What does the result tell you about the variation in the data set?

In this case, the median is 5 (because the tenth and eleventh values of the 20 values are both 5). This value is also referred to as quartile 2. Quartile 1 (Q1) is the median of the lower half of the data. This median is the average of the fifth and sixth value in the data set, which is 3.5. Quartile 3 (Q3) is the median of the upper half of the data set, which in this case falls between 7 and 8, so it is 7.5. The interquartile range is 7.5–3.5, or 4. For hours spent exercising, the interquartile range is fairly small, showing that there is a lot of clustering around the center of the data.

Mean Absolute Deviation. Whereas the range relates to the median, the *mean absolute deviation (MAD)* relates to the mean. It is the mean of how far away each data point is from the mean, telling us the average distance of each data point from the mean for the data set. MAD, therefore, describes how spread out the data are (Kader & Jacobbe, 2013). In other words, a large MAD means that there is a lot of deviation (difference) between data points and the mean, so the data are spread out. In many state standards, MAD is introduced in sixth grade, with the intent to explore it in an informal manner to develop a deeper understanding of variability. Let's use the previous data set to explore the MAD.

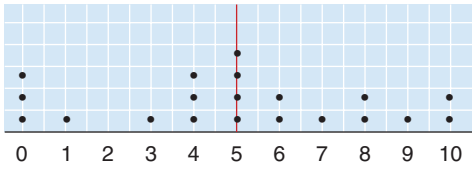


FIGURE 20.24 A dot plot illustrates the data on hours spent exercising over the weekend

Use the following data set to find the mean absolute deviation:

0 0 0 1 3 4 4 4 5 5 5 5 6 6 7 8 8 9 10 10

What does the result tell you about the variation in the data set?

Figure 20.24 places those data in a dot plot. Dot plots can be used to illustrate absolute deviations from the mean (Hudson, 2012/2013). Students draw a vertical line at the mean (in this case 5) and draw or observe how far away each data point is from the mean (absolute deviation).The first step in finding the MAD is to find the deviation (difference) of each data point from the mean. Figure 20.25(a) illustrates these differences in a dot plot. The *absolute* deviation is the distance from the mean, which means the positive difference (see Figure 20.25[b]). Finally, the mean of the absolute deviation is the mean of all these differences (see Figure 20.25[c])—in this case, 2.4. Did you notice that you started with the end of the phrase "mean absolute deviation" by first finding the D, then finding the AD, and, finally, the MAD? Pointing this out can help students focus on the meaning of what they are doing and why.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

In context, this value indicates that the average distance from the mean hours of exercise is around two and a half hours, so not much variability. A good way to have students focus on variability is to have two data sets, one that will have a very small absolute mean deviation and one that will have a large one. Select two sets that have the same mean to emphasize the importance of finding a measure of variation.

(a) Deviation (from the mean)

					0					
–5				–1	0					
–5				–1	0	1		3		5
–5	–4		–2	–1	0	1	2	3	4	5
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

(b) Absolute deviation (from the mean)

					0					
5				1	0					
5				1	0	1		3		5
5	4		2	1	0	1	2	3	4	5
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

(c) Mean absolute deviation (from the mean)

$$\frac{5+5+5+4+2+1+1+1+0+0+0+0+1+1+2+3+3+4+5+5}{20} = \frac{48}{20} = 2.4$$

FIGURE 20.25 Dot plots illustrate the deviation from the mean for each data point of hours spent exercising over the weekend

Just like with measures of center, measures of variability can be resistant (or not) to outliers. For example, the interquartile range (box plots) is resistant to outliers, which is why it often is used in conjunction with the median to describe skewed distributions or distributions with outliers. The mean absolute deviation is not resistant to outliers, which is why it typically is used with the mean to describe symmetric, typically mound-shaped distributions.

With graphs and/or statistics created, students analyze the data, asking questions such as:

- What is the shape of the distribution?
- What are the median and/or mean of the data?
- How do the data vary?

And students interrogate the analysis, for example, asking:

- Is this graph the best way to answer the investigative question?
- Is the mean or median the most sensible measure of center for this data? (Arnold & Franklin, 2021)

Pearson eText
Video Example 20.6
Consider how the essential understandings for measures of center and variability, described by author Karen Karp, inform the teaching of these topics.

Interpreting Results

Learning Outcome 20.7 **Compose questions that help students interpret data.**

Interpreting results is the fourth step in the statistical problem-solving process. Graphs convey factual information (e.g., there is a wide variability in the number of songs sixth-grade students listen to in one day) and provide opportunities to make inferences that are not directly observable in the graph (e.g., most sixth-grade students listen to between 20 and 30 songs a day). Consider these three levels of reading comprehension for graphs (Friel, Curcio, & Bright, 2001):

1. Reading the data (literal)
2. Reading between the data (making comparisons, observing relationships)
3. Reading beyond the data (making inferences)

The difference between *actual facts* and the *inferences that go beyond the data* is an important idea in data analysis. Students' conceptual ability to analyze data and draw conclusions and interpretations is often limited (Tarr & Shaughnessy, 2007); discussing, analyzing, and interpreting data are ways to support this higher-level thinking skill. Students can examine graphs found in print resources or online and discuss the facts in the graphs and the message that may have been intended by the person who made the graph. Activity 20.20 engages students in designing their own data analysis and then interpreting their results to make an accurate estimate of how many words are on the page of a book (Wilburne & Kulbacki, 2014).

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

Activity 20.20 How Many Words on the Page?

CCSS-M: 6.SP.A.1; 6.SP.A.2; 6.SP.B.5

Select a book that is very popular with students. Project a picture of a page where most of the page is covered up, showing only about 7 to 10 lines of text. Ask students to use their statistical reasoning skills to explore this problem and determine how many words are on this page. Ask students to model the statistical process by producing data displays and then interpreting their results. Explain that in the end, they will have to argue for their result, providing statistical evidence and explaining what methods they used to determine their answer. As they are working, and as they present their solutions, ask questions that fall across the three levels of data analysis.

Interpreting data questions should not just focus on the specific question under investigation, but also focus on key ideas of statistics, such as variability, center of the data, and the shape of the data. During interpretation, students might want to create a different data display to get a different look at the data or gather data from a different population to see whether their results are representative.

Different researchers have recommended questions that focus on statistical thinking (Bargagliotti et al., 2020; Friel, O'Conner, & Mamer, 2006; Russell, 2006; Shaughnessy, 2006). Figure 20.26 offers ideas from their lists to get you started on having meaningful discussions about interpreting data. These question prompts apply across many data displays, and they should be a major focus of your instruction. Consider this focus part of the *after* phase of your lesson, although some of these questions will be integrated in the *during* phase as well.

- What do the numbers (symbols) tell us about our class (or other population)?
- If we gathered the same kind of data from another class (population), what might those data look like? If we asked a larger group, what might those data look like?
- How do the numbers in this graph (population) compare to this graph (population)?
- Where are the data clustering? How much of these data are in the cluster? How much are not in the cluster? About what percent is or is not in the cluster?
- What kinds of variability measures might need to be considered in interpreting these data?
- Would the results be different if . . . [change of sample/population or setting]? (Example: Would gathered data on word length in a third-grade book be different from a fifth-grade book? Would a science book give different results from a reading book?)
- How strong is the association between two variables (scatter plot)? How do you know? What does that mean if you know *x*? If you know *y*?
- What does the graph not tell us? What might we infer?
- What new questions arise from these data?
- What is the maker of the graph trying to tell us?

FIGURE 20.26 Interpreting data questions

RESOURCES FOR CHAPTER 20

WRITING TO LEARN

1. How is statistics different from mathematics?
2. Data should be collected to answer questions. What are some examples of questions that students might explore with data at the K–2 level? 3–5? 6–8?
3. How might you engage students in learning to collect data to answer a statistical question?
4. What kinds of graphs are appropriate for categorical data? For numerical data?
5. What do these terms mean: *shape of data*, *bivariate data*, *variability*?
6. Give an example of a problem you might pose to students that would help them explore and compare measures of center.
7. Describe two ways to develop the concept of mean. How can each be developed? Which approach leads to the method of computing the mean?
8. Describe what makes a good question to analyze results.

LITERATURE CONNECTIONS

Literature is full of situations in which things must be sorted, compared, or measured. "Books of Lists" provide fruitful beginnings for data explorations. So many award-winning books can launch a discussion that generates statistical questions. Here is just a beginning list:

We Are Water Protectors

Lindstrom, 2020

This winner of the 2021 Caldecott Medal features Indigenous peoples as they seek to safeguard the earth's water from harm and corruption. A beautiful book that can lead to statistical investigations related to water use and other environmental issues, as well as explorations around culture.

What the Road Said

Wade, 2021

This is a *New York Times*-bestselling book. The child in the story asks the road where it goes, and the road replies, "Be a leader and find out." Besides being just a beautiful story about leadership, this can lead to generating statistical questions like career hopes, favorite places to travel, student-generated examples of leading, and more!

200% of Nothing: An Eye-Opening Tour through the Twists and Turns of Math Abuse and Innumeracy

Dewdney (1993)

This classic book makes points that are never more true than today. As cited in GAISE II, a "healthy skepticism" is part of statistical literacy and means that students ask questions about the data and those collecting, storing, and/or presenting it (Bargagliotti et al., 2020, p. 67). This book describes real ways that "statistics are turned" to mislead people and, thus, is an excellent resource for showing how important it is to be statistically literate. (See Bay-Williams & Martinie, 2009, for more ways to use this book.)

Frog and Toad Are Friends

Lobel (1970)

In this particular classic tale, Frog loses a button. As they search for the button, they find many buttons. Whenever Frog's friends ask, "Is this your button?" Frog responds (with a touch of frustration), "No, that is not my button! That button is _____, but my button was _____." A great fit to introduce sorting activities. Students can model the story directly with sets of buttons, shells, attribute blocks, Woozle cards (see the Woozle Cards activity page), or other objects that have different attributes.

Pearson eText Activity Page: Woozle Cards

RECOMMENDED READINGS

Articles

Hodges, T. E., Johnson, M., & Roy, G. J. (2017). Bridging informal and formal reasoning. *Teaching Children Mathematics*, *23*(6), 368–375.

Through implementing an engaging task, Where Do You Want to Work?, this article shows how to employ informal reasoning to introduce measures of center.

Peters, S. A., Bennett, V. M., Young, M., & Watkins, J. D. (2016). A fair and balanced approach to the mean. *Mathematics Teaching in the Middle School*, *21*(6), 364–372.

Five engaging, high-quality tasks, sequenced developmentally, are described to help students understand the mean as fair shares and as a balance point. Activity pages are provided.

Franklin, C. A., & Mewborn, D. S. (2008). Statistics in the elementary grades: Exploring distribution of data. *Teaching Children Mathematics*, *15*(1), 10–16.

Through excellent examples of activities, including questions and data displays, this article illustrates the process of doing statistics and effective statistics instruction.

Books

Bargagliotti et al. (2020). *PreK-12 guidelines for assessment and instruction in statistics education II* (GAISE II). ASA.

This revised framework provides many excellent examples for teaching statistics, including great tasks to use with students in pre-K–8, and high-quality question prompts.

Curcio, F. (2010). *Developing data-graph comprehension in grades K–8*. NCTM.

This NCTM book shares 30 graphing activities on exploring, investigating, reasoning, and communicating about data. Each activity includes procedures, discussion questions, writing and reading prompts, and ways to use technology.

Kader, G. D., & Jacobbe, T. (2013). *Developing essential understanding of statistics for teaching mathematics in grades 6–8*. NCTM.

This excellent resource helps teachers better understand statistics and provides numerous high-quality tasks to implement in the classroom.

CHAPTER

21

Exploring Concepts of Probability

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **21.1** Describe the probability continuum, including examples from impossible to certain.
- **21.2** Contrast theoretical probability and experiments, including how to integrate both into instruction to better develop a strong understanding of probability.
- **21.3** Illustrate and explain strategies for determining sample space for compound events in a developmental manner.
- **21.4** Explain what a simulation is and how to set up such experiences.

References to probability are all around us: The weather forecaster predicts a 60 percent chance of snow, medical researchers predict that people with certain diets have a high chance of heart disease, investors calculate the risks of specific investments, and so on. Simulations of complex situations are frequently based on probabilities and then used in making decisions about such things as airplane safety under different weather circumstances, highway traffic patterns after new housing has been built, and disaster plans.

Probability is about quantifying randomness; thus, probability is used to make predictions in statistics (ASA, 2020). Students begin to develop probability concepts qualitatively, using words like "This is likely to happen," and then integrate this understanding with quantitative probabilities (a probability of 0.8, for example). Concepts of probability develop as students consider and discuss the outcomes of a wide variety of probabilistic situations. The emphasis should be on exploration rather than on rules and formal definitions.

BIG IDEAS

- Probability is about quantifying randomness; it is based on two foundational ideas: variability (discussed in Chapter 20) and expectation (prediction of what will happen).
- Probability that an event will occur is on a continuum from impossible (0) to certain (1).
- For some events, the exact probability can be determined by analyzing the *sample space* (possible outcomes). A probability determined in this manner is called a *theoretical probability*.

- The relative frequency of outcomes (e.g., from *experiments* or *simulations*) can be used as an estimate of the probability of an event. The larger the number of trials, the more accurate the estimate will be.
- *Simulation* is a technique used for answering real-world questions or making decisions in complex situations in which an element of chance is involved; a model is designed to have the same probabilities as the real situation to explore how likely an event is.

Introducing Probability

Learning Outcome 21.1 Describe the probability continuum, including examples from impossible to certain.

Probability is a ratio that compares the desired outcomes to the total possible outcomes. Probability is closely connected with statistics. Consider these two problems from *GAISE II* (ASA, 2020, p. 10):

> Problem 1: Assume a die is fair. If a die is rolled 10 times, how many times will we see an even number on the top face?
>
> Problem 2: You pick up a die. Is this a fair die? That is, does each face have an equal chance of appearing?

Problem 1 is a probability problem; problem 2 is a statistics problem that uses the probability determined in Problem 1 to seek a solution. Probability does not appear in many state standards until seventh grade. Notions of chance, however, should be developed early in school as students play games and consider events that are likely or not likely (ASA, 2020; Groth, Bergner, & Austin, 2020).

Exploring probability begins with intuitive ideas of whether something is likely to occur (a qualitative statement). Then students begin to quantify probability, connecting the idea of impossible with 0 and certain with 1, and all other probabilities in increments between 0 and 1. But a strict qualitative-to-quantitative progression does not support students' emerging probability vocabulary and related concepts (Groth, Bergner, & Austin, 2020). Hence, the activities here are ordered developmentally, but it is important to continue to emphasize language even when focusing on quantitative probabilities.

Likely or Not Likely

Probability is about how likely an event is. Investigating probability can begin with learning the meanings of qualitative terms such as *possible*, *likely*, *certain*, and *impossible* as a bridge to quantifying these ideas (Confrey et al., 2012; Jones, Langrall, & Mooney, 2007; NCTM, 2000). A good place to begin is with a focus on possible and not possible (Activity 21.1) and then expand to include impossible, possible, and certain (Activity 21.2). Asking students first to predict what they think might happen and then explore through experiments (e.g., games) supports students' emerging ideas about probability (English & Watson, 2016). Such questioning, when students predict first, applies to all the activities throughout this chapter. Start by discussing the meaning of *impossible* and *certain*. These experiences can be woven into discussions across the curriculum, as in Activity 21.1, or connected to science and social studies, as in Activity 21.2.

Activity 21.1 Events in Lyrics: Possible or Not Possible?

CCSS-M: 7.SP.C.5

Create a table, labeling one column "Impossible" and the other "Possible," or download the Events in Lyrics activity page. Use literature or poems that include either possible or impossible occurrences. Nursery rhymes, such as "Hey, Diddle, Diddle," can be fun even for middle school students as they debate whether something in the rhyme is possible. Songs and raps are also engaging. Record each statement in the appropriate column. After sharing some examples, ask students to find more in their favorite songs or poems. Repeat the activity by having them sort the possible into likely or not likely. This is also an opportunity to have multilingual learners share a song or poem from their country or family.

Pearson eText Activity Page: Events in Lyrics

Activity 21.2 Is It Likely?

CCSS-M: 7.SP.C.5

Give students examples of events that are *certain, impossible*, or *possible*, such as the ones on Activity Page: Is it Likely? event cards. A few examples are:

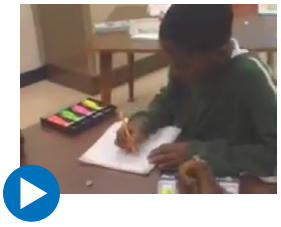
- Drop a rock in water and it will sink.
- A sunflower seed planted today will bloom tomorrow.
- A hurricane/tornado will hit our town.
- In an election, candidate A is elected.
- You will have two birthdays this year.

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Either cut out the event cards in advance or have students cut them out. Have students sort them onto their sorting mats and then add ideas of their own.

This activity is an opportunity to bring in students' identities and cultures. Send a sorting mat home and ask students to work with their families to write down family events that are certain, impossible, or possible. Encourage native language use, as appropriate, for multilingual learners.

Pearson eText Activity Page: Is It Likely? Event Cards
Pearson eText Activity Page: Is It Likely? Sorting Mat



Pearson eText
Video Example 21.1
In this video consider this small group discussion of students' intuitions about what will happen when a coin is flipped ten times and compare their predictions to actually flipping a coin ten times.

After using words to describe probability, students can explore probability concretely, using various random devices (tools) (e.g., spinners, number cubes, coins to toss, colored cubes drawn from a bag). Students make predictions about how likely a particular occurrence is, qualitatively, and then use random devices to experiment and count outcomes, such as tossing a coin. Colored dots can be stuck on the sides of a wooden cube to create different color probabilities, or color tiles (e.g., eight red and two blue) can be placed in opaque bags and students draw a tile from the bag and then return it after each draw.

***TECHNOLOGY* Note.** Probability is based on randomness. Online tools can help generate random values. You can simply search for virtual dice and spinners to see what is new. Possibilities include Nrich Math (https://nrich.maths.org), with dice that can be set to the number of faces and the style; Visnos Demos, with a collection of random spinners (https://www.visnos.com/demos); and Mathigon (mathigon.org), with virtual coins, spinners, and dice. ■

The following dice activities have unequal outcomes. However, students may not initially connect that having more of something means it is more likely. A common initial misconception is that there is a one-in-three chance of each of the values (1, 2, and 3), because each one is possible.

Activity 21.3 1-2-3 How Likely?

CCSS-M: 7.SP.C.5; 7.SP.C.6

Make number cubes with sides labeled as follows: 1, 1, 2, 3, 3, 3. Ask students to predict what number they might get when they roll the cube and record the results in a bar graph. What is likely? What is impossible? Or, more specifically, which row will fill the fastest, or will the rows fill at equal rates? (See the 1-2-3 How Likely? activity page.) Students mark an X in the column for 1, 2, or 3 each time the cube shows that value and stop rolling when one row is full. After they stop, students reflect on how likely each number is.

Pearson eText Activity Page: 1-2-3 How Likely?

Activity 21.4 1-2-3 How Likely Are Sums?

CCSS-M: 7.SP.C.5; 7.SP.C.6; 7.SP.C.8a

This game requires two cubes labeled as in Activity 21.3. It is a more difficult task because it considers the probability of two events (two dice rolls). Students take turns rolling two cubes and recording the sum of the two numbers. Before the game begins, ask students to predict which row will fill the fastest or if the rows will fill at equal rates. Ask students to keep track of their data on the 1-2-3 How Likely Are Sums? activity page. Students roll the cubes until one of the rows is full. Then they discuss with their partner what happened and how likely they now think each number is.

Pearson eText Activity Page: 1-2-3 How Likely Are Sums?

After exploring either 1-2-3 activity, ask, "Which numbers won the most and the least often?" and "If you play again, which number would you pick to win and why?" In Activity 21.4, 1 is impossible, and all outcomes 2 through 6 are possible. A sum of 4 is the most likely. Sums of 2 or 6 are not likely.

Area representations, such as spinners, are more challenging because students cannot count the possible outcomes as readily (Abu-Bakare, 2008). Activity 21.5 uses spinners and counting spins to build this connection.

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Activity 21.5 Race to Ten

CCSS-M: 7.SP.C.6; 7.SP.C.7a, b

Use the Race to Ten activity page. Refer to the spinner and ask: "If we count spins that land on red and ones that land on blue, which color will reach ten first?" Two players take turns spinning the spinner, each time placing an X in the matching column. The activity page includes a spinner that is three-fourths red and one-fourth blue (give students a paperclip to use as a pointer), but this activity can be played with different, more challenging, spinners (see Figure 21.1). Before playing, each student predicts which color will win, red or blue (or gold). Play continues until one color reaches ten. After the activity, discuss which color won and why. Ask students to explain how likely it is for red to win or for blue to win.

Pearson eText Activity Page: Race to Ten

Repeat this activity with a variety of spinners, ones that have two colors with the same area and colors covering different areas, as shown in Figure 21.1. This activity provides an opportunity to explore how likely an event is with an area model, but because 10, the total goal, is a small number of spins, students may have surprising results. This issue is addressed in the "The Law of Large Numbers" section.

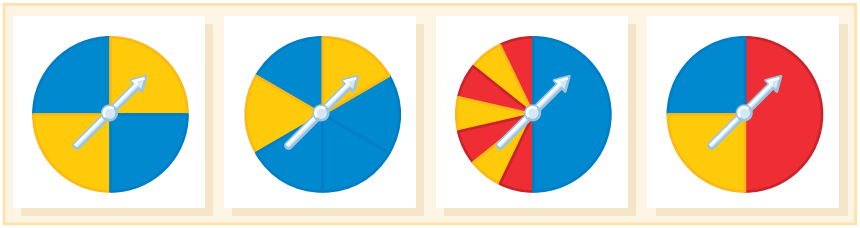


FIGURE 21.1 Possible spinners for Race to Ten

Students do not always see that the first spinner, third spinner, or a spinner partitioned into just two sections (50% blue, 50% red) has the same chance of getting blue (Cohen, 2006; Nicolson, 2005). Therefore, it is important to use spinners that are partitioned in different ways. Spinner faces can easily be made to adjust the chances of different outcomes. Virtual spinners are versatile, allowing you to create multiple sections of different size and can be virtually spun many times. From NCTM's Illuminations, the Adjustable Spinner allows users to run an experiment by setting the number of sections and sizes of each section, entering a number in the number of spins box, and comparing experimental with theoretical probabilities in multiple formats.

FORMATIVE ASSESSMENT Notes. Diagnostic interviews can uncover student misconceptions or preconceptions about the probability of an event. Ask students about the probability of selecting an item from a bag using color tiles or dice (countable objects). For example, ask, "If there are 3 red tiles and 1 blue tile in this bag and I draw one out, what color tile do you think I will get?" and "If I draw four times and put the tile back each time and shake the bag, what do you think I will get?" Ask about the chance of rolling selected outcomes on a die. Some students may think 5 is a more likely outcome on a die than a 2 because 5 is greater than 2. The 2 is not likely compared to the combined possibility of the other choices, but it is *as likely* as any other number (Nicolson, 2005; Watson & Moritz, 2003). Finally, ask about how likely outcomes are in an area representation such as a spinner. These questions will help you know whether you need to focus on counting or area representations, and what kinds of questions or experiences to prepare to help students understand that probability is based on knowing all the possible outcomes and how likely each one is. ■

The Probability Continuum

The number line is an important representation across mathematical concepts, as you have read throughout this book. Probability is no exception. Presenting probability on a number line from 0 (impossible) to 1 (certain) provides a visual representation of how likely an event can be. The number line can be connected to spinners, as illustrated in Figure 21.2. Post the probability continuum in the classroom, where it can be used as a reference for other opportunities to talk about how likely something is. (See the questions in Activity 21.2 for a start to the many things you could ask.) Some things change in their probability; for example, the chance of a snow day could be posted and moved from day to day.

Standards for Mathematical Practice
MP7. Look for structure

The metaphor of a ladder can help students connect probability vocabulary (e.g., certain) to quantitative probabilities (e.g., 1) (Groth et al., 2020; Groth, Butler, & Nelson, 2016). The top rung represents a probability of 1 and the bottom rung represents a probability of 0 (see Figure 21.2). These researchers found that the metaphor of a ladder increased students' use of probability vocabulary and understanding of the interrelatedness of the vocabulary to the numerical values. Notice that some language refers to a point on the ladder, whereas other language refers to a range. Activity 21.6 engages students in creating their own probability ladder.

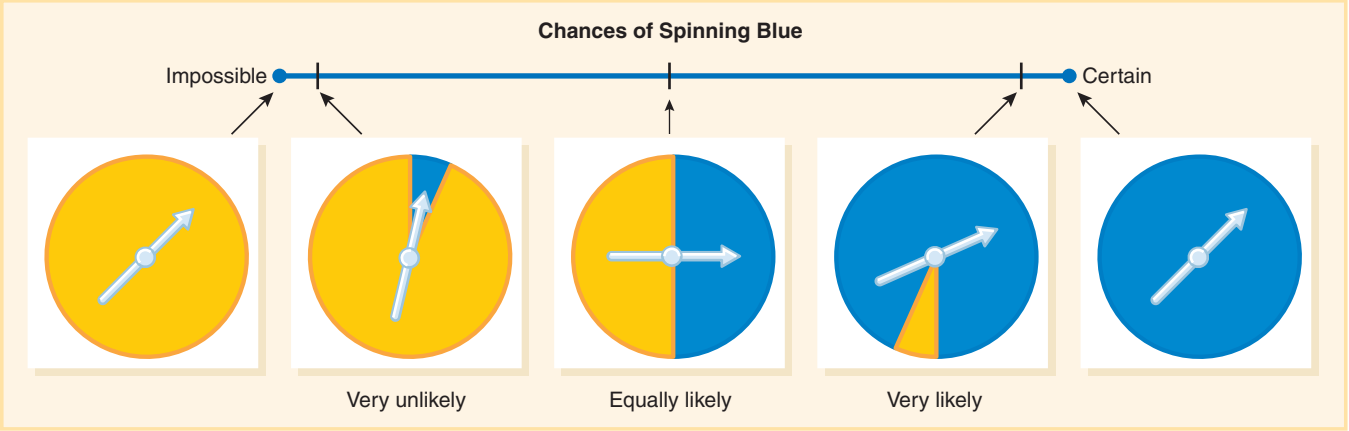


FIGURE 21.2 The probability continuum illustrated on a number line and with related spinners

Activity 21.6 The Likely Ladder

CCSS-M: 7.SP.C.5

Distribute The Likely Ladder activity page, or simply ask students to draw a probability ladder with seven rungs. Students can work individually and later pair and compare, or they can begin with a partner or small group. Have them record words next to the ladder on their activity page, with the most likely words or phrases at the top rung and the least likely words or phrases on the bottom rung. (Their answers will be something like Figure 21.3, though the ranges will vary.)

Another option is to have students cut out and place corresponding spinners on the ladder (provided on page 2 of the activity page). These pictorial images can help multilingual learners with the vocabulary and help all students visualize probabilities. Or, as a challenge, have students sketch five of their own spinners and place them on the ladder.

Pearson eText Activity Page: The Likely Ladder

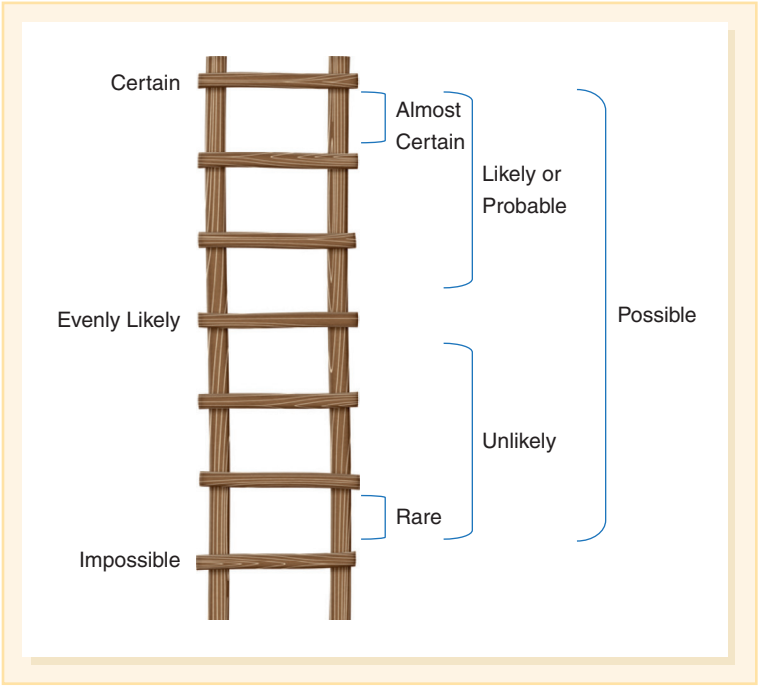


FIGURE 21.3 A ladder as a metaphor for probability language
To the left are words that represent exact values (0, $\frac{1}{2}$, and 1); to the right are words that represent an *approximate* range of values.

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

Beyond area models (spinners), students can explore the probability continuum with counters. Activity 21.7 engages students in conjecturing about how likely an event is, experimenting and comparing their predictions with experimental outcomes as they continue to explore and refine their conjectures. Activity 21.8 extends Activity 21.7 exploring mystery bags.

Activity 21.7 Design and Test Bags

CCSS-M: 7.SP.C.6

Provide each pair or group of students with a copy of the Design a Bag activity page and give each group a value on the probability continuum (e.g., $\frac{1}{3}$, $\frac{3}{4}$, $\frac{1}{6}$). Ask each group to choose a designated color for their tiles so that the probability of selecting that color is the probability they have been assigned. Once they have colored the tiles on the activity page to match their fraction (e.g., $\frac{1}{3}$ are red), the students trade papers. With the new activity page, students do the following:

1. Mark the probability line at the point between impossible and certain that they think matches the number of tiles colored red.
2. Place colored tiles into brown paper bags based on what is colored on the Design a Bag they received. They draw tiles from this bag (with replacement) 50 times. Remind students to shake their bag each time to ensure random sampling.
3. Determine the fraction of times they have drawn a red tile.
4. Return the papers to the group that colored the Design a Bag and find out what the original probability was. (They will have to decide if their fraction is close to the assigned fraction.) For example, is $\frac{17}{30}$ close to $\frac{1}{3}$?

At the end of the activity, have students explain (on the back of the Activity Page or in their journals) how they decided where to place their mark on the probability line. Multilingual learners and students with special needs in math can benefit from sentence starters, such as, "In the bag we received, there were _____ red and _____ blue tiles. We first thought _____. After we did our experiment, we thought _____. We picked this probability because _____."

Pearson eText Activity Page: Design and Test Bags

Activity 21.8 Mystery Bags

CCSS-M: 7.SP.C.6; 7.SP.C.7a,b

Ask each original group to select a probability (e.g., $\frac{1}{4}$, $\frac{1}{6}$, $\frac{3}{4}$, $\frac{3}{8}$) and design a collection of items to place in an opaque bag, using 24 tiles (or cubes) of two different colors, such that $\frac{1}{6}$ of the tiles are red if they selected the probability of $\frac{1}{6}$. Students need to give their bag a name so they can get it back! On a secret card, students record the probability they selected and tuck it away. Groups trade bags. The new group cannot look in the bag. The second group conducts 10 draws (with replacement) and on paper writes the name of the bag, a prediction for how many tiles they think are red, and what they think the probability is of drawing a red. Trade bags again. This time, each group conducts 30 draws (with replacement) and again records the name of their bags and their prediction for how many tiles they think are red and what they think the probability is of drawing a red. Return bags to original owners. During share time, each group holds up their bag, hears the prediction from the group that drew 10 times and from the group that drew 30 times, and then reveals the answer. As a whole class, discuss the connection between probability and number of counters, and how many draws are needed to make a good prediction of what is in the bag.

As students are engaged in exploring probability, using numerical values, continue to reinforce the qualitative probability language. For example, ask, "What words would you use to describe how likely it is to draw a red tile?"

Theoretical Probability and Experiments

Learning Outcome 21.2 **Contrast theoretical probability and experiments, including how to integrate both into instruction to better develop a strong understanding of probability.**

The *probability* of an event is a measure of the chance of that event occurring (ASA, 2020). How do you measure chance of an event? There are two ways: theoretical probability and relative frequency estimates.

Theoretical probability is a term used to describe a probability that can be determined mathematically by analyzing the *sample space* (the set of all possible outcomes for that experiment). Thus, theoretical probability is used when there is a known sample space of equally likely outcomes (e.g., flipping a fair coin or rolling a die). Theoretical probability is calculated by determining the proportion of favorable outcomes compared to the total sample space:

Standards for Mathematical Practice
MP6. Model with mathematics

$$\frac{favorable\ outcomes}{sample\ space} \text{ or } \frac{outcomes\ for\ the\ event}{total\ possible\ outcomes}$$

Oftentimes, the fraction is translated into a decimal or percent.

Example: What is the probability of rolling a value less than three with a fair die?

In this case, the sample space is all outcomes on a die: 1, 2, 3, 4, 5, 6. The favorable outcomes are 1, 2.

The theoretical probability is $\frac{2}{6}$ or about 33% chance of rolling a die and getting a number less than 3.

Now consider these questions and how the probability might be determined.

What is the chance of rain?
What is the probability that Chris Paul will make a free throw?

In these situations, there is no known sample space with equally likely outcomes. Therefore, data on how often it has rained under similar conditions, or previous free-throw records, must be used (Colgan, 2006; Nicolson, 2005). *Relative frequency estimates* are based on data collected through an experiment or simulation related to the topic being analyzed. These estimates are explored through experiments or simulations, and as these experiments are repeated a large number of times, the estimate shows less variability and approaches what might be considered the true probability (Konold et al., 2011). Although this type of probability is less common in the school curriculum, it is most applicable to fields that use probability and therefore important to include in your teaching (ASA, 2020).

Process for Teaching Probability

Probability investigations are an excellent fit to the problem-based *before*, *during*, *after* lesson plan model. In the *before* phase, students make predictions of what they think will be likely; in the *during* phase, students experiment to explore how likely the event is; and in the *after* phase, students compile and analyze the experimental results to determine more accurately how likely the event is. English and Watson (2016) describe three instructional phases, which they found were effective in increasing elementary students' understanding of probability:

1. Concrete exploration: Students make predictions about what they think will happen and then engage in hands-on experiments.

Pearson eText
Video Example 21.2
In this video consider how this experiments build background for theoretical probability.

2. Representation: Students organize and represent the data gathered in phase 1, selecting their own ways to summarize the data, which might include lists, tally charts, tables, dot plots, bar and circle graphs, or pictorial representations.
3. Construction: Students analyze their representations and construct a model describing the probability. This is presented symbolically and diagrammatically.

Notice that this follows the concrete, semi-concrete, abstract (CSA) approach described in Chapter 6, which makes the learning comprehensible and accessible to all students, particularly students with special needs in math. Phase 2 brings in the representations discussed in the previous chapter on statistics. Importantly, statistics and probability are closely related and should be connected. Using data displays helps make sense of how likely an outcome is. (Statistical displays support understanding of probability.) Conversely, analyzing data displays to notice how likely an outcome is makes the creation of data displays more relevant. (Probability supports understanding of statistics.)

Theoretical Probability

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

An excellent concrete exploration to develop an understanding of theoretical probability is to explore fair and unfair games. In the following activities, the results of the game will likely be contrary to students' intuitive ideas. This in turn will provide a real reason to analyze the game in a logical manner and find out why things happened as they did—theoretical probability.

Activity 21.9 Fair or Unfair Coin Games

CCSS-M: 7.SP.C.6; 7.SP.C.7a

For each of these games, ask students first to predict whether they believe the game is fair and then play the game, considering whether the game is fair or unfair (and why). You can play these games within one lesson or on different days.

Coin Game 1: Three students form a group and are given two like coins (e.g., two pennies). For each flip of both coins, one player gets a point, based on the following rules:

Player A: Two heads; Player B: Two tails; Player C: One of each

The game is over after 20 tosses. The player who has the most points wins. Have students play the game two or three times.

Coin Game 2: Partners compete in this game, flipping two like coins (e.g., two pennies). For each flip of both coins, one player gets a point, based on the following rules:

Player A: Same face (e.g., both heads); Player B: One of each

After playing the game, ask students to create data displays of their group's data. Pool the class data and examine their results. Two-coin experiments lend themselves to many representations (e.g., tables, lists, various versions of tree diagrams), and these representations help students understand theoretical probability (English & Watson, 2016). When the full class has played the game several times, conduct a discussion on the fairness of each game. Challenge students to make an argument based on the data and game rules as to whether the game is fair. For multilingual learners, discuss the meaning of *fair* prior to beginning the game and review the term when asking students to create an argument. As another challenge, have students design their own fair game (see Expanded Lesson: Design a Fair Game for details).

Expanded Lesson: Design a Fair Game

A common analysis of Coin Game 1 might go something like this:

At first, students think that because there are three outcomes—two heads, two tails, or one head and one tail—that each has an equal chance, so the game should be fair. However, after playing, students find that player C (the player who gets a point for a mixed result) appears to have an unfair advantage (especially if they have played several games or the class has pooled its data). This observation counters the notion that the outcomes are equally likely, so students begin to discuss why it is that player C has a better chance of winning.

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

When students from fifth through eleventh grades performed a similar two-coin task and were asked for the probability of the same three choices, about 25 percent of the students incorrectly said the probability was $\frac{1}{3}$ (Rubel, 2006, 2007). In fact, there is a 50–50 chance of getting one of each, because two of the four events in the sample space are one of each: HH, TT, HT, TH.

In Coin Game 2, students may initially incorrectly think that player B has an advantage, thinking that different results are more likely than same results (referred to as the *representativeness* misconception (Kustos & Zelkowski, 2013)). Through playing the game and analyzing results, using different representations (e.g., an organized list or tree diagram), students realize this is a fair game (Degner, 2015).

To help students connect how likely an event is to the possible outcomes, encourage students to analyze the situation and generate all the possible outcomes—for example, use a table such as the one in Figure 21.4. Getting a head and a tail happens in two out of the four possible outcomes. Figure 21.5 provides an example of a student's correct explanation for Fair or Unfair. This theoretical probability is based on a logical analysis of the experiment, not on experimental results.

First Coin	Second Coin
Head	Head
Head	Tail
Tail	Head
Tail	Tail

FIGURE 21.4 Four possible outcomes of flipping two coins

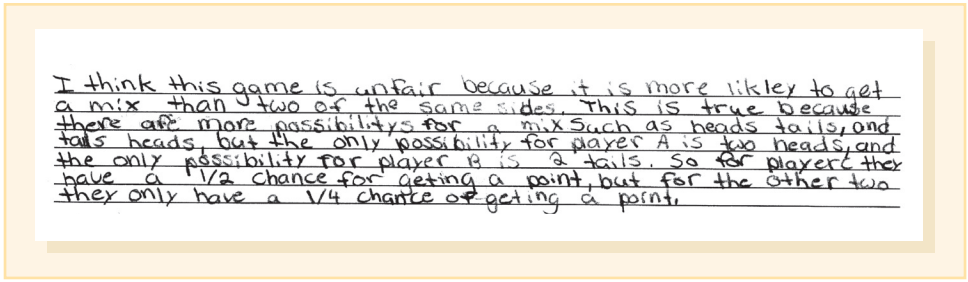

I think this game is unfair because it is more likley to get a mix than two of the same sides. This is true because there are more possibilitys for a mix such as heads tails, and tails heads, but the only possibility for player A is two heads, and the only possibility for player B is 2 tails. So for player C they have a 1/2 chance for geting a point, but for the other two they only have a 1/4 chanke of geting a point.

FIGURE 21.5 Student reasoning for Fair or Unfair? that connects outcomes to probability

Activity 21.10 is based on the TED-Ed lesson, "The Last Banana: A Thought Experiment in Probability" (Barichello, 2015), which you can find on YouTube and in Sullivan (2020).

Activity 21.10 The Last Banana Contest

CCSS-M: 7.G.B.4; 7.SP.C.6; 7.SP.C.7b

Use The Last Banana activity page, or share the background for the game:

Two hungry students stay after school. They go to the cafeteria and find only one banana. Rather than share and still both be hungry, they decide to compete for it by rolling dice. Here are the rules to win the banana:

- If the largest number rolled is a 1, 2, 3, or 4, Student A wins.
- If the largest number rolled is a 5 or 6, Student B wins.

Ask students to predict who has the better chance of winning the game and then give students two dice (preferably in different colors because using two different-colored dice helps student notice that 3 and 4 are two different outcomes (Sullivan, 2020)). Have them compete 50 times for the last banana, keeping track of the number of wins for each partner. Discuss with the class what happened in each group. Compile class results and ask students to determine possible outcomes (sample space) and justify who is more likely to win (or are they equally likely?).

Pearson eText Activity Page: Last Banana

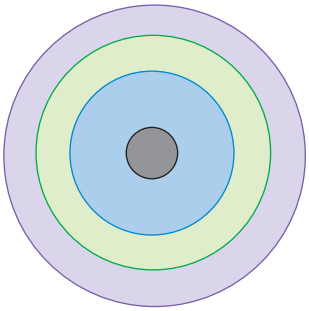
Rock-paper-scissors is another fun context for exploring fair games and possible outcomes. It can be played in the normal way or adapted so that "same" scores 1 point for one player and "different" scores 1 point for the other player. Challenge students to determine whether this version is a fair game (Ellis, Yeh, & Stump, 2007–2008).

Area models support probability concepts and help students analyze theoretical probabilities (Lischka & Stephens, 2020; Lovin, 2020). In many state standards, students explore both the area of circles and probability in grade 7. The following activity is an excellent way to integrate these two important ideas.

Activity 21.11 Chance of Hitting the Target?

CCSS-M: 7.G.B.4; 7.SP.C.6; 7.SP.C.7b

Project a target such as the one illustrated here with concentric circles having radii of 2 inches, 6 inches, 8 inches, and 10 inches, each region shaded a different color. Ask students to determine the fraction and percent of each colored region in the circle.



Ask students to discuss what the probability would be for landing on the center (assuming all throws land on the circle and are thrown randomly). Ask students to discuss why data may or may not match the percentage of the area that is covered (e.g., people with good aim will be able to hit the smaller areas more often), and then have students propose what point values they would assign to each region. Students may assign values in various ways. For example, they may think the skinny outer circle is harder to land on and give it more points than other sections, even though the area of that region may be more. Allow students time to share their reasoning and critique others' ways of assigning points.

Dartboards can be made in many ways, not just in the traditional way (Williams & Bruels, 2011). For example, by using a 10×10 grid, any combination of shapes can be drawn in such a way that the area can be calculated. Students can determine the area of various regions and use the areas to determine the probability of different outcomes, an excellent connection between area measurement and probability.

Pearson eText Blackline Master: 10×10 Grid

Experiments

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Experiments are worthwhile whether or not the sample space can be determined theoretically. Actually conducting experiments and examining outcomes in teaching probability are important in helping students address false assumptions about probability and build a deeper understanding for why certain things are more likely than others.

Specifically, experiments:

- Provide a connection to counting strategies (i.e., lists, tree diagrams) to increase confidence that the probability is accurate.
- Provide an experiential background for examining the theoretical model. (When you begin to sense that the probability of two heads is $\frac{1}{4}$ instead of $\frac{1}{3}$ through experiments, the analysis in Figure 21.5 seems more reasonable.)
- Help students see how the ratio of a particular outcome to the total number of trials begins to converge to a fixed number. (For an infinite number of trials, the relative frequency and theoretical probability would be the same.)
- Help students learn more than students who do not engage in doing experiments (Gurbuz, Erdem, Catlioglu, & Birgin, 2010).

But when probabilities cannot be determined theoretically, using data is the only option to determine relative frequency estimates. These data may be preexisting or need to be established through an experiment, with a sufficiently large number of trials conducted to become confident that the resulting relative frequency is an approximation of the theoretical probability. Activity 21.12 provides engaging situations in which doing an experiment and using the results of a large number of trials is the only way to establish how likely an outcome is.

Activity 21.12 Drop It!

CCSS-M: 7.SP.C.7b

In this activity, students predict how likely a landing position will be. Then, they gather small-group and class data to examine relative frequencies. Here are a few ideas to try:

1. *Falling Kisses.* This experiment involves dropping Hershey's Kisses to determine the probability of two identified outcomes—landing on its base or on its side. Give each student a kiss. Alternative foods include Hershey's Rolo Caramels or, for a healthier option, consider fish crackers (the direction in which the fish faces). (See also Gallego, Saldamando, Tapia-Beltran, Williams, & Hoopingarner, 2009.)
2. *Cup Toss.* This time, students determine the possible options (upside down, right side up, or on the side), predict, and then conduct the experiment. Each pair of students needs a cup to toss, and they must determine a uniform way to toss the cup.
3. *Toy Animal Drop.* Bring in small plastic animals that can land in different ways. When students have different animals, there can be more variability in their own tallies and the class data. (See Nelson & Williams, 2009 or Young, 2016, both which provide a series of engaging activities with tossing toy pigs, and include activity pages.)

In each of these activities, the relative frequency of the combined data should approximate the actual probability. Students can compare their individual/small-group data relative frequency to the class relative frequency. Ask, "Are the relative frequencies (more or less) the same? If yes, why? If no, why not?" Discuss with students what might lead to different relative frequencies of these dropping experiments.

Pearson eText Activity Page: Falling Kisses
Pearson eText Activity Page: Cup Toss
Pearson eText Activity Page: Toy Animal Drop

In these experiments, there is no practical way to determine the results before you start. However, once you have results for 200 tosses (empirical data), you will feel more confident in predicting the results of the next 100 tosses. After gathering data for 1000 trials, you will feel even more confident. In other words, the more tosses that are made, the more confident you become. For example, in dropping the cup, after 100 or so trials, you may have determined a probability of $\frac{4}{5}$, or 80 percent, for the cup to land on its side.

***TECHNOLOGY* Note.** PBS Learning Media (pbslearningmedia.org) offers many videos about probability! One interesting one is titled, "Probability Risk Analysis," wherein a risk analysis underwriter explains how he determines odds. Events such as halftime basketball shooting and hole-in-one golf contests are among those in which the underwriter uses probability experiments, including historical and theoretical probability, to determine the risk one can assume of providing a pay-off for winning such events. ■

The Law of Large Numbers. The phenomenon in which the relative frequency of an event becomes a closer approximation of the actual probability or the theoretical probability as the size of the data set (sample) increases is referred to as the *law of large numbers*. The larger the size of the data set, the more representative the sample is of the population. In thinking about statistics, a survey of 1000 people provides more reliable and convincing data about the larger population than does a survey of five people.

Although critical to understanding probability, this concept may initially be difficult for students to grasp. Students commonly think that a probability should play out in the short term, a false assumption sometimes referred to as the law of small numbers (Flores, 2006; Tarr, Lee, & Rider, 2006). Comparing small data sets to large data sets is one way to help students think more deeply about the fact that the number of trials matters. The next two activities are designed with this purpose in mind.

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

Activity 21.13 Get All 6!

CCSS-M: 7.SP.C.6

Distribute the Get All 6! activity page or ask students to prepare a frequency table with the numbers 1 through 6 at the bottom. Students roll a die and mark an X over each number until they have rolled each number at least once. Have students repeat Get All 6! three times. Discuss how the frequency charts compare in each case. Students will see that in some cases there were many fours, for example, or that it took 25 rolls before all numbers were rolled, whereas in other cases they got all the numbers in only 10 rolls. Next, have students combine their own data and analyze the relative frequency. Finally, gather class data and discuss relative frequencies. Focus discussion on the fact that in the short run, data varies a lot—it is over the long run that the data evens out.

Pearson eText Activity Page: Get All 6!

Truly random events often occur in unexpected groups; a fair coin may turn up heads ten times in a row. A 100-year flood may hit a town twice in 10 years. Using random devices such as spinning spinners, rolling dice, or drawing cubes from a bag (physically or virtually with an app) gives students an intuitive feel for the imperfect distribution of randomness. The next activity is designed to help students with this sometimes difficult idea.

Activity 21.14 What Are the Chances?

CCSS-M: 7.SP.C.6

Distribute the What Are the Chances? activity page. Provide each pair of students with a spinner face that is half red and half blue. Mark the halfway point on the continuum of impossible to certain and draw a vertical line down through all of the lines below this point. Discuss the chances of spinning blue. Then have the students in each pair spin their spinner 10 times, tallying the number of red and blue spins. Mark the number of blue spins on the second line. For example, if there are 3 blue and 7 red spins, place a mark at about 3 on the 0-to-10 number line. If the result of the 10 spins is not exactly 5 and 5, discuss possible reasons this may be so. Repeat this process, but each time, include the results from the previous spins and mark the location on the continuum. There are various options for virtual spinners, including NCTM's adjustable spinner (https://www.nctm.org/adjustablespinner/).

Pearson eText Activity Page: What Are the Chances?

Each of the successive number lines used in What Are the Chances? has the same length and each represents the total number of trials. When the results are plotted on any one number line, the position of the point on the continuum shows the fraction of the total spins as a visual portion of the whole line. With more trials, the marks will get closer and closer to the $\frac{1}{2}$ mark. Note that 240 blue spins out of 500 is 48 percent, or very close to one-half. This is relatively close to an even chance of red or blue, although there are 20 more red spins (260) than blue.

The process of accumulating data in stages can be used for other experiments, for example, the Drop It experiments. Rather than drawing a vertical line before collecting data, decide on the best estimate of the actual probability after the number of trials has become large and then draw the vertical line in the appropriate position. Observe and record on the number lines 10 additional trials, 20 additional trials, and 50 additional trials. Compare these smaller data sets with the larger data set. Write the probabilities as fractions and as percents to show the connection between these representations.

FORMATIVE ASSESSMENT Notes. Pose the following performance assessment to assess students' ideas about long-run results versus short-run results.

Margaret spins the spinner 10 times. Blue turns up on three spins. Red turns up on seven spins. Margaret says that there is a 3-in-10 chance of spinning blue. Carla then spins the same spinner 100 times. Carla records 53 spins to blue and 47 spins to red. Carla says that the chance of spinning blue on this spinner is about even.

Who do you think is more likely to be correct: Margaret or Carla? Explain. Draw a spinner that you think they may have been using.

Look for evidence that students understand that the larger number of spins is more reliable (less subject to variability).

Duane has a lucky coin that he has tossed many, many times. He is sure that it is a fair coin with an even chance of heads or tails. Duane tosses his coin six times, and heads come up six times in a row. Duane is sure that the next toss will be tails because he has never been able to toss heads seven times in a row. What do you think the chances are of Duane tossing heads on the next toss? Explain your answer.

Look for evidence that students understand the idea that each toss of the coin is independent of prior tosses, and that chance has no memory. ■

Use of Technology in Experiments

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Random numbers and random outcomes can be readily generated through various virtual tools. Random numbers can be related to the possible outcomes in an experiment. For example, if the final digit is odd, you can assign it to represent one outcome, and, if it is even, to represent a second outcome. If there are four outcomes, you can look at the remainder when the last two digits are divided by 4 (i.e., the remainder will be 0, 1, 2, or 3) and assign a remainder to each outcome. In addition, handheld graphing calculators, like the TI-84 or the Casio fx-8750GIII, can run the free Probability Simulation app, an interactive tool that simulates tossing coins, rolling number cubes, using spinners, picking cards, and generating random numbers.

Computer interactives can be used to flip coins, spin spinners, or draw numbers virtually from a hat. An advantage of digital simulations is that you can run a large number of trials in a short amount of time. PhET Interactive Simulations (https://phet.colorado.edu/) offers a simulation tool, Plinko Probability, for exploring the probability of a ball dropping through a triangular grid of pegs and into one of multiple containers underneath. Data can be viewed in

a histogram to examine the distribution connected to the containers in which the balls land. NCTM's Illuminations website has a series of lessons, "Probability Explorations," in which students explore probability through virtual experiments and can graph the results. Also, in a virtual world, dice can be loaded and used to challenge students' thinking ("Are these fair dice? How can you find out?") (Beck & Huse, 2007; Phillips-Bey, 2004).

Probability of Compound Events

Learning Outcome 21.3 **Illustrate and explain strategies for determining sample space for compound events in a developmental manner.**

In probability, an *event* is a subset of the sample space. For example, if a bag contains two red, three yellow, and five blue tiles, the sample space (all possible outcomes) consists of all ten tiles, and an event is drawing a yellow tile. Thus, a compound event involves two (or more) actions to determine an outcome. Examples include rolling two dice, drawing two tiles from a bag, and the combination of the occurrence of rain and forgetting your umbrella.

When two-event experiments are explored, there are new issues to consider: Does the occurrence of the event in one stage have an effect on the occurrence of the event in the other? How do I count all the outcomes? Does the order of the events matter? The following sections address independent events; dependent events; and counting, combinations, and permutations.

Independent Events

In Activity 21.9 (Fair and Unfair Coin Games), students explored the results of tossing two coins. Each toss is independent of what happened before and what happens after. Tossing a coin twice is an example of *independent events;* the occurrence or nonoccurrence of one event has no effect on the other.

Suppose that your students gather data on the sums that they get for rolling two dice. The results might be recorded in a dot plot (see Figure 21.6[a]). These compound events (sums) do not appear to be equally likely, and in fact, the sum of 7 appears to be the most likely outcome. To explain this, students might look for the combinations that equal 7: 1 and 6, 2 and 5, and 3 and 4. But there are also three combinations for 6 and for 8. It seems at first as though 6 and 8 should be just as likely as 7, and yet they are not.

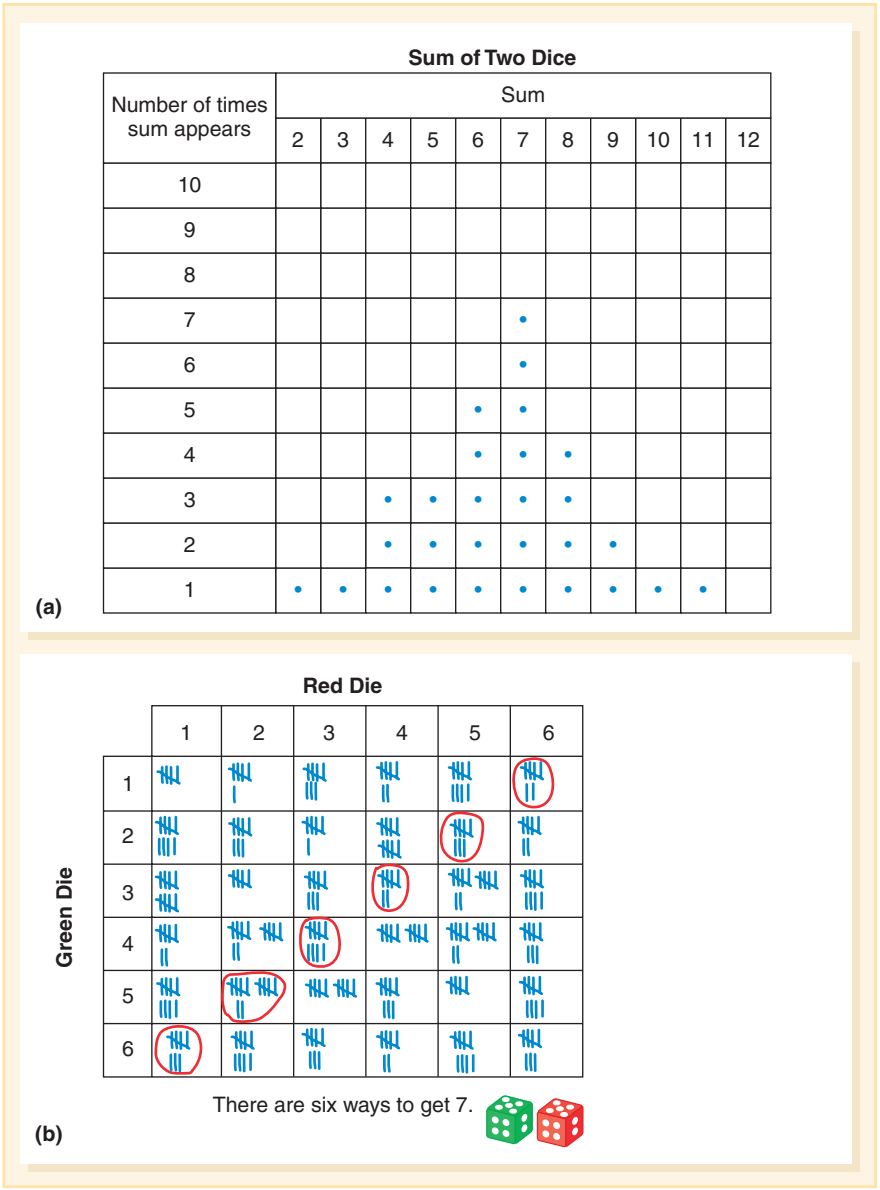
Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Now suppose that the experiment is repeated, but this time students roll two dice of different colors and keep tallies in a matrix (see Figure 21.6[b]). Using class data to get a large data set shows that each cell is equally likely. Compare the sums of 6, 7, and 8, the most common sums. Notice that six cells have a sum of 7 out of a total of the sample space (36), for a probability of $\frac{6}{36}$ or $\frac{1}{6}$. The sums for 6 and for 8 each include a double, and therefore each sum has five outcomes (not 6), for a probability of $\frac{5}{36}$. The color-coded dice can help students see how possibilities are counted, addressing the fact that while $3 + 4$ is the same as $4 + 3$, these are each separate outcomes in rolling dice.

Determining Sample Space for Independent Events. There are several ways to determine the sample space for compound events. One way is to make a list of all possible outcomes and count the number of outcomes that make up the event. This is effective but has some limitations: (1) lists can get tedious when there are many possibilities, (2) students can lose track of which possibilities they have included, and (3) a list implies that all outcomes are equally likely. As you can see in Figure 21.7, a matrix works well when there are two events. Two other important options include area models and tree diagrams.

Standards for Mathematical Practice
MP4. Model with mathematics

Area Models. Area models address the limitations of making lists, and students are familiar with area models from their work with multiplication and fractions. Area models are less abstract than equations or tree diagrams, and they can help students make sense of these other models. In particular, area models help students understand when and why they multiply or add probabilities (Lovin, 2020).



Sum of Two Dice

Number of times sum appears	Sum										
	2	3	4	5	6	7	8	9	10	11	12
10											
9											
8											
7						•					
6						•					
5					•	•					
4					•	•	•				
3			•	•	•	•	•				
2			•	•	•	•	•	•			
1	•	•	•	•	•	•	•	•	•	•	

(a)

Red Die

Green Die	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	4	5	6	7
3	3	4	5	6	7	8
4	4	5	6	7	8	9
5	5	6	7	8	9	10
6	6	7	8	9	10	11

There are six ways to get 7.

(b)

FIGURE 21.6 Exploring the frequencies (dot plot) and possible outcomes (matrix) for the sum of two dice

Words and phrases such as *and*, *or*, *at least*, and *no more than* may cause students some confusion and therefore require explicit attention. Of special note is the word *or*, because its meaning in everyday usage is generally not the same as its strict logical meaning in mathematics. In mathematics, *or* includes the case of both. For example, in this next activity, being born in spring *or* in the year of the dog means either or both could be true. But being born in spring and in the year of the dog means both must be true.

Activity 21.15 Are You a Spring Dog?

CCSS-M: 7.SP.C.8a, b

Before doing this activity, determine which Chinese birth year animals are likely to be represented in the classroom (e.g., the dog and the rooster). Spend some time discussing the Chinese birth year animals with students. Begin by finding out which season each student was born and then ask students what their animal is. Determine what percentage of the class is represented by each animal. Ask, "If I name one of the Chinese birth year animals, what is the probability it will be *your* birth

year animal?" Illustrate this percentage by partitioning a rectangle, as in Figure 21.7(a). (This particular illustration finds that 64 percent of the students in the class were born in the year of the dog, and 36 percent were born in the year of the rooster.) Next, ask, "If I name a season, what is the probability it will be *your* season? Ask students to illustrate their response by partitioning and shading a rectangle. See Figure 21.7(b). Then ask, "What is the probability of being both spring *and* a dog?"

Standards for Mathematical Practice

MP8. Look for and express regularity in repeated reasoning

In Figure 21.7(b), you can visually see that students in the year-of-the-dog and spring groups make up $\frac{1}{4}$ of 64 percent, or 16 percent of the population of this class of students. This should look familiar because the same process is used for multiplying fractions (and whole numbers).

The area model is also effective in solving *or* situations. Consider the question, "What is the probability you were born in summer or fall, *or* that you are a rooster?" This is the probability of A or B, or P(A or B).

The shading for this example is illustrated in Figure 21.7(c). Half of the students are born in summer or fall (P(A) = 50), and 36 percent are born in the year of the rooster (P(B) = 36). Students can add up the percentages in the boxes, or they can think about the two situations separately: 50 percent are born in summer or fall and 36 percent are born in the year of the rooster. The sum of these two results would be 86 percent ((P(A) + P(B)), but some students are "both" and have therefore been double-counted (see overlap in diagram). In this example, the overlap (students falling in "both") is 18 percent (P(A and B)). Therefore, the population that is born in summer or fall *or* born in the year of the rooster is $50 + 36 - 18 = 68$ percent of the population. This generalizes to this formula:

$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$$

A = probability of being born in spring or fall; B = probability of being a rooster

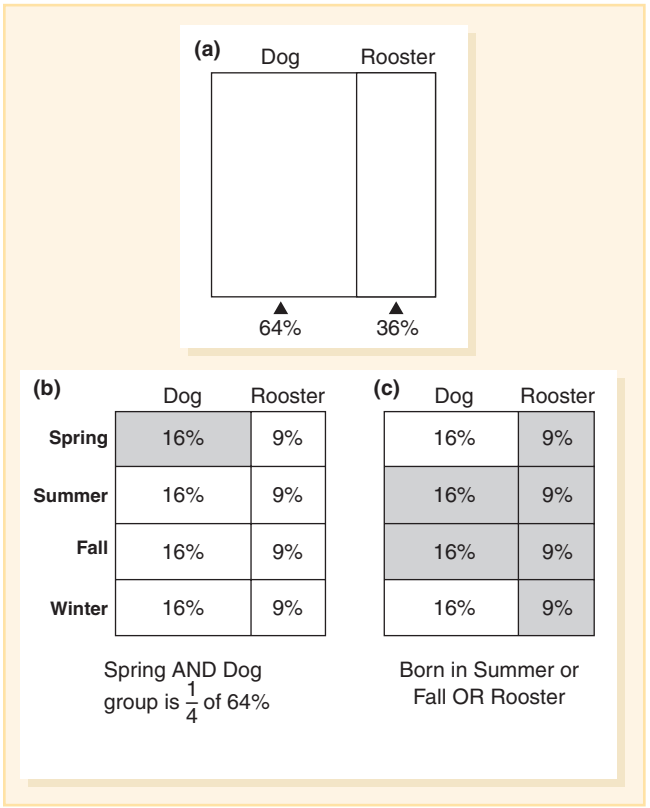


FIGURE 21.7 An area model for determining probabilities

Tree Diagrams. A tree diagram (Figure 21.8) is another method of determining a sample space, and it can be used with any number of events. The tree diagram is more abstract than the area model but efficient for finding probabilities. Connecting a list to a tree diagram can help students make sense of a tree diagram. For example, consider this ice cream-building problem.

At the county fair, cones are made with a waffle cone or a regular cone, ice cream that is dipped or not dipped, and then one of three flavors. If you are handed a cone at random, what is the probability of getting:

- A waffle cone, dipped, with strawberry ice cream?
- A waffle cone with chocolate ice cream?
- Or chocolate, dipped, in a regular cone? [and so on]
- Getting *your* favorite ice cream cone?

Note: The final question is truly open-ended. If the student says their favorite flavor is coffee, then the probability is 0. And they might not care about the cone type, which would also change the probability.

Tree diagrams can help students answer these questions, and simulations can also give students an opportunity to estimate the probability. This problem can be simulated with coins and a spinner, as illustrated in Figure 21.9.

Heads = Waffle Cone	Tails = Regular cone
Heads = Dipped	Tails = Not dipped
Red = strawberry	Green = vanilla; Blue = chocolate

Pause & Reflect

Use a chart or tree diagram to analyze the sum of two number cubes, each with sides 1, 1, 2, 3, 3, and 3. (These were the same cubes used in Activities 21.3 and 21.4.) What is the probability of each sum, 1 through 6? How might lists and tree diagrams support student understanding of sample space and the probability of independent events? ●

Once students have learned to use area models and tree diagrams, pose challenging tasks that allow them to choose representations. As you have seen in earlier activities, a process to help students understand sample space with probability is to (1) make a prediction of the probability of the event; (2) conduct an experiment with a large number of trials, and (3) determine the sample space and see how it compares with the prediction and the results of the experiment. With compound events, games provide an excellent context for such explorations. Here are a few favorites.

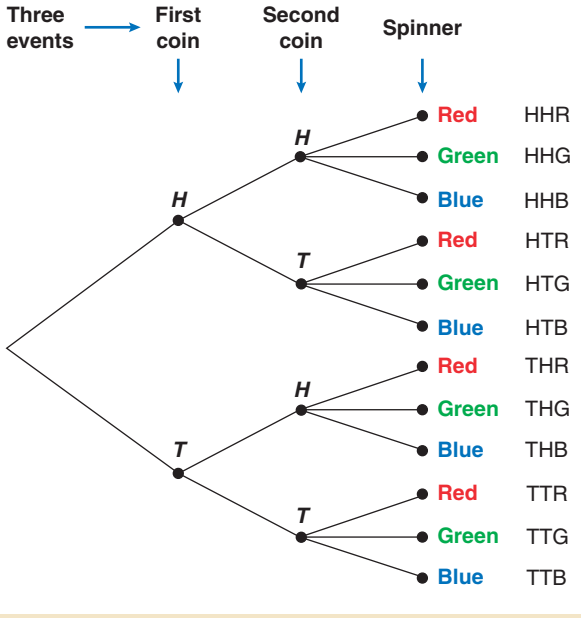


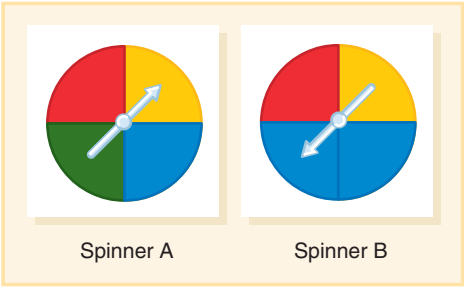
FIGURE 21.8 A tree diagram showing all possible outcomes for two coins and a spinner that is $\frac{1}{3}$ red, $\frac{1}{3}$ green, and $\frac{1}{3}$ blue

Activity 21.16 Best Chance of Purple

CCSS-M: 7.SP.C.7a; 7.SP.C.8a

Distribute the Best Chance of Purple activity page, or simply pose this task:

Samuel, Susan, and Sandu are playing a game. The first one to spin purple wins! Purple means that one spin lands on red and one spin lands on blue for these spinners:



Each person chooses to spin each spinner once or one of the spinners twice. Samuel chooses to spin spinner A twice; Susan chooses to spin spinner B twice; and Sandu chooses to spin each spinner once. Who has the best chance of purple? (Based on Lappan & Even, 1989.)

Have students predict which person they think has the best chance (and explain why), and then ask students to experiment—use a paper clip with the spinners on the activity page, or use a virtual spinner. Once students have experimented, pose questions such as:

- For Sandu's turn (spinner A, then spinner B), would it matter if he spun B first and then A? Why or why not?
- How might you change one spinner so that Susan has the best chance at purple?

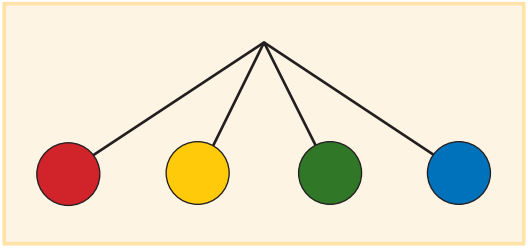
Pearson eText
Video Example 21.3
In this video consider how the students use tree diagrams to determine probabilities of two events (coin toss and pick a color cube where three were yellow and one was green).

Have students create a visual to determine the theoretical probabilities (see example tree diagram and area models in Figure 21.9). As students work, pose questions such as:

- How do the tree diagrams/area models relate to the spinners?
- Which branches on your tree diagram or regions in your area model represent getting purple?
- How could you make tree diagrams or area models for each player's choices?
- How is the area model alike and different from the tree diagram?

Pearson eText Activity Page: **Best Chance of Purple**

Tree Diagram



Area Models

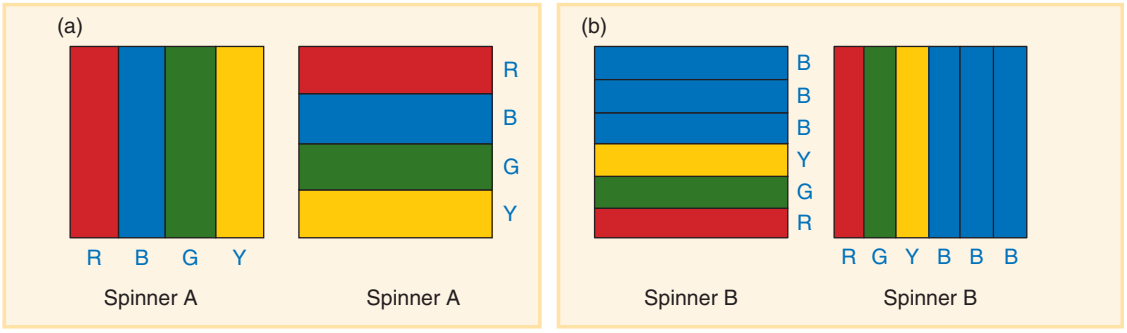


FIGURE 21.9 Tree diagrams and area models for Best Chance of Purple

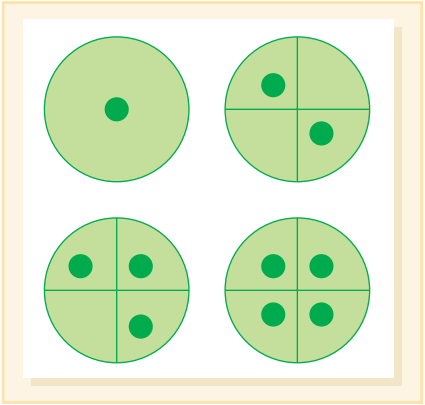
Activity 21.17 Lu-Lu

CCSS-M: 7.SP.C.7a; 7.SP.C.8a

This Hawaiian game involves taking turns tossing four stones and calculating the resulting score. (You can create stones like the ones pictured here by getting glass stones from a craft store and marking dots on one side.)

Player 1 tosses the four stones (marked on only one side). If all 4 are face up, they score 10 and take a second turn, adding 10 to their second sum. If all 4 are not face up, Player 1 re-tosses the face-down stones and adds any face-up stones to their first toss. Repeat for each player. Play several rounds, recording scores each time. High score wins.

After the students have played, ask what they notice about the sums they are getting.



What sums are possible?

What sums are common?

What are all the outcomes (possible combinations of stones)?

What is the probability of each score?

Students from other countries may also have games to share that include probability, and these games can be explored in a similar manner. (See McCoy, Buckner, & Munley, 2007, for more on this game.)

Designing a spinner is a challenging and engaging way for students to think about the probability of independent events (Ely & Cohen, 2010). The following activity can be challenging, and an area representation can be used to help students reason about how to design their spinner.

Activity 21.18 Design a Winning Spinner

CCSS-M: 7.SP.C.6; 7.SP.C.7a; 7.SP.C.8a, b

Explain that each student is going to create a winning spinner, which means that when it is spun twice, the sum will be on a number strip with values 2, 3, 4, 5, 6, 7, and 8. Students create their own spinner, partitioning the area of the circle however they like and writing a number in each sector. Once students have their spinner, they pair with someone else and play the game with their own spinner: Student A spins twice and adds the two values. If the sum is 5, Student A covers 5 on their number strip. Student B takes a turn. The first partner to cover all numbers on their strip wins. Play three rounds. Next, ask students to redesign their spinner, find a new partner, and play three more rounds. If possible, repeat a third time. Afterward, discuss how they designed a winning spinner.

In thinking backward (from the desired outcome to the spinner), students can build a deeper understanding of how to determine the probability of independent events.

The following are additional examples of probabilities of independent events. Any one of these could be explored as part of a full lesson.

- Rolling an even sum with two dice
- Spinning blue twice on a spinner
- Having a tack *or* a cup land up when each is tossed once
- Getting at *least* two heads from tossing four coins
- Rolling two dice and getting a difference that is *no more than* 3

Dependent Events

A *dependent event* is a compound event in which the second event outcome depends on the result of a first event. For example, suppose that there are two identical boxes. One box contains one genuine dollar bill and two counterfeit bills, and the other box contains one genuine and one counterfeit bill (you do not know which box is which). You may choose one box and from that box select one bill without looking. What are your chances of getting a genuine dollar bill? Here there are two events: selecting a box and selecting a bill. The probability of getting a dollar in the second event depends on which box is chosen in the first event. Activity 21.19 involves another dependent situation.

Activity 21.19 Keys to a New Car

CCSS-M: 7.SP.C.8a, b

Use the Keys to a New Car activity page, or pose the following problem: In a game show, you can win a car—if you make it through the maze to the room where you have placed the car key. You can place the keys in either Room A or Room B (see maze in Figure 21.10). At the start and at each fork in the path, you must spin the indicated spinner and follow the path that it points to. Once you've reached Room A or Room B, the game is over—there is no going back through the maze. In which room should you place the key to have the best chance of winning the car?

Pearson eText Activity Page: Keys to a New Car

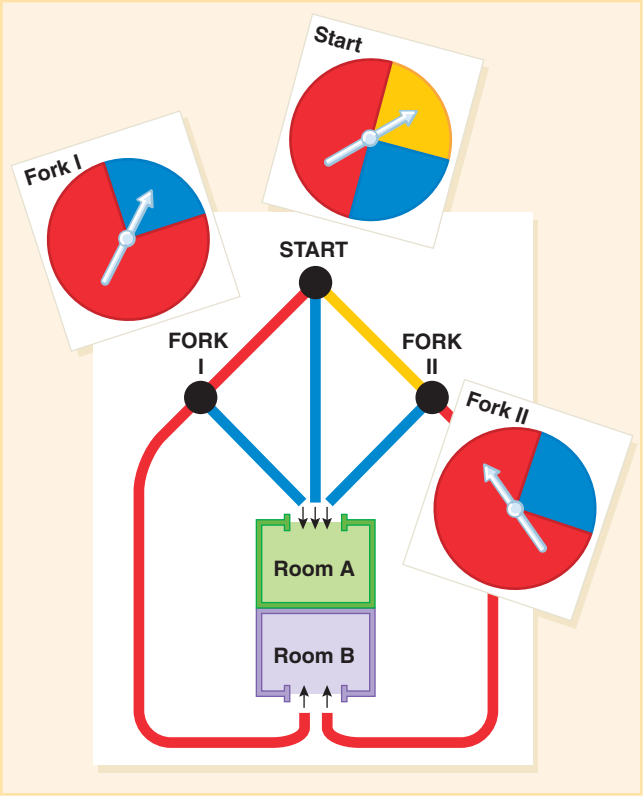


FIGURE 21.10 Should you place your key in Room A or Room B to have the best chance at winning a new car?

Like with independent events, area models and tree diagrams can help students make sense of the sample space and the probability (see Figure 21.11 and Figure 21.12). Having students describe the connection between the area and the tree diagram representations can help build meaning for the tree diagram approach, which can be used in any multiple-events probability task.

Pause & Reflect

How would the area representation for the car problem be different if the spinner at Forks I and II were $\frac{1}{3}$ A and $\frac{2}{3}$ B spinners? What questions like this one can you ask students to help them think about dependent events? ●

A classic task involving dependent probability is the Monty Hall Problem from the game show, *Let's Make a Deal*. At the end of the show, the final contestant is shown three doors, and when they choose a door, they win everything behind it. One of the three doors has a very valuable prize. The contestant guesses one door, but before they see what is behind that door, Monty shows the contestant a goat behind one of the not-selected doors. The contestant can (1) keep their original choice or (2) switch doors. Question: Which is the better option (or does it not matter)? Try to answer this question and then google the task to learn more about the interesting solution.

Counting possibilities includes finding combinations and permutations. Because this is beyond typical middle school mathematics, it is not addressed here, but if you are teaching this topic, the Combinations and Permutations Overview can be downloaded.

Standards for Mathematical Practice
MP4. Model with mathematics

Pearson eText Teacher Resource: Combinations and Permutations

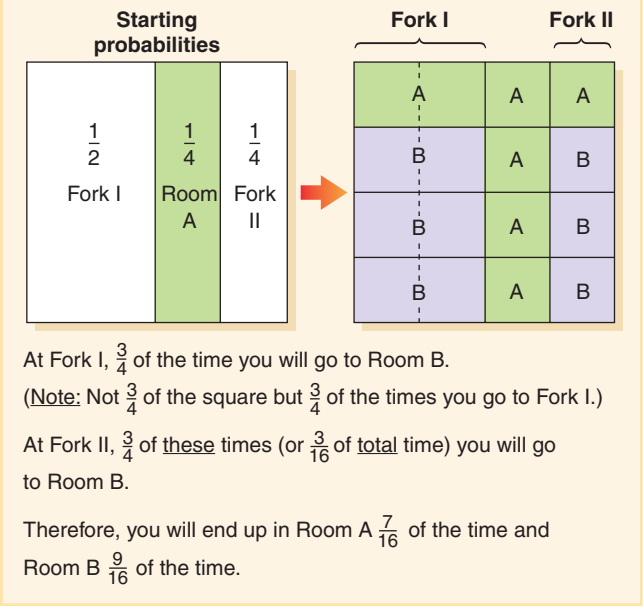

Starting probabilities
Fork I
Fork II
1/2
Fork I
1/4
Room A
1/4
Fork II
A A A
B A B
B A B
B A B
At Fork I, 3/4 of the time you will go to Room B.
(Note: Not 3/4 of the square but 3/4 of the times you go to Fork I.)
At Fork II, 3/4 of these times (or 3/16 of total time) you will go to Room B.
Therefore, you will end up in Room A 7/16 of the time and Room B 9/16 of the time.

FIGURE 21.11 An area model to represent the Keys to a New Car problem

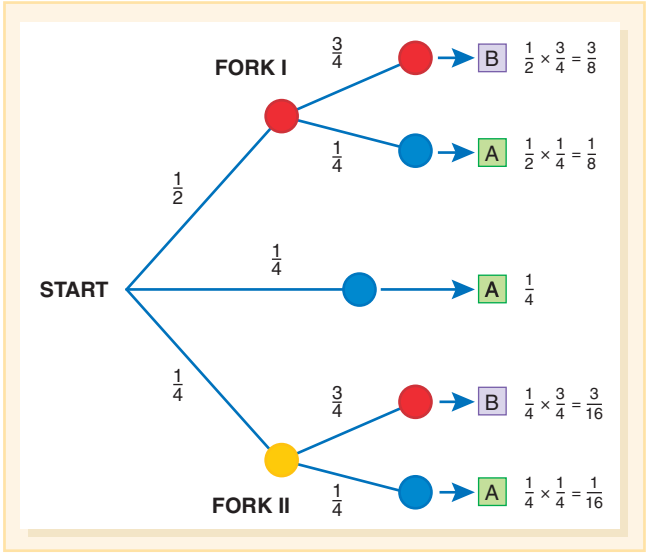


FIGURE 21.12 A tree diagram to represent the Keys to a New Car problem

Simulations

Learning Outcome 21.4 **Explain what a simulation is and how to set up such experiences.**

Simulation is a technique used for answering real-world questions or making decisions in complex situations where an element of chance is involved. Many times, simulations are conducted because it is too dangerous, complex, or expensive to manipulate the real situation. To see what is likely to happen in the real event, a model must be designed that has the same probabilities as the real situation. For example, in the designing of a rocket, a large number of related systems all have some chance of failure that might cause serious problems. Knowing the probability of serious failures will help determine whether redesign or backup systems are required. It is not reasonable to make repeated tests of the actual rocket, because that is very costly. Instead, a model that simulates all of the chance situations is designed and run repeatedly with the help of a computer. The computer model can simulate thousands of flights and generate an estimate of the chance of failure.

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Activity 21.20 Probability of Getting Water

CCSS-M: 7.SP.C.8b, c

Show students an illustration of a water pump system like the one illustrated in Figure 21.13. Explain that the five pumps that connect A and B are aging. It is estimated that at any given time, the probability of pump failure is $\frac{1}{2}$. If a pump fails, water cannot pass that station. For example, if pumps 1, 2, and 5 fail, water flows only through pumps 4 and 3. Ask students to discuss how likely they think it is that water will make it through the system. Have students mark how likely on a probability continuum. Follow the steps for teaching a simulation (described next). After students have completed their simulation, revisit the important probability questions:

- What is the probability that water will flow at any time?
- On the average, about how many stations need repair at any time?

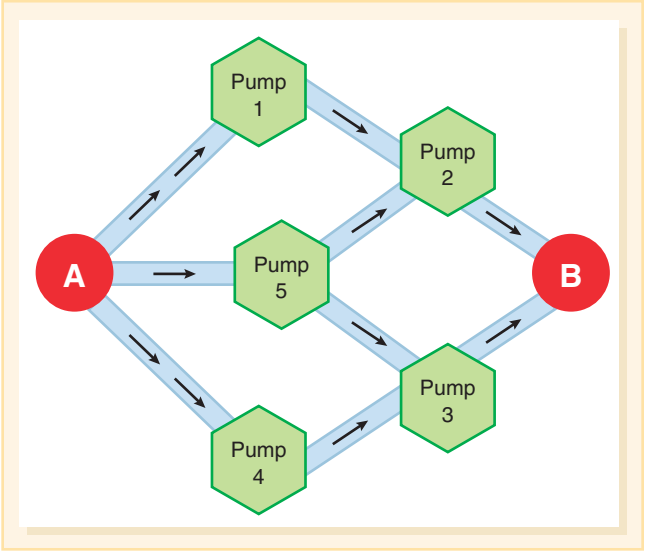


FIGURE 21.13 Each of these five pumps has a 50 percent chance of failure. What is the probability that some path from A to B is working?

***TECHNOLOGY* Note.** In their Classroom Resources, NCTM (nctm.org) offers an exploration titled "Pandemics: How Are Viruses Spread?" It is set up to show that a virus is influenced by four factors. Students can manipulate those variables to create different outcomes. Results are modeled in a logistic curve, an S-shaped curve that describes population growth—of both viruses and people—as well as other phenomena in economics and science. ■

For any simulation, the following steps can serve as a useful guide. Here the steps are explained for use with Activity 21.20.

1. *Identify key components and assumptions of the problem.* The key component in the water problem is the condition of a pump. Each pump is either working or not working. In this problem, the assumption is that the probability that a pump is working is $\frac{1}{2}$.
2. *Select a random device for the key components.* Any random device can be selected that has outcomes with the same probability as those of the key component of the simulation—in this case, the pumps. Here a simple choice might be tossing a coin, with heads representing a working pump and tails representing a pump that has failed. Students may initially

have trouble selecting a random device for simulations. Virtual spinners and dice can often be set up to reflect the number of outcomes you want to represent.

3. *Define a trial.* A *trial* consists of simulating a series of key components until the situation has been completely modeled one time. In this problem, a trial could consist of tossing a coin five times, each toss representing a different pump.
4. *Conduct a large number of trials and record the information.* For this problem, it would be useful to record the number of heads and tails in groups of five, because each set of five is one trial and represents all of the pumps.
5. *Use the data to draw conclusions.* There are four possible paths for the water, each flowing through two of the five pumps. As they are numbered in the drawing, if any one of the pairs 1-2, 5-2, 5-3, and 4-3 are working, it makes no difference whether the other pumps are working. By counting the number of trials in which at least one of these four pairs of coins both come up heads, we can estimate the probability of water flowing through the system. To answer the second question, the number of tails (pumps not working) per trial can be averaged.

Here are a few more examples of problems for which a simulation can be used to gather empirical data.

In a true-or-false test, what is the probability of getting 7 out of 10 questions correct by guessing alone?
Key component: **Answering a question.**
Assumption: **Chance of getting the answer correct is $\frac{1}{2}$.**
Simulation option: **Flip a coin 10 times for one trial.**

Standards for Mathematical Practice
MP4. Model with mathematics

In a group of five people, what is the chance that two were born in the same month?
Key component: **Month of birth.**
Assumption: **All 12 months are equally likely.**
Simulation option: **Use 12 cards or 12-sided dice. Draw/roll one, replace, and draw/roll again.**

Casey's batting average is .350. What is the chance he will go hitless in a complete nine-inning game?
Key component: **Getting a hit.**
Assumptions: **The probability of a hit for each at-bat is .35. Casey will get to bat four times in the average game.**
Simulation option: **Use a spinner that is 35 percent shaded. Spin four times for one trial.**

Krunch-a-Munch cereal packs one of five games in each box. About how many boxes should you expect to buy before you get a complete set of games?
Key component: **Getting one game.**
Assumption: **Each game has a $\frac{1}{5}$ chance.**
Simulation option: **Use a spinner with five equal sections. Spin until all sections occur at least once. Record how many spins it took (this is one trial). Repeat. The average length of a trial answers the question.**
Extension: **What is the chance of getting a set in eight or fewer boxes?**

In this activity, consider how you would design a simulation.

Activity 21.21 Chance of Triplet Girls

CCSS-M: 7.SP.C.8b,c

Ask students, "What is the chance that a woman having triplets will end up with all girls?" Record estimates. Ask students to create a simulation to model this problem, using the five steps previously described. Encourage students to use various random devices to simulate (flipping three coins, using a random number generator, spinning a two-color spinner three times, etc.) the key component. After examining experimental results, ask students to create a sample space, for example, with a list or a tree diagram, to make sense of the results. See also Goodwin and Ortiz (2015) for ideas and activity pages related to this activity.

Whether doing simulations and experiments, or calculating theoretical probability, it is important for students to use many representations (lists, area, tree diagrams) and discuss developing probability conceptions explicitly. In addition to being more interesting, teaching probability in this way allows students to understand important concepts that have many real-world implications. This final activity is designed to apply different representations as well as simulations.

Activity 21.22 Money in Two Piggy Banks

CCSS-M: 7.SP.C.8a,b,c

Use the Money in Two Piggy Banks activity page or post the following problem for students.

Use an area representation and a tree diagram to determine the probability for the following situation:
In a game at the county fair, the game leader puts one $5 bill and three $1 bills in Piggy Bank 1. In Piggy Bank 2, he puts one $5 bill and one $1 bill. To play the game, you get to take one bill from Piggy Bank 1 (without looking) and put it in Piggy Bank 2. After mixing Piggy Bank 2, you get to take one bill from that bank. The game costs $2 to play. Should you spend your money?

Ask students to illustrate the theoretical probability, using (1) an area representation and (2) a tree diagram. This can be done by having some groups work on one, other groups work on the other, and then share and compare their diagrams. Or everyone can create both. After these illustrations are complete, ask students how they might design a simulation to test this game. Invite students to explain their simulation and present it. Ask other students to determine whether the simulation accurately models the problem.

Pearson eText Activity Page: Money in Two Piggy Banks

Student Assumptions Related to Probability

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

Tasks like Chance of Triplet Girls can lead to interesting follow-up questions: "Are three girls less likely or more likely than two girls and a boy?" and "If a family already has two girls, what do you think they will have for their third child?" (Tillema, 2010). These questions are focused on addressing some (incorrect) assumptions students have that are related to probability. These challenges, including two shared earlier, are summarized in Table 21.1.

TABLE 21.1 COMMON FALSE ASSUMPTIONS (INTUITIONS) RELATED TO PROBABILITY

False Assumptions or Intuitions	What It Looks Like	How to Help
1. *Commutativity confusion.* Knowing that $3 + 4$ is the same as $4 + 3$, student thinks that the chance of an outcome such as one boy and two girls is one event, not three events (BGG, GBG, GGB).	Student thinks that the probability of having two girls and a boy is the same as having three girls.	• Focus on sample space: Have students think of all the ways an event can occur. • Discuss this confusion explicitly. • Use different-colored dice in simulations to highlight this issue. • Do experiments and discuss the variability across students' data sets. • Use tree diagrams with lists of possible outcomes (see Figure 21.8 and Figure 21.12).

(*Continued*)

TABLE 21.1 ***(CONTINUED)***

False Assumptions or Intuitions	What It Looks Like	How to Help
2. *Gambler's fallacy.* The notion that what has already happened influences the event.	Student argues if there are two sisters, then a son is more likely. Similarly, if a coin has had a series of four heads, student thinks that tails are more likely for the fifth flip (Ryan & Williams, 2007).	• Point out that a coin has no memory, and the probability of heads or tails is always 50%.
3. *Law of small numbers.* Students expect small samples to be like the greater population (Flores, 2006; Tarr, Lee, & Rider, 2006).	Student thinks that having five tails in a row when flipping a coin should not happen.	• Engage in activities that challenge this expectation, such as Activities 21.13 and 21.14. • Point out that in the case of the coins, it is not so unusual—it is just a very small data set, so it is not likely to resemble the larger population.
4. *Possibility counting.* Students assume that each possibility is equally likely.	Student assumes a spinner that is 75% red and 25% blue has a 50% chance of each color because there are two regions.	• Do experiments to challenge their predictions. • Use Activities 21.3, 21.4, and 21.5, which are designed to help students focus on frequency rather than possible outcomes.
5. *Representativeness.* The (incorrect) idea that outcomes that seem more random are more likely to happen (Kustos & Zelkowski, 2013).	Student thinks that a Pick 3 lottery ticket such as 10, 21, 35 is more likely than 10, 11, 12.	• Use experiments and simulations. • Explicitly discuss representativeness.

RESOURCES FOR CHAPTER 21

WRITING TO LEARN

1. What is the probability continuum and why is it important? What probability vocabulary and quantities belong on the continuum (and where)?
2. What is the relationship between experiments and theoretical probability, and what implications does this have for teaching probability?
3. Explain the law of large numbers. Describe an activity that might help students grasp this idea.
4. Describe the difference between an independent and a dependent event. Give an example of each.
5. What is a simulation? What decisions must be made and what tools are needed to do one?

LITERATURE CONNECTIONS

The books described here offer both fanciful and real-life data for investigating probability. These books can also be paired with activities in this chapter.

A Totally Cool Book about Numbers

Ball (2005)

This wonderful book could be placed in every chapter of this text. About 40 topics are covered, one of which is called "Take a Chance." This two-page spread is full of interesting contexts for probability, including a match-dropping experiment and genetics.

A Very Improbable Story

Einhorn (2008)

Ethan wakes up with a cat on his head. He must do something improbable to remove the cat before his soccer game. This book, written to explore probability, is an enjoyable read for elementary and middle school students.

Do You Wanna Bet? Your Chance to Find Out about Probability

Cushman (2007)

Danny and Brian become involved in everyday situations both in and out of school, each involving an element of probability. For example, two invitations to birthday parties are for the same day. What is the chance that two friends would have the same birthday? Students can create simulations to explore these ideas.

Harry Potter and the Sorcerer's Stone

Rowling (1998)

The game of Quidditch can lend itself to creating a simulation to explore the probability of winning. *Wagner and Lachance (2004)* suggest that sums of two dice be linked to Quidditch actions. For example, a roll of 7 means a player scores

a goal, which is worth 10 points. Rolls of 2 or 12 mean the player catches the Snitch, etc. Students play and then explore the probability of winning.

My Little Sister Ate One Hare

Grossman (1996)

This counting story involves a girl eating one rabbit, two snakes, three ants, and so on, including bats, mice, worms, and lizards. Upon eating ten peas, she throws up everything she ate. With everything now on the floor, the probability of selecting an item (e.g., a worm) can be explored. Students can use cards representing things eaten and approach the task experimentally.

Pigs at Odds

Axelrod (2003)

This is one of a series of Pigs books, this one with a focus on probability through looking at fair games. Mr. Pig tries to win games at the county fair, but the odds seem to be against him winning.

That's a Possibility: A Book about What Might Happen

Goldstone (2013)

With colorful photographs and interactive examples, this book introduces the language and concepts of impossible, possible, probable, and certain.

RECOMMENDED READINGS

Articles

Lim, V., Rubel, L., Shookhoff, L., Sullivan, M., & Williams, S. (2016). The lottery is a mathematics Powerball. *Mathematics Teaching in the Middle School*, *21*(9), 526–532.

This article describes an innovative and high-quality exploration Color Pick that is then used to analyze the lottery. Additional excellent activities help students explore social justice issues through the lens of mathematics.

McCoy, L. P., Buckner, S., & Munley, J. (2007). Probability games from diverse cultures. *Mathematics Teaching in the Middle School*, *12*(7), 394–402.

As the title suggests, this article includes games from African, Hawaiian, Jewish, Mexican, and Native American cultures. Games include probability connections, with handouts, and questions to pose to students.

McMillen, S. (2008). Predictions and probability. *Teaching Children Mathematics*, *14*(8), 454–463.

This article provides a series of high-quality probability lessons—various contexts and representations are used, as well as calculators. The lessons include a number of key concepts discussed in this chapter, and two handouts are provided.

Books

Bright, G. W., Frierson, D., Tarr, J. E., & Thomas, C. (2003). *Navigating through Probability in Grades 6–8*. NCTM.

With explanations, full lesson plans, and ready-to-go, high-quality activities on important concepts of probability, this book is a user-friendly, excellent resource.

Gage, J., & Spiegelhalter, D. (2016). *Teaching Probability*. Cambridge, UK: Cambridge University Press.

This book provides many excellent probability investigations and has a companion website (teachingprobability.org) with downloadable pages to support the activities.

CHAPTER

22

Exploring Exponents, Negative Numbers, and Real Numbers

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **22.1** Describe strategies to develop student understanding of exponents, including order of operations and scientific notation.
- **22.2** Explain meaningful contexts and representations to develop understanding of positive and negative numbers.
- **22.3** Illustrate and explain conceptual approaches to teaching operations with positive and negative numbers.
- **22.4** Explain ways to help students see relationships among different types of numbers (irrational, whole, etc.).

In elementary school, students explore whole numbers and positive fractions and decimals; in middle school, students expand these concepts and learn new ways to represent numbers, including scientific and exponential notation, negative numbers, and irrational numbers. The ideas presented in this chapter build on ideas that have been already developed throughout this book, such as: Exponents are used in algebraic expressions. Scientific notation expands how large and small numbers are represented, building on place-value concepts. Integers move beyond the positive counting numbers to numbers less than 0 and therefore extend the number line (as well as operations) to include negative values.

BIG IDEAS

- Our number system includes whole numbers, fractions, decimals, and integers, all of which are rational numbers. Every rational number can be expressed as a fraction.
- Integers are the negative and positive counting numbers and 0. Positive and negative numbers describe quantities having both magnitude and direction (e.g., temperature above or below zero).

- Exponential notation is a way to express repeated products of the same number. Specifically, powers of 10 express very large and very small numbers in an economical manner.
- Real numbers include rational numbers and irrational numbers. Rational numbers can be expressed as fractions and include various subsets, such as whole numbers. Irrational numbers can be expressed only symbolically or approximately, using a close rational number. Examples include $\sqrt{2} \approx 1.41421\ldots$ and $\pi \approx 3.14159$.

Exponents

Learning Outcome 22.1 **Describe strategies to develop student understanding of exponents, including order of operations and scientific notation.**

Many real contexts require very small or very large numbers. Exponential notation is more efficient for conveying numeric or quantitative information. The learning progression used in many state standards, based on the Common Core State Standards (NGA & CCSSO, 2010), is summarized in Table 22.1.

TABLE 22.1 PROGRESSION OF EXPONENT CONCEPTS IN GRADES 5–8

Grade 5	Grade 6	Grade 8
Use whole-number exponents to denote powers of 10.		Use scientific notation; perform operations with numbers in scientific notation Understand and use square roots and cube roots.
	Write and evaluate numerical expressions involving whole-number exponents. Use order of operations with expressions involving exponents.	Know and apply the properties of integer exponents to generate equivalent numerical expressions. Example: $3^2 \times 3^{-5} = 3^{-3} = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$.

Exponents in Expressions and Equations

Students need to explore exponents with whole numbers before they use exponents with variables. By looking at whole-number exponents, they can notice patterns in solving problems and generate (and understand) the rules of exponents themselves. A whole-number exponent is simply shorthand for repeated multiplication of a number times itself with the exponent governing the number of times—for example, $3^4 = 3 \times 3 \times 3 \times 3$.

Symbols for exponents are abstract and likely new to students. They must understand these foundational ideas:

1. An exponent applies to its immediate base. For example, in the expression $2 + 5^3$, the exponent 3 applies only to the base of 5, so the expression is equal to $2 + (5 \times 5 \times 5)$. However, parentheses can change this. In the expression $(2 + 5)^3$, the 3 is an exponent of the quantity $2 + 5$ and is evaluated as $(2 + 5) \times (2 + 5) \times (2 + 5)$.
2. Exponents are the first operation in the order of operations. Thus, the expression $2 + 5^3$ becomes $2 + 125$, and the answer is 127.

As with any topic, introduce it with what is familiar and concrete. With exponents, this means beginning with exploring powers of 2 and 3—operations that can be represented geometrically.

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

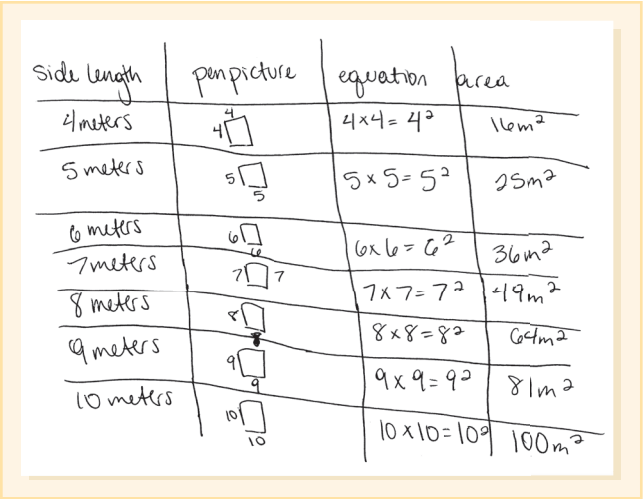


FIGURE 22.1 A student records possibilities for making a square pen

Minia knows that square animal pens are the most economical for the amount of area they provide (assuming straight sides) as compared to the amount of fencing. Can you provide a table for Minia that shows the areas of square pens that have between 4 meters and 10 meters of fence on each side?

Students may set up a table similar to the one in Figure 22.1, showing possible areas for the square pens with different side lengths.

Another way to explore exponents is to explore algebraic growing patterns, involving squares and/or cubes. The classic Painted Cube Problem is one such activity.

Activity 22.1 Painted Cubes

CCSS-M: 5.MD.C.4; 5.MD.C.5; 6.EE.A.2c; 6.GA.4; 7.EE.B4

Distribute the Painted Cubes activity page. The painted cube is made up of centimeter blocks (called blocks here for clarity); each face of each cube is painted. As the cubes grow in size, so does the size of each square face, as well as the number of blocks hidden inside the large painted cube (see Figure 22.2).

Distribute blocks or sugar cubes for students to build cubes. In partners, have them first build a $2 \times 2 \times 2$ cube and determine how many blocks have 0 sides painted, 1 side painted, 2 sides painted, and 3 sides painted (recording their data in a table). Continue in the same manner with a $3 \times 3 \times 3$ cube and a $4 \times 4 \times 4$ cube. As the painted cube gets bigger, there are blocks inside that cannot be seen (and that have 0 sides painted). Students can remove layers to peek inside. As students explore the growing cubes, they will see that one of their counts in the columns on the recording sheet stays the same, one is linear, one is quadratic, and one is cubic.

Pearson eText Activity Page: Painted Cubes

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

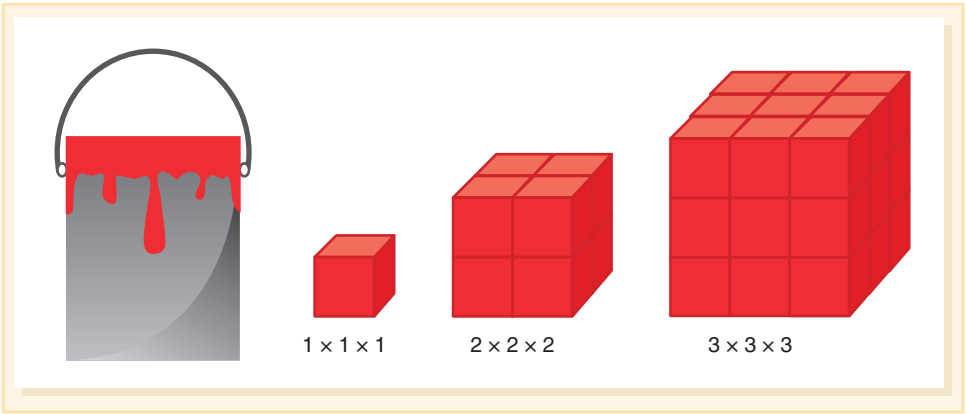


FIGURE 22.2 The Painted Cubes Problem asks students to figure out how many sides of the little blocks are painted when the cubes are dipped in (red) paint

***TECHNOLOGY* Note.** Exponential growth is very interesting to explore in real-world contexts. A powerful exploration of exponential growth is to look at Powers of 10 on the Florida State University Molecular Expressions website. The "Secret Worlds: The Universe Within" (https://micro.magnet.fsu.edu/primer/java/scienceopticsu/powersof10/) sequence of images starts way out in the universe (10^{23}) and then continues to zoom in until it reaches earth (with real screen shots and ends with zooming in on cells to a quark at 10^{-14}. ■

Order of Operations

Working with exponents is one of the operations in the *order of operations*. When students first learn the order of operations in about third grade, they focus only on the four operations. Exponents are attended to before these four operations, unless grouping symbols (e.g., parentheses or bracket) indicate something else must be computed first. Although the order of operations follows conventions (e.g., working from left to right, using parentheses), the order of the computations is actually a result of interpreting multiplication as repeated addition; exponentiation as repeated multiplication can be developed conceptually and is not as rigid as it is often portrayed (Bay-Williams & Martinie, 2015; Dupree, 2016; Zazkis & Rouleau, 2018). A context can make this point clearer, which is the focus of Activity 22.2.

Activity 22.2 Stacks of Coins

CCSS-M: 3.OA.D.8; 6.EE.A.2c

Select a story situation that includes such things as stacks of coins, bricks, or notebooks. If you have the book *Two of Everything* by Lily Hong (1993), you can show the Haktaks' stacks of coins from their magic pot and use those as the context. Tell stacks of coins stories and ask students to (1) write an expression and (2) tell you how many. For example: "Angela had one stack with seven coins and four stacks with ten coins. How many coins did she have?" (Students write $7 + 4 \times 10$ or $4 \times 10 + 7$ for the expression.) Ask, "Could we write it either way? Why or why not?" and "Could we add the seven to the four and then multiply by 10? Why or why not?" Then, switch the order so that you post an expression with addition and multiplication and ask students to tell their own stories and solve the problem.

Pause & Reflect

Simplify these expressions:

(a) $24 \div 2 \times 3$

(b) $12 - 2 \times 4 + 8 \times 3 - 12 + 8$

(c) $12 \times 15 \div 6$ ●

How did you solve the first problem? Is the answer 36 or 4? The correct answer is 36, because division is done first, because it is on the left. If you got this wrong, you are in good company—a study of elementary teachers found that a third got this problem wrong (Glidden, 2008). One cause of thinking multiplication should be first is the unfortunate (and popular) mnemonic PEMDAS (sometimes remembered as **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally). Much research shows that this mnemonic misleads students into thinking multiplication precedes division and addition precedes subtraction (Ameis, 2011; Glidden, 2008; Jeon, 2012; Watson, 2010). Finally, the mnemonic varies from country to country (e.g., BEDMAS is more common in Canada, starting with brackets rather than parentheses).

Let's look at (b) and (c). You may have first multiplied to get $12 - 8 + 24 - 12 + 8$. What did you do next? Dupree (2016) found that many teachers combined the opposites ($12 + {}^{-}12$ and ${}^{-}8$ and 8) to simplify, getting the answer of 24. This is efficient and correct, but notice that it breaks the rule of the order of operations. For task (c), you might have thought of it as $\frac{12 \times 15}{6}$ and then divided first (because it is easier that way, and still correct). Finally, notice that polynomial expressions (e.g., $8x^2 - 2x^2$), wherein subtraction does come before

Standards for Mathematical Practice
MP7. Look for and make use of structure

multiplication and exponents, do not blindly follow the order of operations (Taff, 2017). The point is that the order of operations is not as rigid as it is sometimes interpreted and must be considered alongside properties such as the commutative property, inverse operations, and the associative property.

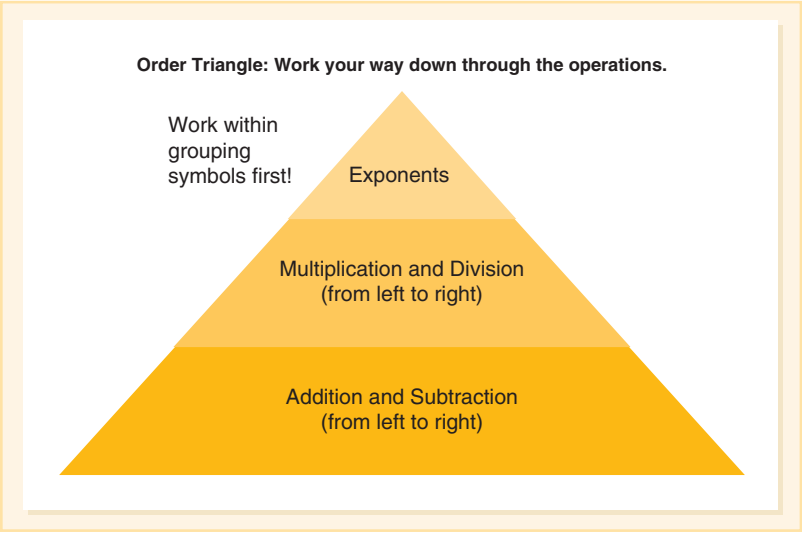

Order Triangle: Work your way down through the operations.
Work within grouping symbols first!
Exponents
Multiplication and Division (from left to right)
Addition and Subtraction (from left to right)

FIGURE 22.3 A visual to represent the order of operations (notice *no* mnemonic)

Pause & Reflect

How would you calculate $7(5 + 10)$? Would you use the order of operations and add first and then multiply by 7, or apply the distributive property and then add $35 + 70$? Do both produce the same answer? Is one way more efficient? Will one way always be more efficient? ●

Due to the various issues with mnemonics, researchers suggest not using any mnemonic for the order of operations (Dupree, 2016; Kalder, 2012; Watson, 2010). A visual can more effectively illustrate that multiplication and division are at the same level and addition and subtraction are at the same level. Figure 22.3 illustrates a triangle that has three levels (Ameis, 2011). These are the three categories of operations. (Parentheses are used to change the customary order so are referred to next to the triangle.)

You can also strengthen students' understanding of the order of operations by having them use appropriate symbols to record expressions in a fun context such as I'm Thinking of a Number.

Activity 22.3 I'm Thinking of a Number . . .

CCSS-M: 6.EE.A.2a, b, c; 6.EE.B.6

This activity is a routine in which you share the operation(s) you have performed on your secret number and tell the result. Students think backward to figure it out and then record related equations. For multilingual learners and students with special needs in math, provide the statements in writing and verbally. Students record equations, using parentheses appropriately to reflect the actions you describe.
Examples:

- I'm thinking of a number; I divide it by 8 and get 4. [student records: $\frac{n}{8} = 4$]
- I'm thinking of a number; I add 5, double it, and get 22. [student records: $(n + 5) \times 2 = 22$]
- I'm thinking of a number; I subtract 2, square it, and get 36. [student records: $(n - 2)^2 = 36$]
- I'm thinking of a number; I double it, add 2, cube the result, and get 1000. [student records: $(2n + 2)^3 = 1000$]

After writing their equations and getting their guess of the secret number in each case ready, have students talk to a partner to compare. For students with special needs in math, you may want to start with a known number rather than an unknown number—for example, start with 5, square it, add 11, divide the result by 6. They should write $(5^2 + 11) \div 6 = n$.

Alternatively, you can place the I'm Thinking of a Number activity page in a learning center and students can solve and write the related equations.

Pearson eText Activity Page: I'm Thinking of a Number

FORMATIVE ASSESSMENT Notes. "I'm thinking of a number" can serve as an illuminating formative assessment. Use it to see whether students can use grouping symbols and can simplify expressions correctly. Another option is to pose three prompts, like the ones in the preceding Pause and Reflect. Once solved, ask students whether there are other options for how to simplify the expression. In this way, you learn about their knowledge of the order of operations as well as about their fluency with the properties of the operations. ■

True–false tasks set up an excellent opportunity for students to debate, justify, and critique the justifications of their peers.

Activity 22.4 True or False Equations

CCSS-M: 3.OA.D.8; 6.EE.A.3; 6.EE.A.4

True or False equations are designed to uncover and address commonly misunderstood aspects of the order of operations. For example, $24 \div (4 \times 2) = 24 \div 4 \times 2$. If students think that multiplication comes before division, or if they are not aware of the left-to-right prioritizing, they will write "true." Examples with exponents include:

$$2 + 5^3 = 7^3$$
$$3.2^2 + 3.2^2 = 3.2^4$$

There are two ways to use the menu of equations on the order of operations in the True or False Equation activity page. First, you can create a stack of cards and have students work in partners to sort the cards. Second, you can choose from among these for a daily or weekly True or False? routine. The activity page also includes examples involving variables. For example, $4y - y = 4$. Whether a sorting activity or a routine, ask students to justify their answers.

Pearson eText Activity Page: Order of Operations: True or False Equations

Exploring Exponents on the Calculator

Standards for Mathematical Practice
MP3. Construct viable arguments and critique the reasoning of others

Handheld calculators can be powerful tools for exploring the impact of operations. For example, to evaluate 3^8, press $3 \times$ =======. (The first press of = will result in 9, or 3×3.) Students will be fascinated by how quickly numbers grow when exponents are involved. Enter any number, press ×, and then repeatedly press =. Try two-digit numbers. Try 0.1. Scientific calculators often use algebraic logic (automatically follow the order of operations) and include parentheses keys so that both $3 + 2 \times 7$ and $(3 + 2) \times 7$ result in the correct answer of 17. Basic calculators, however, will often calculate in the order the values are entered, in this case adding $3 + 2$ first and getting an answer of 35. The fact that calculators are programmed differently can be an investigation in and of itself but is certainly something students need to know about their own calculator so that they enter expressions correctly.

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Activity 22.5 Entering Expressions

CCSS-M: 6.EE.A.3; 6.EE.A.4

Provide students with the Simplify Expressions Using Technology activity page, which includes pairs of numeric expressions like these:

1. $3 + 4 \times 8$ $\quad$ $4 \times 8 + 3$
2. $3 + 5^2(3 + 5)^2$
3. $2^4 - 15 + 8$ $\quad$ $2^4 - (15 + 8)$

If students are using a basic calculator, they will find that they need to enter the numbers and operations differently than if they are using a scientific calculator. Ask, "How will you enter these expressions to preserve the order of operations?" Listen to suggestions and record them on the board. Have students work individually on the first few. Then have them pair-compare with someone else. Revisit the question of how they must enter the expressions. Have students continue to solve the rest of the problem set and again compare results with their peers.

Pearson eText Activity Page: Simplify Expressions Using Technology

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Even though scientific calculators with algebraic logic will automatically produce correct results (i.e., follow the order of operations), students must know the order of operations, including when they have to do one operation before another, and when it doesn't matter which goes first. This flexibility and awareness become the foundation for symbolic manipulation in algebra.

Integer Exponents

What does 2^{-4} mean? This is a good question to ask students who have only been working with positive exponents. The following two related options can help students explore the possibilities of negative exponents. First, looking for patterns as the power of 10 changes directly relates to place value and helps students see the regularity in the base 10 system.

Standards for Mathematical Practice
MP8. Look for and express regularity in repeated reasoning

Ask students to continue the pattern of 10^n as follows:

$10^4 = 10{,}000$
$10^3 = 1000$
$10^2 = 100$
$10^1 = 10$
$10^0 = ?$
$10^{-1} = ?$

To continue the pattern, 10° would be 1, and it is! (This is the definition of 10^0; in fact, any number to the zero power is 1.) The next value would be one-tenth of 1, and each successive number is one-tenth of the one that comes before it.

$$10^{-1} = 0.1 = \frac{1}{10}$$
$$10^{-2} = 0.01 = \frac{1}{100} = \frac{1}{10^2}$$
$$10^{-3} = 0.001 = \frac{1}{1000} = \frac{1}{10^3}$$

Students may notice that the effect of the negative exponent results in the reciprocal of the value it would be without the negative sign.

***TECHNOLOGY* Note.** Students can explore negative exponents by using technology that includes negative exponents. RapidTables (rapidtables.com) has a plethora of calculators, including an exponents calculator. The screen shows the answer and how it was computed, showing the movement of exponents above and below the fraction bar. ■

Because there are a lot of exponent rules, students tend to make mistakes if those rules are not well understood. Identifying a mistake in someone else's work (i.e., a worked example) is an effective way to help students think about the correct (and incorrect) ways to solve problems.

Activity 22.6 Find the Error

CCSS-M: 8.EE.A.1

Distribute a copy of the Find the Error activity page for each student, or create your own set of worked examples that are solved incorrectly. Ask students to explain what the student did wrong and how to simplify it correctly. Two examples are provided here:

$$\textit{Wilma: } \frac{20x^8}{5x^2} = 4x^4 \qquad \textit{Yoli: } 3^3 \times 3^{-5} = 3^{-2} = {}^{-}9$$

Examples can be increasingly challenging and mixed with correct solutions. For additional examples, see Johnson and Thompson (2009).

Pearson eText Activity Page: Find the Error

Scientific Notation

The more common it becomes to find very large or very small numbers in our daily lives, the more important it is to have convenient ways to represent them. Numbers can be written in common form, but when this becomes cumbersome, a better option is scientific notation. Scientific notation means a number is changed to be the product of a number greater than or equal to 1 and less than 10 multiplied by a power of 10. For example, 3,414,000,000 can also be written as 3.414×10^9.

Different notations have different purposes and values. For example, the population of the world on 6/1/21 was estimated to be 7,869,723,288 (https://www.worldometers.info/world-population/). (You can google this and find the actual population on the day/minute you do this activity). This number can be expressed in various ways:

7,869 million

7.9×10^9

About 7.9 billion

Each way of stating the number has value and purpose in different contexts. Rather than spend time with exercises converting numbers from standard form to scientific notation, consider large numbers found in newspapers, magazines, and atlases. How are they written? How are they said aloud? When are they rounded and why? What forms of the numbers seem best for different purposes? What level of precision is appropriate for the situation, and how do these numbers relate to other numbers? How does the population of the world relate to the population in your state, country, or continent?

***TECHNOLOGY* Note.** The U.S. Census Bureau (and other sites) makes population data readily available. The World-0-Meter website (https://www.worldometers.info/) has interesting data. For example, the home page in 2021 posts the second-by-second COVID-19 data. Search for world population, and that is also updating constantly. ■

Contexts for Very Large Numbers. The real world is full of very large quantities and measures. We see references to huge numbers in the media all the time. Unfortunately, most of us have not developed an appreciation for extremely large numbers, such as the following examples:

- Exploring lottery combinations and then probabilities can help students understand social justice issues by using mathematics (Lim, Rubel, Shookhoff, Sullivan, & Williams, 2016). A state lottery with 40 numbers has $40 \times 39 \times 38 \times 37 \times 36 \times 35$ possible ways in which the balls could come out of the hopper (2,763,633,600). But the order is not important, so we divide by the different arrangements of six numbers ($6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$), and that equals 3,838,380 possible lottery numbers (about 4 million possibilities).

- The estimated size of the universe is 40 billion light-years. One light-year is the number of miles light travels in one year. The speed of light is 186,281.7 miles per second, or 16,094,738,880 miles in a single day.
- The human body has about 100 billion cells.
- The distance to the sun is about 150 million kilometers.
- The changing population of the world. (Various population estimates can be found by seeking worldometers on the Internet.)

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Connect large numbers to meaningful points of reference to help students get an idea of their true magnitude. For example, suppose students determine that the population in their city or town is about 500,000 people. They can then figure that it would take approximately 15,000 cities of the same population size to generate the population of the world. You can extend this context by asking, "What would a list of 15,000 city names look like? Are there that many cities of that size in the United States? The world?

Suppose students want to make sense of the distance to the sun. As a benchmark, they find the distance across the United States (San Francisco, California, to Washington, D.C.) (4600 km). They discover it would take more than 32,000 trips back and forth between these two cities to equal the distance between the earth and the sun.

Lim et al. (2016) helped students visualize the number of possible combinations in the New York lottery (about 4 million) by having students figure out how tall a stack of pennies (1.52 mm each) would be and relate that to the height of the Empire State Building. The result was 14 Empire State Buildings tall! What's the chance you could pick the one right penny out of that collection? That's about your chance of winning the lottery. Building from such familiar or meaningful reference points helps students develop benchmarks to work with, make sense of large numbers, and better understand the world in which they live.

The following activity uses real data to develop an understanding of scientific notation and the relative size of numbers.

Activity 22.7 How Far Away from the Sun?

CCSS-M: 8.EE.A.4

Use the How Far Away? activity page (or have students look online for the data).

Mercury	57,909,000 km	Jupiter	778,400,000
Venus	108,200,000 km	Saturn	1,423,600,000
Earth	149,600,000 km	Uranus	2,867,000,000
Mars	227,940,000 km	Neptune	4,488,400,000

Explain to students that they are going to compare planetary distances from the sun (in km), record the data in scientific notation, and create a scaled illustration of the distances. Encourage students to develop strategies to figure out the relative distance between two planets.

Distribute a long strip of adding machine paper to each group and have the students mark the sun on one end and Neptune on the other, locating the other planets in their relative distance from the sun. For multilingual learners, reinforce the names of thousands, millions, and billions. (Note: Billion can mean one million millions in some countries, not one thousand millions as it does in the United States.)

Pearson eText Activity Page: How Far Away?

Standards for Mathematical Practice
MP7. Look for and make use of structure

Contexts for Very Small Numbers. It is also important to use real examples of very small numbers. As with large numbers, connecting very small numbers to points of reference can help students conceptualize how tiny these numbers really are, as shown by the following real-world examples:

- The chance of winning the lottery. Continuing the aforementioned example, there is a 0.00000026 or $\frac{26}{100{,}000{,}000}$ or 2.6×10^{-7} chance of winning.

- The length of a DNA strand in a cell is about 10^{-7} m. This is also measured as 1000 angstroms. (Based on this information, how long is an angstrom?) For perspective, the diameter of a human hair is about 2.54×10^{-5} m.
- Human hair grows at the rate of 10^{-8} miles per hour.
- The mass of one atom of hydrogen is 0.000 000 000 000 000 000 000 001 675 g. The mass of one paper clip is about 1 g.
- Sound takes 0.28 second (2.8×10^{-1}) to travel the length of a football field. In contrast, a TV signal travels a full mile in about 0.000005368 second, or 5.3×10^{-6} second. A TV viewer at home hears the football being kicked before the receiver on the field does.

Finding real data that are very, very small or very, very large can build the meaning of small and large numbers *and* insights into the world.

Activity 22.8 At a Snail's Pace

CCSS-M: 8.EE.A.4

The speed of garden snails has been clocked at about 3×10^{-2} mph. Ask students to estimate how long it will take a snail to travel one mile. To explore, have them record the decimal equivalent of 3×10^{-2} (0.03). They can use the calculator's counting function (enter .03 + .03 =). On many calculators, when you hit = repeatedly, the calculator counts by the last value entered (.03). Each = represents one hour. Ask students to count or use other strategies to determine the time it will take the snail to go one mile. Share solutions and strategies.

Extend this investigation by repeating the problem, but for a snail moving at a rate of 3×10^{-3}. Students should conclude that it will take the snail ten times longer.

Scientific Notation with a Calculator Tool. Students may learn how to multiply by 10, by 100, and by 1000 by simply moving the decimal point. Help students expand this idea by examining powers of 10 on a calculation tool that handles exponents. Students need to become familiar with the power-of-ten expressions in written forms and the calculator form. For example, when using some calculation tools, the product of $45{,}000{,}000 \times 8{,}000{,}000$ is displayed as 3.6E14 (with E indicating what follows is the exponent), meaning 3.6×10^{14}, or 360,000,000,000,000 (360 trillion).

Activity 22.9 Exploring Powers of 10

CCSS-M: 8.EE.A.3

Have students use the Exploring Powers of 10 activity page and any calculator that permits entering exponents to explore powers of 10, using the calculator. Questions from the activity page include:

- Explore 10^n for various values of n. What patterns do you notice? What does 1E15 mean? (1E15 is the typical calculator form of 1×10^{15}.) What does 1E–09 mean?
- What can you find out about adding numbers written in scientific notation, such as $(4.5 \times 10^n) + (27 \times 10^n)$? What about multiplying numbers in scientific notation?

Pearson eText Activity Page: Exploring Powers of 10

Operations with Scientific Notation. The goal of doing operations with scientific notation is that students understand how to use both expanded form and scientific notations and *can select the form* that lends itself to solving the problem efficiently. For example, compare these two multiplication problems:

$30{,}000{,}000 \times 900{,}000$ $\qquad$ $3 \times 10^7 \times 9 \times 10^5$

The answer is 27×10^{12} or 2.7×10^{13}. In this case, either form can be done mentally. If a problem is presented in scientific notation, students need to be able to understand the structure. For example $(4.5 \times 10^7) \times (8 \times 10^6)$ can be solved mentally by multiplying 4.5×8, which equals 36 and then multiplying $10^7 + 10^6$, which equals 10^{13}. Combining these products, the total is 36×10^{13} or 3.6×10^{14}. Writing each of these out in expanded notation is not efficient.

Positive and Negative Numbers

Learning Outcome 22.2 **Explain meaningful contexts and representations to develop understanding of positive and negative numbers.**

Generally, negative numbers are introduced with integers—the whole numbers and their negatives or opposites—instead of with fractions or decimals. However, it is a mistake to focus only on integer values, because students must understand where numbers like $^{-}4.5$ and $^{-}1\frac{1}{4}$ are positioned on the number line in relation to integer values. In many states, positive and negative numbers are introduced and developed in sixth grade and continue through middle school and beyond.

Learning about negative numbers is developmental, just as any topic. Initially, students may ignore the negative sign (minus sign), but through experiences students formalize their understanding of negative numbers, seeing the number line as symmetric around 0 (Bofferding, 2014). Looking for and using underlying structures to reason about positive and negative numbers is important in supporting students' understanding of integers and integer operations (Bishop et al., 2014). For example, students need opportunities to make sense of situations new to them, such as subtracting more than are there (e.g., $5 - 8$), and recognizing that some generalizations they have constructed or been told (e.g., that addition always makes larger) are not always true (Bishop et al., 2014; Karp, Bush, & Dougherty, 2014).

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Contexts for Exploring Positive and Negative Numbers

As with any new topic or type of number, it is important to start with familiar contexts so that students can use prior knowledge to build meaning. With integers, students are often not sure which direction they are moving when they compute, so having a context is particularly important. For many students, in particular multilingual learners, it is important to include visuals with the contexts to support language development (Swanson, 2010). As students learn to compare and compute, they can use the contexts to ground their thinking and justify their answers. Students have interactions with many contexts that involve negative numbers.

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Quantity Contexts. Quantity contexts provide an opportunity for students to combine opposites (4 and $^{-}4$) to equal zero. Quantity contexts can be illustrated with two-color counters or other counting objects.

Golf Scores. In golf, scores are often written in relationship to a number considered par for the course. So, if par for the course is 70, a golfer who ends the day at 67 has a score of $^{-}3$, or 3 strokes under par. Consider a player in a four-day tournament with day-end scores of $^{+}5$, $^{-}2$, $^{-}3$, $^{+}1$. What is their final result? How did you think about it? You could match up the positives and the negatives (in this case, $^{+}5$ with $^{-}2$ and $^{-}3$ to get a net score of 0) and then see what is left (in this case, $^{+}1$). The notion that opposites (5 and $^{-}5$) equal zero is an important concept in teaching about positive and negative numbers. You can post a mixed-up leaderboard of golf scores and ask students to order the players from first through tenth place. Emphasize that first place is the *lowest* score—and therefore the *smallest* number.

Money: Payments and Deposits. Bank accounts show how many dollars are in your account (positive) or debt you have (negative). This is a good context for exploring addition and subtraction of positive and negative numbers, as in the example illustrated in Figure 22.4.

Linear Contexts. Many of the real contexts for negative numbers are linear. The number line provides a good tool for understanding the ordering of negative numbers and can support reasoning in doing operations with positive and negative numbers (Bishop et al., 2014). Many state standards emphasize the need for students to be able to represent integers on a number line as well as on a coordinate axis.

Item	Payments or Deposits	Balance
Mowing lawn	+12.00	$34.00
Phone bill	–55.00	–21.00
iTunes downloads	–9.00	–30.00
Paycheck	+120.00	90.00

FIGURE 22.4 A checkbook as a context for adding and subtracting positive and negative numbers

Temperature. The number line measuring temperature is vertical. This context demonstrating negative numbers may be the most familiar to students, because they have either experienced temperatures below zero or know about temperatures at the North or South Pole. A good starting activity for students is finding where various temperatures belong on a thermometer. For example, Figure 22.5 displays a thermometer marked in increments of five degrees, and students are asked to place on the number line the following temperatures from a week in North Dakota: 8°F, $^{-}$2° F, $^{-}$12° F, 4° F, $^{-}$8° F. Ask students to order the temperatures from the coldest to the warmest (least to greatest). Temperatures as a context are advantageous because you can also use fractional and decimal values.

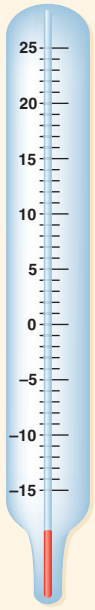


FIGURE 22.5 Thermometers provide an excellent tool for exploring positive and negative numbers

Altitude. Another vertical number line model, altitude, is also a good context for positive and negative numbers. The altitudes of sites below sea level are negative, such as the town of Dead Sea, Israel (with an altitude of $^{-}$1371 feet), and Badwater, California, in Death Valley (which has an altitude of $^{-}$282 feet). Positive values for altitude include Mount McKinley (the tallest mountain in North America) at 20,322 feet. Students can order the altitudes of various places around the world (data easily found through the Internet) or find the difference between the altitudes of two places—a good context for subtraction of integers.

Hot-Air Balloons. Hot-air balloons are a great context for exploring altitude. The idea is that as you add more hot air (e.g., little balloons or bags of hot air), the hot-air balloon goes up. If you add bags of sand, the balloon goes down. If you add two hot air bags and two sand bags, you have a net gain of 0, so the altitude stays the same.

Timelines. Asking students to place historical events on a timeline is an excellent interdisciplinary opportunity. The timeline is useful for examples with larger values (e.g., 1950) as well as with negative values (e.g., thinking of 3000 BCE as $^{-}$3000 on a number line). Or students can explore their own personal timeline, with their birthday at 0 on the number line (Weidemann, Mikovch, & Hunt, 2001). Students find out key events that happened before they were born (e.g., the birth of an older sibling) and have happened since they were born (e.g., the move to a new house) and place them on their personal number line. By partitioning a year into months, students can gain experience with rational numbers on the number line.

Football. A statistic reported on every play in a football game is yards gained or lost, which provides a good context for exploring integers, especially when it comes to comparing and adding integers. Students can be asked questions like these: "If our team started their drive on the 20-yard line and the first three plays were recorded as $^{-}$4, $^{+}$9, $^{+}$3, did they get a first down?"

Activity 22.10 Football Statistics

CCSS-M: 6.NS.C.5; 6.NS.C.6a

Look up the average yards gained for some of the best running backs in the NFL or from college teams popular with your students. Ask students to use average yards gained per down to create a possible list of yardage gains and losses for each player. For example, if a player had an average of 4 yards per carry in a game, the following could have been the data for ten carries:

10, $^{-}$3, $^{-}$2, 21, $^{-}$5, 3, $^{-}$1, 5, $^{-}$1, 13

You may want to do this problem together and then have students create their own. For students with special needs in math, you can scaffold this idea by starting with three carries (not 10) and using a visual of a football field marked in ten-yard intervals (see the Football Field activity page). The football context provides an excellent way to use integers meaningfully, integrated with the important concept of averages. Multilingual learners may not be familiar with American football, because football in most countries is what the United States calls soccer, so yards may need to be explained (related to meters). Role-playing the game with students is a fun way to be sure the game is understood by all.

Pearson eText Activity Page: Football Field

***TECHNOLOGY* Note.** Go to PBS Learning Media (pbslearningmedia.org) and search for The Number Line: Rational Numbers and Football for a fun app that uses football as the context to add integers. This site has many other quality videos, apps, and resources as well. Just search for what you are teaching and check out the options. ■

Meaning of Negative Numbers

Negative numbers are the numbers less than 0. More formally, they are defined in relation to their positive counterparts. The set of *integers* is defined as the positive whole numbers, the opposites of the whole numbers (or negative integers), and 0, which is neither positive nor negative.

Absolute Value. Absolute value is introduced in sixth grade in many states. The *absolute value of a number* is defined as the distance between that number and zero. Knowing the distance between two points, either on a number line or on a plane, is often needed in applications of mathematics. For example, we need to be able to determine how far a helicopter is from a hospital, regardless of its direction. The notation for absolute value consists of two vertical bars on either side of the number. Thus, the absolute value of a number n is $|n|$. Opposites, such as $^{-}12$ and 12, are the same distance from zero and therefore have the same absolute value.

When students' absolute value experiences are limited to simplifying expressions like $|^{-}8|$ or $|6 - 10|$, they do not connect the procedure with the meaning of absolute value or see real purpose for doing this. Add in a context to make it meaningful. For example, $|6 - 10|$ can be the distance between the 10-mile marker and the 6-mile marker on a highway. In this example, you can see that both $10 - 6$ and $6 - 10$ can lead to the answer, and distance is positive (absolute value), so the answer is 4 miles.

Minus Sign Notation. Through their early years in school, students recognize the "–"sign to mean "subtract." They need a deeper understanding of other meanings of this symbol to solve equations and make sense of variables (Lamb et al., 2012). There are three meanings of the minus symbol (Bofferding, 2014; Lamb et al., 2012). Each is illustrated with examples here:

Standards for Mathematical Practice
MP6. Attend to precision

Subtraction: $25 - 12 =$ _____ or $9 -$ _____ $= 4.5$
Negative number: $25 + {}^{-}12 =$ _____ or $9 -$ _____ $= {}^{-}4.5$
The opposite of: ${}^{-}({}^{-}5) = 5$ or ${}^{-}x$

The different heights at which the negative sign may appear (e.g., -7 and $^{-}7$) may be confusing to students. The minus sign can also switch meanings in the middle of simplifying an equation. For example, consider the equation $3.5 - x = {}^{-}0.6$. A step in simplifying might be to add $^{-}3.5$ to both sides, leaving $^{-}x = {}^{-}4.1$. Here, the equation reads, "The opposite of x is equal to negative 4.1 (or the opposite of 4.1)." Parentheses are sometimes placed around the number so that it is separate from the operation—for example, $8 - ({}^{-}5)$. Students have not seen parentheses used in this way and may think they should multiply. It is important to connect to their prior knowledge and explicitly build meaning for the new use of the minus symbol and parentheses. Activity 22.11, based on Lamb and colleagues (2012), is designed to do this.

Activity 22.11 Greater, Less, Equal, or Don't Know?

CCSS-M: 6.NS.C.5; 6.NS.C.6a

Use the Greater, Less, Equal, or Don't Know activity page or write an equation on the board that addresses different meanings of the minus symbol and/or parentheses. Have students vote on whether it is equal or not equal. Encourage students to work with a partner to prepare a rationale for whether they think the equations are equal or not equal. Facilitate a debate as appropriate. Examples include:

$2 - (^{-}3)$_____1 $\quad ^{-}x$_____x $\quad ^{-}(^{-}8)$_____8

In these examples, you might ask students, "When do we use parentheses in mathematics?" Students might say they are used for grouping a series of computations to show what to do first and that it can also mean multiplication. Point out that parentheses are also used to make a number sentence more readable—separating the negative number from the operation symbol for clarity. Multilingual learners may benefit from additional vocabulary support to understand the distinctions between words such as minus, negative, and opposite. Pointing and saying the words as a class can help all students learn the appropriate terminology. Also, not all countries use the same notations; if you see different ways of notating expressions, invite students to share their notations and compare them.

Pearson eText Activity Page: Greater, Less, Equal, or Don't Know

Tools for Illustrating Positive and Negative Numbers

Standards for Mathematical Practice
MP5. Use appropriate tools strategically

Two tools are popular for helping students understand comparisons and the four operations with negative and positive numbers. Two-color counters can be used for integer values (counting numbers, including 0), and number lines are needed to illustrate positive and negative non-counting numbers.

Counters. Two-color counters are a great fit for showing positive and negative integers, because one side (e.g., yellow) can represent positive counts, and the other side (e.g., red) can represent negative counts. Combining one counter of each type results in zero ($^{+}1 + {}^{-}1 = 0$), illustrating that they are opposites. Consider money: If yellow counters are credits and red counters are debits, having 5 yellows and 7 reds is the same as having 2 reds or 2 debits and is represented as $^{-}2$ (see Figure 22.6). It is important for students to understand that it is always possible to add to or remove from a collection of any number of pairs consisting of one positive and one negative counter without changing the value (i.e., it is like adding equal quantities of debits and credits).

***TECHNOLOGY* Note.** LearnZillion (learnzillion.com) offers various math instructional videos, including "Add Integers Using Chips." Brainingcamp's Two Color Counter app (brainingcamp.com) allows you to pick an integer option so that students can explore the negative and positive relationship as well as model integer operations. ■

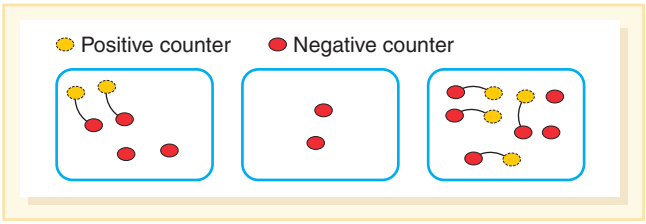


FIGURE 22.6 Each collection is a representation of $^{-}2$

Number Lines. The number line has a number of advantages when teaching positive and negative numbers. A number line:

1. Is familiar to students from their computation with whole numbers, fractions, and decimals.
2. Is an excellent tool for representing the operations conceptually. To add a context, consider using small cutouts of grasshoppers that move up and down the line (Swanson, 2010). Or use your school's mascot!
3. Shows the distance from 0 (or the absolute value of the number).

4. Allows students to explore non-integer negative and positive values (e.g., $^{-}4\frac{1}{4} + 3\frac{1}{3}$ or $^{-}9.2 - 4.5$), which is very difficult to do with counters.
5. Connects to the coordinate axis, which involves two perpendicular number lines.
6. Students can physically walk a long number line (e.g., taped on the floor) to model problems (Nurnberger-Haag, 2007) (see Activity 14.13).

Arrows can be used in operations with negative numbers to show distance and direction. For example, 4 can be modeled with an arrow 4 units long, starting at any location on the number line and pointing to the right, and $^{-}3$ can be modeled with an arrow 3 units long, starting at any location on the number line and pointing to the left (see Figure 22.7). The arrows help students think of integer quantities as directed distances. A positive arrow never points left, and a negative arrow never points right. Furthermore, each arrow is a quantity with both length (magnitude or absolute value) and direction (sign). These properties are constant for each arrow regardless of its position on the number line.

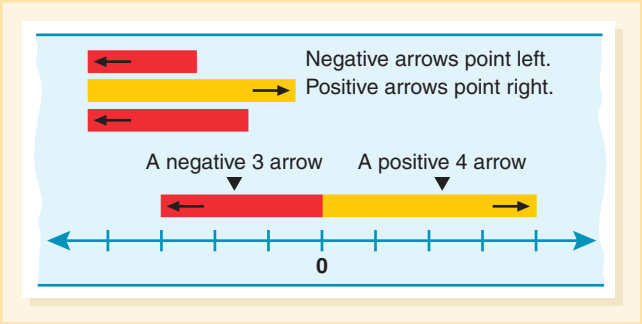


FIGURE 22.7 Number line visual for negative and positive numbers

Which Tool to Use. Seeing how positive and negative numbers are represented across these two tools while making connections between the visuals can help students develop the intended concepts and operation sense. The context may decide the representation (e.g., if the context is height, the number line is a better fit), and if the values are not integers, the number line is likely a better fit.

Operations with Positive and Negative Numbers

Learning Outcome 22.3 **Illustrate and explain conceptual approaches to teaching operations with positive and negative numbers.**

Standards for Mathematical Practice
MP7. Look for and make use of structure

Exploring operations with positive and negative numbers must not be about memorizing rules (e.g., when you see subtraction, change it to addition and change the sign of the second number). This approach is not effective and deprives students of the opportunity to reason quantitatively. As students extend their understanding of the operations to include negative numbers, they need opportunities to consider how underlying structures and generalizations of arithmetic might be applied with negative numbers (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2016a). Just like with whole numbers, fractions, and decimals, operations with negative numbers need to focus on developing fluency, which includes developing reasoning strategies and focusing on flexibility.

Addition and Subtraction

To begin adding and subtracting, use tools and familiar contexts. For example, consider adding golf scores, which can be modeled with two-color counters:

Phil golfs twice every weekend in May. What were his weekend total scores for the two games? On which weekend did he do the best?

Weekend 1: $^{+}3$ and $^{-}2$; Weekend 2: $^{+}3$ and $^{-}5$; Weekend 3: $^{-}4$ and $^{-}2$

Golf tournaments usually are 4 days long, so this context works well for adding a series of small integers, such as $3 + 2 + {}^{-}5 + {}^{-}2$.

For adding and subtracting, consider altitude changes, such as with a hot-air balloon, and have students use a vertical number line and a balloon cutout to model the problems and write equations.

Activity 22.12 Height of Hot-Air Balloons

CCSS-M: 7.NS.A.1

Explain to students that the balloon height can be manipulated with two types of bags:

- **Hot-air bags**. One hot-air bag raises the balloon 1 unit; removing one hot-air bag lowers the balloon 1 unit.
- **Sand bags**. Adding one sand bag lowers the balloon 1 unit. Removing one sand bag raises the balloon one unit.

Give each student the Hot-Air Balloon Moves activity page or provide students with a vertical number line (see Figure 22.8) and pose problems. Here are three types.

1. **Adding stories.**
   The balloon starts at ⁻1, and 5 hot-air bags are added. Where will the hot-air balloon be?
   The balloon starts at ⁻3 and 2 sand bags are added. Where will the hot-air balloon be?
2. **Subtracting stories**
   The balloon starts at ⁻3 and 2 sand bags are removed. Where will the hot-air balloon be?
3. **Missing-value problems (which have various solutions)**
   The balloon starts at ⁻3 and ends up at ⁻6. What might have happened? (Options include: 3 air bags were removed; 3 sand bags were added; 5 air bags and 8 sand bags were added; etc.)

Students can work individually and then compare their work with a partner, or students can work together to solve.

Pearson eText Activity Page: Hot-Air Balloon Moves

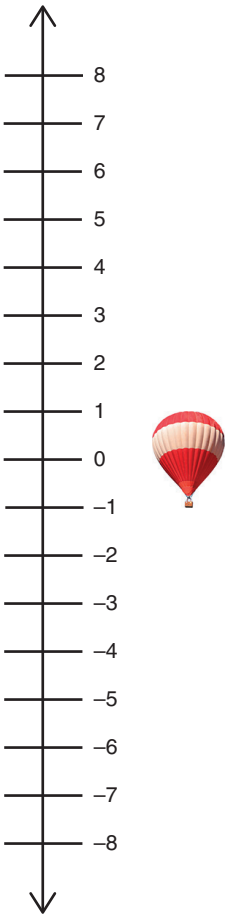


FIGURE 22.8 Adding and subtracting with a hot-air balloon context
Image source: Zilu8/Shutterstock

***TECHNOLOGY* Note.** The hot-air balloon context has become very popular for teaching addition and subtraction of positive and negative numbers; thus, there are various online resources. Desmos has a series of lessons on adding and subtracting, GeoGebra has a Balloons and Sandbags applet, and Texas Instruments (education.ti.com) has a hot-air balloon classroom activity. In addition, various sites offer more explorations for download, for example, LearnAlberta.ca (a resource site for teachers in Alberta, Canada, that uses the balloon context in a resource for teaching integer subtraction). ■

Additionally, ask students to author their own stories for integer operations. One way to scaffold this is to ask students these three prompts (Swanson, 2010):

Where did you start?

How far did you go?

Where are you now?

So, for $^{-}3 - 5 = ?$ a student might write, "I was 3 feet under water, then dove down 5 feet. Where am I now?" ($^{-}8$)

Contexts are an important place to begin, but students also need to be able to reason about the numbers themselves and draw on their experiences and understanding of addition and subtraction. Rather than tell students a rule for solving, pose open number sentences involving negative numbers and ask students what they think the answer might be. Examples include:

$$5 - 11 = n \quad {}^{-}3 + n = 7 \quad n - {}^{-}3 = 5$$

Bishop and colleagues (2016b) found that students could apply their understandings of the structure of numbers and the meaning of addition and subtraction to these problems. A student might reason about the first example, thinking they are going down and therefore

Standards for Mathematical Practice
MP1. Make sense of problems and persevere in solving them

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

the answer is $^{-}6$ and reason that they start at $^{-}7$ on the number line and *go up* 12 to 5. Another student might apply the commutative property, changing the equation (mentally) to $12 + {}^{-}7 = n$ and then reason that they *go down* 7 to 5. Students use several reasoning strategies to determine answers (Bishop et al., 2016b; Lamb, et al., 2018):

1. Emergent reasoning (using whole-number reasoning)
2. Order-based reasoning (e.g., to solve $^{-}4 + 5$, a student moves up to 0, then up to 1)
3. Analogy-based reasoning (e.g., student uses a context (like balloons) to reason or to solve $^{-}4 - {}^{-}2$, a student thinks about the difference $4 - 2$
4. Formal reasoning (e.g., to solve $5 + n = 3$, a student reasons it must be a negative number because the sum is less than the start value)
5. Computational reasoning (e.g., to solve $5 - 11$, student converts to addition problem $5 + {}^{-}11$, following the operation rules they have been taught)

These researchers suggest asking questions that help students focus on the structure of numbers, such as:

- How do these two problems compare (e.g., $^{-}4 + 5 = n$ and $^{-}4 - 5 = n$)?
- How might you explain why you can change $7 - {}^{-}3$ to $7 + 3$?
- How does knowing $5 + 4$ help you think about $5 + {}^{-}4$?
- I see that you counted up for this problem; why did you count up (and not count back) (e.g., $^{-}9 - {}^{-}3$ [correct to count up] or $^{-}9 - 3$ (incorrect to count up)?

Contrast this reasoning approach with telling students, "When you see subtraction, change it to addition and change the sign of the second number." Although these rules take less time to explain, they rob students of the opportunity to understand negative numbers, making it impossible for them to determine whether an answer is reasonable or for them to apply mental, logical strategies for arithmetic involving negative numbers.

A focus on opposites can help students reason about addition and subtraction with negative numbers (Friedmann, 2008). Activity 22.13 develops the idea of opposites.

Activity 22.13 Find the Zero

CCSS-M: 7.NS.1a, b, c

Before beginning the activity, ask students to find the sums of several opposites (e.g., $4 + {}^{-}4$). Distribute the Find the Zero activity page and ask students to look at a sum that is not opposites (e.g., $7 + {}^{-}4$) and ask whether they can find a zero by decomposing one of the numbers (e.g., $(3 + 4) + {}^{-}4$) and solve. Using a number line, students can find the difference between these numbers. Students, particularly students with special needs in math, may benefit from creating a zero box below each problem as they solve it to highlight the zero pair, as illustrated in the following.

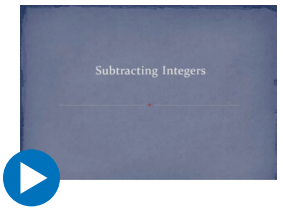
$$12 + {}^{-}5 =$$

Zero Box: $\boxed{5 + {}^{-}5}$

$$(7 + 5) + {}^{-}5) = 7 + (5 + {}^{-}5) = 7 + 0 = 7$$

Students must continue to illustrate what is happening when they are adding and subtracting negative numbers. If they do not, they will get lost in the symbols and which direction to head on the number line. See Expanded Lesson: Find the Zero for details on teaching this activity.

Pearson eText Activity Page: Find the Zero
Pearson eText Expanded Lesson: Find the Zero



Pearson eText
Video Example 22.1
In this video consider how counters illustrate subtraction involving integers with a focus on using opposites.

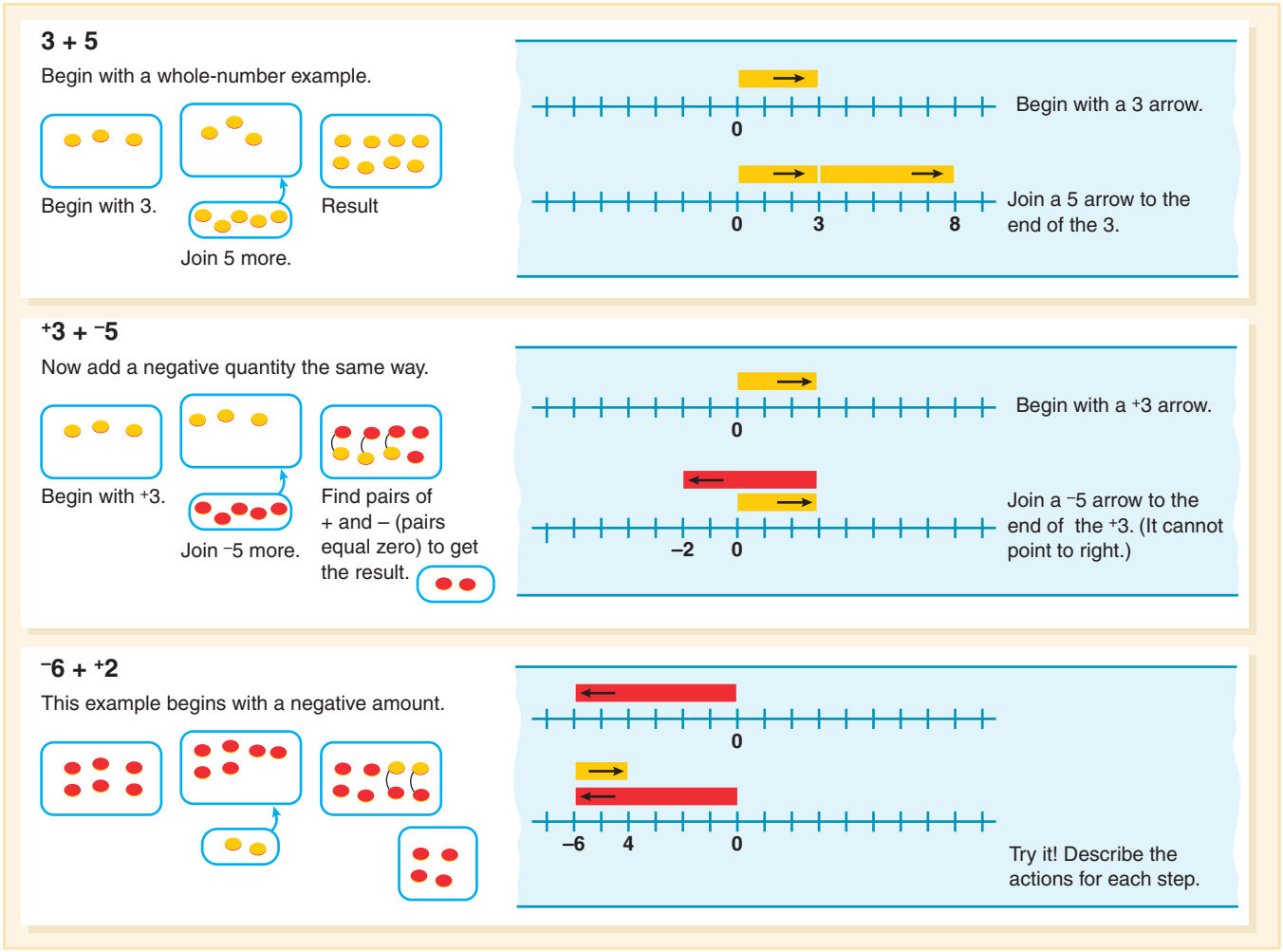

3 + 5
Begin with a whole-number example.
Begin with 3.
Join 5 more.
Result
0
Begin with a 3 arrow.
0 3 8
Join a 5 arrow to the end of the 3.
+3 + −5
Now add a negative quantity the same way.
Begin with +3.
Join −5 more.
Find pairs of + and – (pairs equal zero) to get the result.
0
Begin with a +3 arrow.
–2 0
Join a −5 arrow to the end of the +3. (It cannot point to right.)
−6 + +2
This example begins with a negative amount.
0
–6 4 0
Try it! Describe the actions for each step.

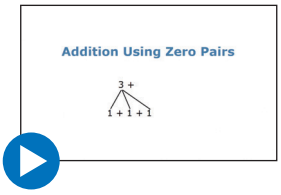
FIGURE 22.9 Addition with counters and number lines

Addition and subtraction can be modeled with counters or number lines (see Figure 22.9). To add, using the number-line model, note that each added arrow begins at the arrowhead end of the previous arrow.

Subtraction can be interpreted as separate (takeaway) or compare (the difference or distance between the values). Let's explore this idea with the problem $^{-}5 - {}^{+}2$, the second example modeled in Figure 22.10.

If using counters, the context could be money, such as, "I start with a debit of $5 and then withdraw (take out) $2 more from my account. What balance will my bank account show (if no fees have been charged yet for my overdrawn account)?" To model this problem, you start with the five red counters. To remove two positive counters from a set that has none, two zero pairs (one positive, one negative) must be added so that two positive counters can be removed. Then, 2 positive counters (yellow) can be taken away. The net effect is to have a debt of $7 or $^{-}7$ (less money because $2 was taken away). This is a takeaway interpretation.

With the number line, subtraction can be illustrated using arrows for separate and comparison situations. A takeaway interpretation means starting at $^{-}5$ and going 2 to the left. The result is $^{-}7$. A compare interpretation of $^{-}5 - {}^{+}2$ asks how far to get *from* 2 *to* the starting amount of $^{-}5$. The distance is 7 units, and the direction is negative, so $^{-}7$. Although this example may sound more comprehensible as a separate (takeaway) interpretation, sometimes comparisons of the distance between the two points makes more sense to students (Tillema, 2012). Consider subtracting two negatives: $^{-}7 - {}^{-}4$. In words, how do you get *from* $^{-}4$ to the starting point of $^{-}7$? You count back 3, so $^{-}3$.



Pearson eText
Video Example 22.2
In this video consider how to add integers on the number line and by finding zero pairs.

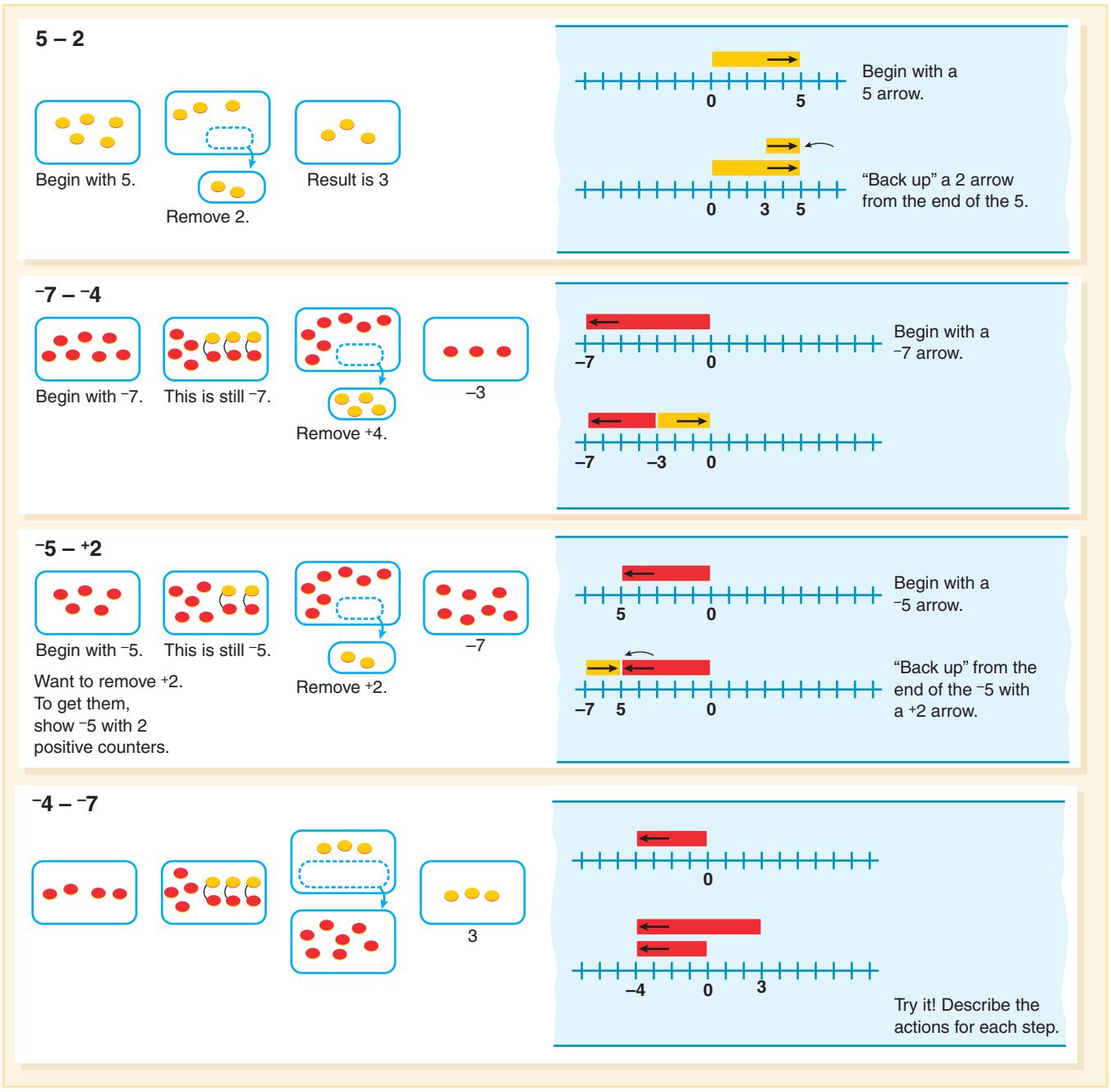


FIGURE 22.10 Subtraction illustrated with counters and a number line

Pause & Reflect

Try to explain the problems in Figures 22.9 and 22.10, using both a quantity and a linear context. For subtraction, explain each, using a separate situation and a comparison situation. What kinds of stories or contexts might fit with these different ways to think about addition and subtraction? ●

To help students is to connect visual representations to their symbolic computations; ask students to create basic open number line illustrations with their equations, as illustrated in Figure 22.11. Encourage students to use a context to help them reason (e.g., hot-air balloons or temperature change).

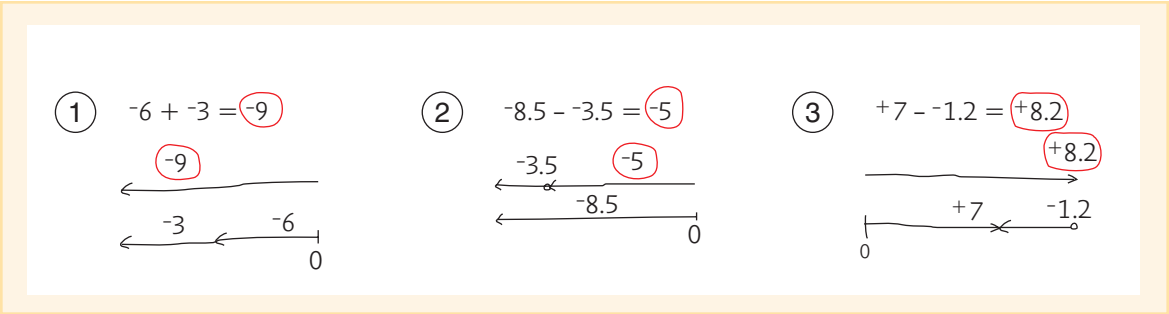
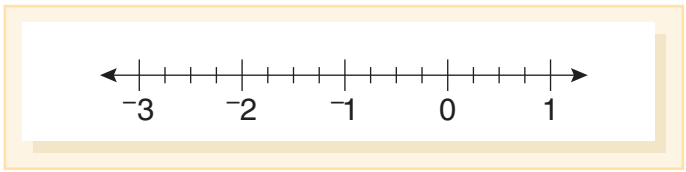


FIGURE 22.11 Students use arrow sketches to represent addition and subtraction

It is important for students to see that $^{+}3 + {}^{-}5$ is the same as $^{+}3 - {}^{+}5$ and that $^{+}2 - {}^{-}6$ is the same as $^{+}2 + {}^{+}6$. Illustrating addition and subtraction problems on the number line and explicitly discussing how the two expressions are related will help students see the connections between these expressions.

Standards for Mathematical Practice
MP4. Model with mathematics

Many of the examples in this section have focused on integers, but with the use of a ruler or any number line partitioned into fractional amounts, the same arrow illustrations can help students reason about rational number addition and subtraction of positive and negative numbers (with decimals or with fractions).



***TECHNOLOGY* Note.** Several sites offer ways to readily make your own number lines. Math Learning Center has a free number line app (mathlearningcenter.org) that allows you to design a number line with negative values and then use virtual pens to draw your jumps. Brainingcamp (brainingcamp.com) also has build-your-own number lines. You can select decimals or fractions and select your range (e.g., −5 to 5) and the partitions you want to have shown. Or use their open number line option.

PhET Interactive Simulations has an app that involves adding and subtracting numbers, and students can pick the visuals (counters or number line) or context (debts and assets). ■

Multiplication

Multiplication of integers is an extension of multiplication of whole numbers, fractions, and decimals. Just like with addition and subtraction, students need opportunities to apply their existing understanding of number and operations to consider how those structures apply to negative numbers. For example, students challenged to solve and make sense of $\frac{1}{2} \times -6$ were able to use a number-line model and apply their knowledge of fraction operations of positive numbers, dispelling one of the students' initial assumption that finding half should make the answer smaller (Carter, Prince, & Schwartz, 2017). These students thought of this problem as scaling. Scaling translates to integer multiplication quite readily when the first factor is positive, regardless of the sign of the second factor. For example, for $3 \times {}^{-}4$, the question is "How much is three groups of $^{-}4$?" With a context, options include: "If I put 4 sand bags in the hot-air balloon three times, how is the height of the balloon affected?"; "Three days in a row, Hans scored $^{-}4$ in his golf tournament; what is his total score?" or, "Three times today, the temperature dropped four degrees. How much has the temperature changed?" Connecting to contexts and looking at repeated examples can lead to students creating conjectures about the rules for multiplying (and dividing) with negative numbers (Choppin, Clancy, & Koch, 2012).

Activity 22.14 Conjectures for Operations with Negative Numbers

CCSS-M: 7.NS.A.2a, b

This activity is described here for multiplication but could be replicated for any of the operations with negative numbers. Ask students to make a conjecture about what they think the sign will be on a product of a positive factor times a negative factor. Give students a set of multiplication problems with a positive-times-a-negative equation (keep first factor a whole number to see the pattern better), such as:

$$3 \times {}^{-}10 = \qquad 5 \times -\tfrac{1}{4} = \qquad 8 \times {}^{-}0.5 =$$

Invite students to create stories or visuals to model each of the equations. Students with special needs in math benefit from having a menu of contexts and tools. After telling stories for each, ask students to make a generalization about a positive number times a negative number.

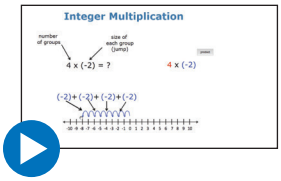
Repeat this process for a negative number times a positive number:

$$^{-}3 \times 4 = \qquad ^{-}5 \times \tfrac{1}{2} = \qquad ^{-}10 \times 0.25 =$$

Ask students to revisit and revise (as needed) their conjectures regarding the product of a negative factor times a positive factor. Connect their conjectures to the contexts that were selected. If there are multilingual learners in the class, you may select one context to explore, rather than several, to limit unnecessary language demands.

Standards for Mathematical Practice
MP7. Look for and make use of structure

Looking at patterns, as we did to establish the meaning of negative exponents, is another way to explore the meaning of signs when multiplying negative numbers. Consider the patterns in Figure 22.12. With a positive number multiplier (the number of groups), the meaning can be thought of as equal addition. With a negative multiplier, the related meaning is equal subtraction (e.g., $^{-}2 \times 3$), subtracting two groups of 3, or $^{-}6$.



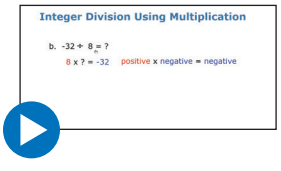
Pearson eText
Video Example 22.3
In this video consider how to use number lines to illustrate the meaning of multiplication involving integers.

$4 \times 3 = 12$	$3 \times (^{-}2) =$ _____	$(^{-}3) \times 3 = {}^{-}9$
$4 \times 2 = 8$	$(^{-}2) \times 3 =$ _____	$(^{-}3) \times 2 = {}^{-}6$
$4 \times 1 = 4$	$5 \times (^{-}2) =$ _____	$(^{-}3) \times 1 = {}^{-}3$
$4 \times 0 = 0$	$(^{-}2) \times 5 =$ _____	$(^{-}3) \times 0 =$ _____
$4 \times (^{-}1) =$ _____	$5 \times (^{-}7) =$ _____	$(^{-}3) \times {}^{-}1 =$ _____
$4 \times (^{-}2) =$ _____	$(^{-}7) \times 5 =$ _____	$(^{-}3) \times {}^{-}2 =$ _____
$4 \times (^{-}3) =$ _____	$(^{-}4) \times 9 =$ _____	$(^{-}3) \times {}^{-}3 =$ _____

FIGURE 22.12 What patterns do you notice in each series (column) of equations?

Division

Understanding division with negatives numbers rests on a good conceptual understanding of multiplication problems in which the first factor is negative and knowledge of the relationship between multiplication and division. One option is to create a series of related problems, as was illustrated in Figure 22.12, but to use division. Another common option is to justify the rules for division, using the inverse relationship with multiplication. For example, $^{-}24 \div {}^{-}4 = 6$ *because* 6 groups of $^{-}4$ is $^{-}24$. When the divisor is a whole number, sharing contexts can help make sense of the situation and why the answer of a negative divided by a positive is, in fact, negative. For example, for this equation, $^{-}12.5 \div 5$ might be about losing weight over 5 weeks and finding the weekly average.



Pearson eText
Video Example 22.4
In this video consider how the meaning of multiplication of integers is used to understand the division of integers.

Pause & Reflect

Try talking through examples and drawing pictures for $^{-}8 \div {}^{+}2$, $^{+}9 \div {}^{-}3$, and $^{-}12 \div {}^{-}4$. What contexts can fit these equations? ●

Encourage students to think first about how to visualize the whole-number situation and then connect that understanding to negative numbers. Extend these explorations to decimal and fraction values and continue to have students draw illustrations to accompany their equations.

***TECHNOLOGY* Note.** DreamBox (dreambox.com) teacher tools include various apps to meet grade-level standards. The Integer Operations app integrates algebraic thinking and order of operations, having students choose tiles to make an equation with a given answer. ■

Real Numbers

Learning Outcome 22.4 Explain ways to help students see relationships among different types of numbers (irrational, whole, etc.).

Whole numbers, fractions and decimals, and integers are all rational numbers because they can all be written as a fraction with an integer over a nonzero integer. An interesting fact is that there are infinitely many rational numbers between any two numbers. Exploring this idea can deepen students' understanding of rational numbers.

Activity 22.15 How Many in Between?

CCSS-M: 6.NS.C.5, 7.NS.A.2d

Post two fractions for students, such as $\frac{1}{2}$ and $\frac{9}{10}$. Ask students to locate those fractions on a number line and then to insert four additional fractions *between* the two given fractions. Encourage students to use different strategies to reason about what fractions come in between. Students may convert one-half to five-tenths and readily find $\frac{6}{10}$, $\frac{7}{10}$, and $\frac{8}{10}$ but be challenged to find the fourth equivalence (there are many, e.g., $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$ and $\frac{8}{9}$). Or, they may convert to decimal equivalencies to find decimals and then convert the decimals back to fractions. See Expanded Lesson: How Many in Between? for a full lesson exploring rational numbers between two rational numbers.

Pearson eText Expanded Lesson: How Many in Between?

Standards for Mathematical Practice
MP2. Reason abstractly and quantitatively

Irrational numbers are numbers such as $\sqrt{2}$ —numbers whose value cannot be written as a fraction and whose exact value can only be estimated. Eighth-grade students begin to explore irrational numbers and find their rational approximation (NGA Center & CCSSO, 2010). All these numbers are part of the *real numbers*, which are the only types of numbers students explore until high school, where they consider the square roots of negative numbers, called *imaginary numbers*. These sets of numbers are interrelated, and some are subsets of other sets. Figure 22.13 provides an illustration of the types of numbers and how they are interrelated.

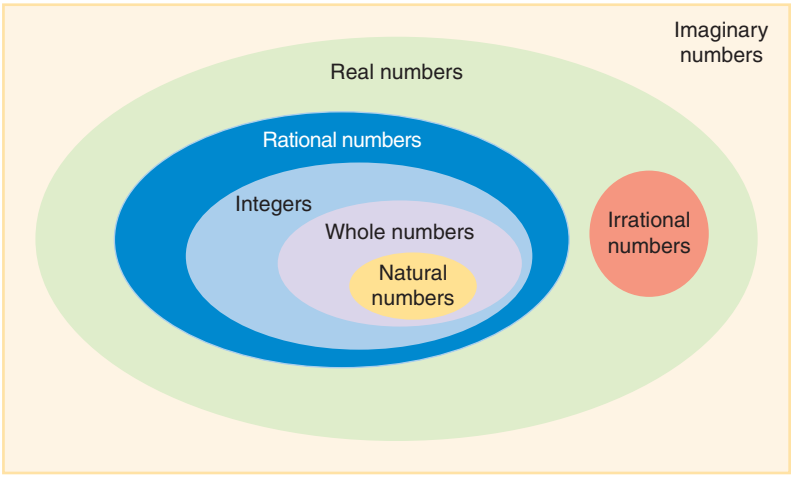


FIGURE 22.13 An illustration of the organization of the real numbers

***TECHNOLOGY* Note.** The Math Page (https://themathpage.com/) offers a wonderful resource, called *The Evolution of the Real Numbers*, which goes from number type to number type within the real-number system. Although mostly text, the pages are filled with interactive questions. And a very old but fun video is The Weird Number, which is available on YouTube (https://www.youtube.com/watch?v=pSO66sL9SmY). ■

Rational Numbers

Rational numbers comprise the set of all numbers that can be represented as a fraction—or a ratio of an integer to an integer. Even when numbers are written as whole numbers or as terminating decimals, they can also be written as fractions and thus are rational numbers. In fact, in school mathematics, the term *rational numbers* is often used to refer to fractions, decimals (terminating and repeating), and percents. These are rational numbers, but so are integers, including whole numbers.

Moving among Representations. Students should be able to recognize a rational number as a point on a number line. In most states, in middle school, students develop a unified understanding of number, recognizing fractions and finite or repeating decimals, and percents are different interchangeable representations of rational numbers. This means that given any value, students can think about it and operate it on it across representations, moving flexibly among different representations. For example, students should be able to explain equivalence, as noted here:

Standards for Mathematical Practice
MP7. Look for and make use of structure

- $4\frac{3}{5}$ is equivalent to 4.6, because $\frac{3}{5}$ is six-tenths of a whole, so 4 wholes and six-tenths is 4.6.
- $4\frac{3}{5}$ is equivalent to $\frac{23}{5}$, and that is the same as $23 \div 5$, or 4.6 if I use decimals.
- 4.6 is read "four and six-tenths," so I can write that as $4\frac{6}{10} = 4\frac{3}{5}$.

Similarly, compare these three expressions:

$$\frac{1}{4} \text{ of } 24 \qquad \frac{24}{4} \qquad 24 \div 4$$

This discussion can lead to a general development of the idea that a fraction can be thought of as division of the numerator by the denominator, or that $\frac{a}{b}$ is the same as $a \div b$.

When a fraction is converted to a decimal, the decimal either terminates (e.g., 3.415) or repeats (e.g., 2.5141414…). Is there a way to tell in advance whether a given fraction is a terminating decimal or a repeating decimal? The following activity can be used to discover whether that prediction is correct.

Activity 22.16 Repeater or Terminator?

CCSS-M: 7.NS.A.2d

Have students generate a table, listing in one column the first 20 unit fractions $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \frac{1}{21})$. The second column is for listing the prime factorization of the denominators; and the third column is for listing the decimal equivalents of the fractions (use calculators to get the precise decimal form).

After they have completed the table, ask students to make a conjecture that will tell in advance whether the decimal will repeat or terminate. Have students test their conjecture on other fractions. For example, ask students to explore what happens when the numerator is not 1. To challenge students, ask them to test their conjecture with fractions that have denominators beyond 21.

As students work on this task, they will notice various patterns, as can be seen in the student work provided in Figure 22.14. As this student has discovered, the only fractions with terminating decimals have denominators that factor with only combinations of twos and fives. Why is this the case?

Middle school students must understand that any rational number, positive or negative, whole or not whole, can be written as a fraction and as a decimal. So, $^{-}8$ can be written as the fraction $\frac{-8}{1}$ or $\frac{-16}{2}$ or as the decimal $^{-}8.0$. In fact, there are infinite ways to write equivalencies for $^{-}8$. Fluency with equivalent representations is critical and requires much more than teaching an algorithm for changing from one representation to another.

Irrational Numbers

As noted earlier, irrational numbers are not rational, meaning they cannot be written in fraction form. The irrational numbers together with the rational numbers make up the *real numbers*. The real numbers fill in all the holes on the number line, even when the holes are infinitesimally small. This topic is often an eighth-grade standard. Students' first experience with irrational numbers typically occurs when exploring square roots of whole numbers. The following activity provides a good introduction to square roots and cube roots.

Activity 22.17 Edges of Squares and Cubes

CCSS-M: 8.NS.A.1; 8.EE.A.2

Show students pictures of three squares (or three cubes) as in Figure 22.15. The sides (squares) and edges (cubes) of the first and last figure are consecutive whole numbers. The areas or volumes of all three figures are provided. Give students tiles (for squares) or centimeter cubes or blocks (for cubes) and ask them to build the shapes shown in the figure. Students will realize they cannot make the middle figures. Then, have students explore the problem, using a calculator to find the length of the sides (squares) or edges (cubes) of the figure in the center. Explain to students that they are not to use the square root or cube root key, but to estimate what they think the length of the side would be and test it by squaring or cubing it. Ask students to continue to estimate until they have found a value to the hundredths place that gets as close to the area of 45 as possible (or the volume of 30 in the case of the cube). Solutions will satisfy these equations:

$$\square \times \square = 45, \text{ or } \square^2 = 45$$

and

$$\square \times \square \times \square = 30, \text{ or } \square^3 = 30$$

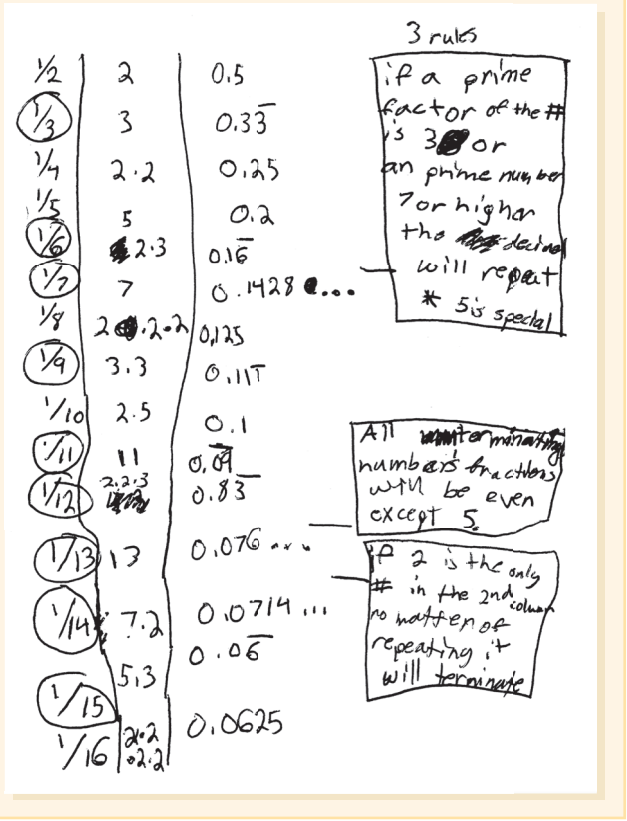

3 rules
if a prime factor of the # is 3 or an prime number 7 or higher the decimal will repeat
* 5 is special
All terminating numbers/fractions will be even except 5
if 2 is the only # in the 2nd column no matter of repeating it will terminate

FIGURE 22.14 Jacob notes patterns as he explores the Repeater or Terminator activity

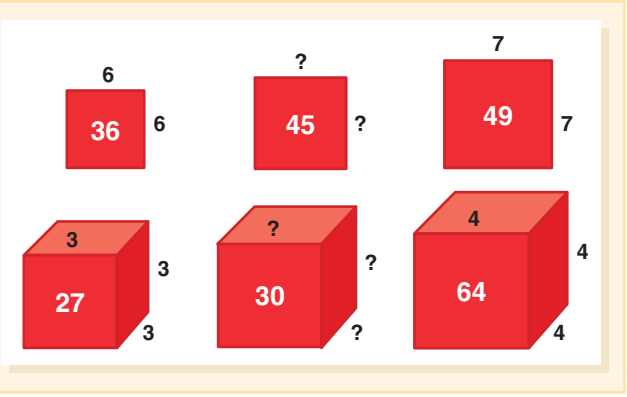


FIGURE 22.15 A geometric interpretation of square roots and cube roots

To solve the cube problem, students might start with 3.5 and then find that 3.53 is 42.875, which is much too large. Through trial and error, they will find that the solution is between 3.1 and 3.2. Continued use of strategic trial and error will lead to a close approximation. Although a calculator can find these square or cube roots quickly, the estimation activity strengthens students' understanding of squares and square roots and the relative sizes of numbers.

From this introduction, students can be challenged to find solutions to equations such as ($\square^2 = 8$). At this point, students are ready to understand the general definition of the nth root of a number x as the number that when raised to the nth power equals x. The square and cube roots are simply other names for the second and third roots. It is important to point out that $\sqrt{6}$ is a *number*, not a computation (because it looks so much like a division problem). The cube root of eight is the same as $\sqrt[3]{8}$, which is equivalent to 2.

In middle school, students encounter irrational numbers primarily when working with the Pythagorean theorem ($a^2 + b^2 = c^2$), which is used to find the distance between two points (the distance being the diagonal, or c) (see Chapter 19 for more). If $a = 3$ and $b = 4$, then $c = 5$. All sides are rational numbers. But this case is the exception to the rule. More often, sides will be something like 4 and 7 units, in which case, $c = \sqrt{16 + 49} = \sqrt{65}$. This is an irrational number.

An engaging middle school project applying the Pythagorean theorem and irrational numbers is a Wheel of Theodorus, as described in Activity 22.18. Theodorus was one of the early believers in the existence of irrational numbers. (This was quite a contentious issue for the Pythagoreans, who reportedly threw another mathematician, Hippasus, off a boat to keep the idea of irrationals a secret (Havil, 2012)).

Activity 22.18 Wheel of Theodorus

CCSS-M: 8.NS.A.2; 8.EE.A.2

Ask students to construct a right triangle that measures 1 centimeter on each side adjacent to the right angle and then draw the hypotenuse and record its measure. They then use the hypotenuse as side *a* of a new right triangle, drawing side *b* 1 cm long. Connect the endpoints of side *a* and side *b*. Draw and record the new hypotenuse $(\sqrt{3})$. Create the next triangle, which will have sides of $\sqrt{3}$ and 1 and a hypotenuse of $\sqrt{4}$ or 2, and so on. Doing this about 30 times will form a spiral-like wheel. (See Bay-Williams & Martinie, 2009, for a complete lesson, or search online for instructions and diagrams of a Wheel of Theodorus.)

Common Challenges Related to Exponents, Negative Numbers, and Real Numbers

Hopefully, a theme that has come through this chapter is that students can reason and understand all kinds of numbers. As they expand the types of numbers they encounter to negative values, very large, very small, and irrational numbers, they must have the opportunity to connect what they already know about number and operations to the new types of numbers. This requires a strong focus on reasoning and looking at structure, which must be part of daily instruction on these topics. Students who do not have these opportunities will likely struggle to make sense of and correctly solve problems involving rational numbers, which leads to difficulties throughout high school and beyond. Table 22.2 provides common challenges related to concepts throughout this chapter, which can serve as a guide to ensure that you are supporting student conceptual understanding and procedural fluency.

TABLE 22.2 COMMON CHALLENGES RELATED TO EXPONENTS, ORDER OF OPERATIONS, NEGATIVE NUMBERS, AND REAL NUMBERS

Common Challenge	What It Looks Like	How to Help
1. Not distinguishing between exponents and multiplication.	Student simplifies 2^3, thinking it is 2×3, and gets an answer of 6 instead of 9.	• Have students write the related multiplication linked to the exponents out in long form. • Avoid confusing language like saying, "It is two three times." • Explore correct and incorrect worked examples, such as in Activity 22.6, to explicitly discuss this error (and others).
2. *Order of Operations.* Thinking addition precedes subtraction.	Simplifies the expression $15 - 10 + 5$ by first adding, getting an incorrect answer of 0.	• Do not use PEMDAS as the way to remember the order of operations. • Give many examples where subtraction appears before addition in the expression. • Use worked examples with these errors in it. • Use a visual, like the triangle in Figure 22.3
3. *Order of Operations.* Thinking multiplication precedes division.	Simplifies the expression $60 \div 10 \times 2$ by first multiplying, getting an incorrect answer of 3.	

Common Challenge	What It Looks Like	How to Help
4. In scientific notation, thinking the exponent tells the number of zeros to add on to a number.	Student writes expanded form of 3.6×10^{14} as 3,600,000,000,000,000, with 14 zeros.	• Avoid giving short cuts that don't make sense to students or are not conceptually grounded. • Ask questions to bring attention to this challenge. For example, "Why are there 13 zeros and not 14? "Is there a relationship between the exponent and the number of zeros?" • Explore much smaller numbers (e.g., 5.9×10^3), writing the number different ways and noticing patterns that are true/not true.
5. Locating negative non-integer values on the number line.	Student places $^-1\frac{1}{4}$ between $^-1$ and 0 instead of between $^-2$ and $^-1$.	• Use more non-integer values through instruction of negative numbers. • Use the number line frequently to illustrate location of values and to do operations. • Don't forget to use negative numbers when lesson goals focus on decimal and fraction operations.
6. Multiple meanings of the minus sign.	Student does not know the meaning of $-(^-8)$ because they don't see a number to subtract from; student thinks they should multiply an expression such as $8.6 - (^-4)$	• Explicitly teach the three meanings of the minus sign. • In simplifying expressions, repeatedly ask, "What does this symbol mean here?" • Use Activity 22.11
7. Making sense of the meaning of operations involving negative numbers.	Student thinks that $9 - 18$ is impossible; Student does not know whether $^-5.7 - 9$ should be less than or more than $^-5.7$. Student memorizes that a positive times a negative is a negative without understanding why this is true.	• Use many contexts in teaching integer operations. • Use both counters and number lines, but particularly number lines. • When students get different answers, provide time for students to debate which is correct and why.
8. Thinking square root notation indicates an operation.	Student thinks $\sqrt{6}$ means some number is being divided into 6.	• Emphasize that $\sqrt{6}$ is actually a number in and of itself and, relatedly, $\sqrt[3]{8}$ is equivalent to 2. Both are numbers.

RESOURCES FOR CHAPTER 22

WRITING TO LEARN

1. What strategies can you use to help students understand and appropriately use the order of operations?
2. Explain contexts that can be used for very small and very large numbers. Why use contexts in working with scientific notation?
3. Use a context to explain each of the following:

$$^-10 + {}^+13 = {}^+3 \qquad {}^-4.5 - {}^-9.2 = {}^+4.7$$
$$4 \times {}^-3 = {}^-12 \qquad {}^-82.5 \div 5 = {}^-16.5$$

4. How would you explain the difference between a rational and an irrational number to a middle school student?
5. Students oftentimes get the same answer for these three expressions:

$$2^3 \qquad 2 \times 3 \qquad 3^2$$

For each expression: (a) explain the relationship between the 2 and the 3 and (b) draw a representation.

LITERATURE CONNECTIONS

The Boy Who Dreamed of Infinity

Alznauer (2020)

This is a beautiful story of the famous Indian mathematician Ramanujan (a number theorist) as a child, thinking about small and large numbers and writing his ideas in journals. His ideas, so original, are still used today in thinking about mathematics. This book can be a good connection with any activities on small or large numbers, or just to learn about an Indian mathematician.

Why Pi? and Go Figure!

Ball (2016)

These two nonfiction books offer interesting facts on a range of mathematical topics, including very big numbers. *Why Pi?* uses powers of 10 to describe the galaxy and *Go Figure!* explores exponential growth connected to historical and classic problems.

Your Place in the Universe

Chin (2020)

This award-winning book shares wonderful comparisons and engages students in thinking about very large numbers. Beautifully illustrated!

The Number Devil

Enzensberger (1997)

Full of humor and wit, *The Number Devil* presents a collection of interesting ideas about numbers. Robert, a boy who hates mathematics, meets up with a crafty number devil in each of 12 dreams (one per chapter). On the fourth night's dream, Robert learns about infinitely repeating decimals and the "rutabaga of two" (the square root of two), providing a connection to rational and irrational numbers.

Oh, Yikes! History's Grossest, Wackiest Moments

Masoff and Sirrell (2006)

In this picture-rich reference book, the authors describe important historical events and people with facts that are interesting to middle schoolers (e.g., "Aztec Antics," "Cruel Constructions," "Humongous Hoaxes"). Selecting a topic, such as brushing teeth, the author describes how this was handled across all of history—an opportunity for timelines that include dates such as 2500 B.C.

RECOMMENDED READINGS

Articles

Dupree, K. M. (2016). Questioning the order of operations. *Mathematics Teaching in the Middle School*, *22*(3): 152–159.

Mnemonics perpetuate misconceptions. This article provides examples of student misconceptions and describes a more effective way to teach this critically important topic.

Swanson, P. E. (2010). The intersection of language and mathematics. *Mathematics Teaching in the Middle School*, *15*(9), 516–523.

Weaving in many strategies to support multilingual learners, the author shares how she engaged students in real contexts to learn integer operations. Visuals and language support are used to ensure that students understand.

Appendix A

Standards for Mathematical Practice

The Standards for Mathematical Practice found in the *Common Core State Standards* describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with long-standing importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands for mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations, and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently, and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems and try special cases and simpler forms of the original problem to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, and graph data and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems by using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments by using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships, using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently and express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school, they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Source: Council of Chief State School Officers. (2010). *Common Core State Standards*. Copyright © 2010 National Governors Association Center for Best Practices and Council of Chief State School Officers. All rights reserved.

Appendix B

NCTM Mathematics Teaching Practices from *Principles to Actions*

Establish mathematics goals to focus learning.

Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

Implement tasks that promote reasoning and problem solving.

Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

Use and connect mathematical representations.

Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.

Facilitate meaningful mathematical discourse.

Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Pose purposeful questions.

Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.

Build procedural fluency from conceptual understanding.

Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

Support productive struggle in learning mathematics.

Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

Elicit and use evidence of student thinking.

Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

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Appendix C

Guide to Blackline Masters

This Appendix has images of all 45 Blackline Masters (BLM) to engage your students in activities throughout the book (and beyond). You can create full-sized masters from these images or print them from this URL: https://media.pearsoncmg.com/ab/TD11E-resources/ir_page/ir_index.html

A good way to have many materials made quickly and easily for students is to have them duplicated on card stock, or copied on colored paper and laminated, and then cut into smaller pieces if desired. Once cut, materials are best kept in clear freezer bags with zip-type closures. Punch a hole near the top of the bag so that you do not store air.

Blackline Master	Number
Addition and Subtraction Recording Sheets	20
Bar Diagrams – Blank	41
Base-Ten Materials	32
Circular Fraction Pieces	33
Clock Faces	31
Coordinate Grid – Quadrant I	22
Coordinate Grid – 4 Quadrants	23
Degrees and Wedges	30
Dot Cards	45
Dot Paper – 1-Centimeter	8
Dot Strips	43
Five-Frame	12
Geoboard Pattern (5 by 5)	26
Geoboard Pattern (10 by 10)	28
Geoboard Recording Sheets (5 by 5)	27
Geoboard Recording Sheets (10 by 10)	29
Grid Paper – 0.5 Centimeter	7
Grid Paper – 1-Centimeter	6
Grid Paper – 2-Centimeter	5
Grid Paper – Base-Ten	18
Grid Paper – 10,000	19
Grids 10 by 10	25
Hundred Chart – Blank	2
Hundred Chart	3
Hundred Charts – Four Small	4
Hundred Chart – Bottom-Up	36
Isometric Dot Paper – 1-Centimeter	10
Isometric Grid Paper – 2-Centimeter	9
Multiplication and Division Recording Sheets	21

(*continued*)

Blackline Master	Number
Multiplication Array 10 × 10	16
Number Cards 0–10	1
Number Lines: 0–2 and 0–3	39
Number Lines – Blank	40
Open Array Template – 2-Digit	35
Place-Value Mat (with Double Ten-Frames)	17
Rational Number Wheel	24
Sidebar Chart	34
Smiling Decimal Point	37
Square/Diagonal Grid Paper – 1-Centimeter	11
Tangram Pieces	38
Ten-Frame	13
Ten-Frame (Horizontal)	14
Ten-Frame – Double	15
Ten Frames – Little	42
Two-Column Cards	44

0	1	2	3
4	5	6	7
8	9	10	

Number Cards 0–10— 1

Blank Hundreds Chart—2

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Hundreds Chart—3

Four Small Hundreds Charts—4

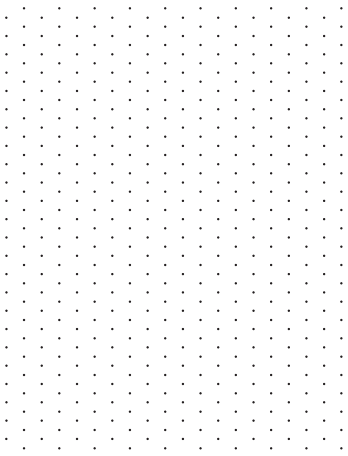
2-Centimeter Grid Paper—5

1-Centimeter Grid Paper—6

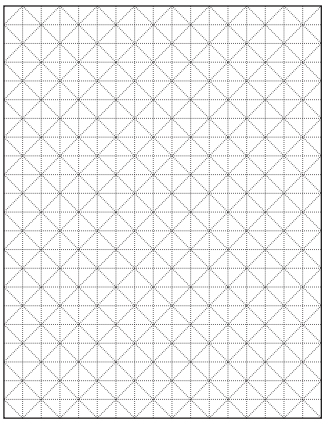
0.5-Centimeter Grid Paper—7

1-Centimeter Dot Paper—8

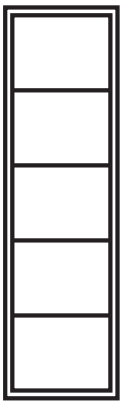
2-Centimeter Isometric Grid Paper—9



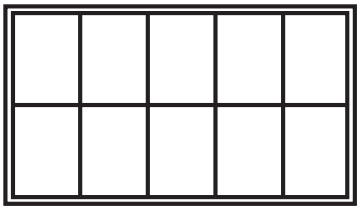
1-Centimeter Isometric Dot Paper—10



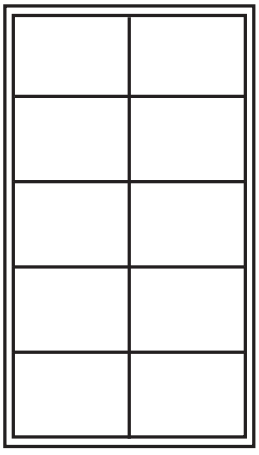
1-Centimeter Square/Diagonal Grid Paper—11



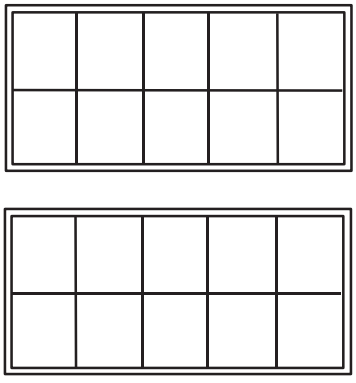
Five-Frame—12



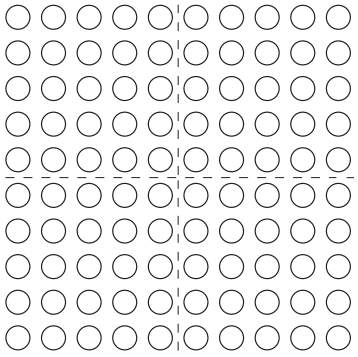
Ten-Frame—13



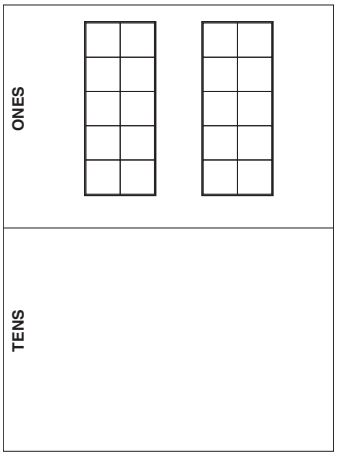
Ten-Frame Horizontal—14



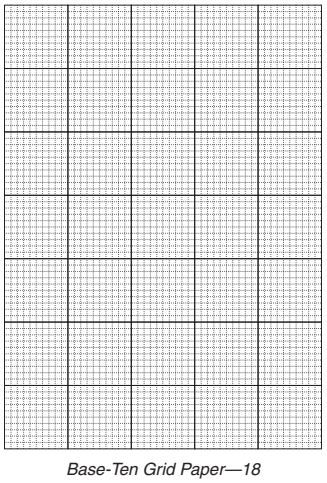
Double Ten-Frame—15



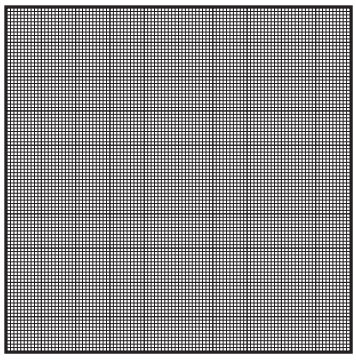
10 × 10 Multiplication Array— 16



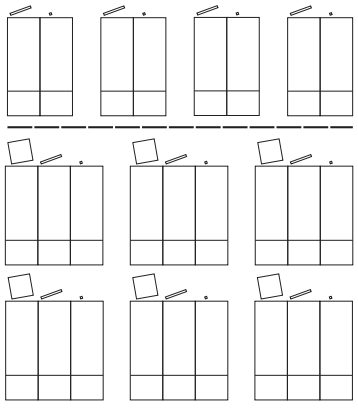
Place-Value Mat (with Ten-Frames)—17



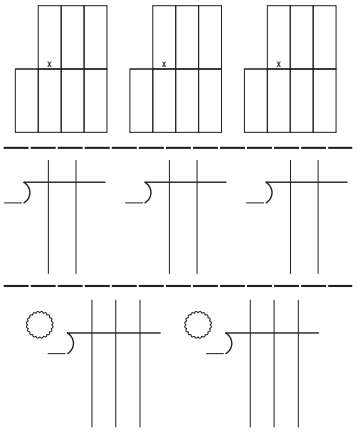
Base-Ten Grid Paper—18



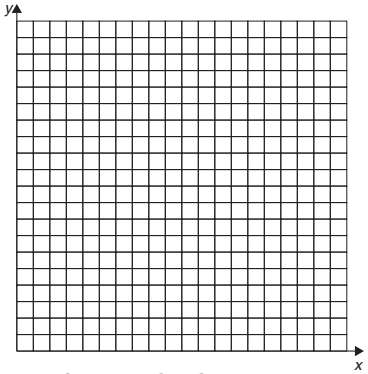
10,000 Grid Paper—19



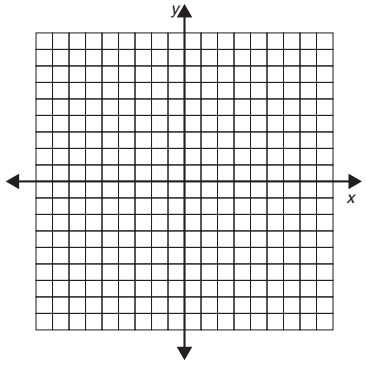
Addition and Subtraction Recording Sheets—20



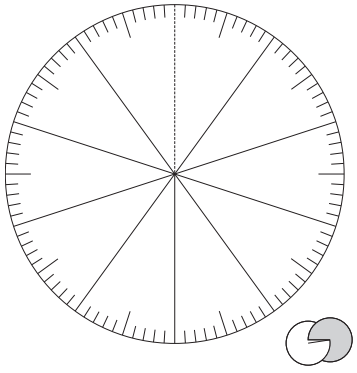
Multiplication and Division Recording Sheets—21



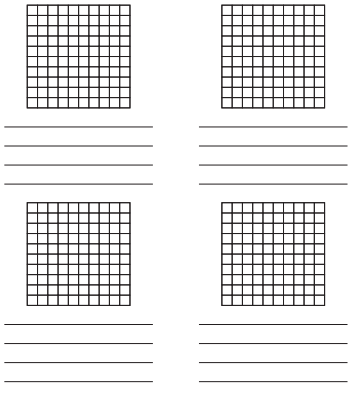
Coordinate Grid-Quadrant I—22



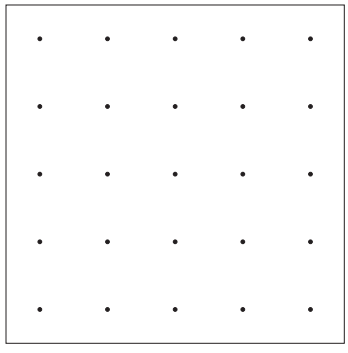
Coordinate Grid-4 Quadrants—23



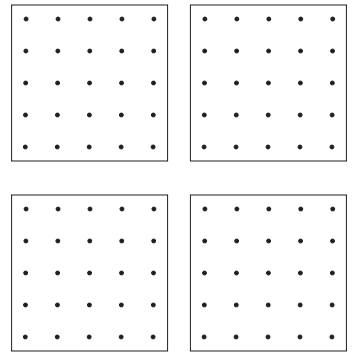
Rational Number Wheel—24



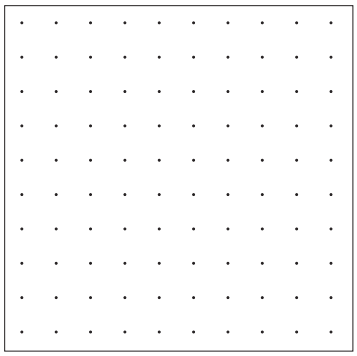
10 × 10 Grids—25



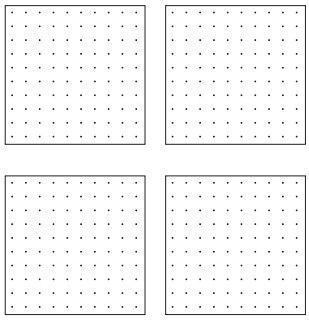
Geoboard Pattern (5 × 5)—26



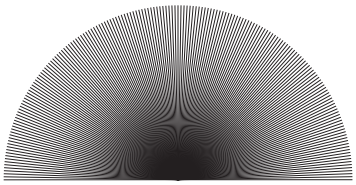
Geoboard Recording Sheets—27



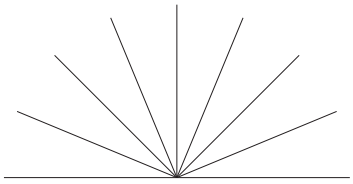
Geoboard Pattern (10 by 10)—28



Geoboard Pattern Recording Sheets (10 by 10)—29

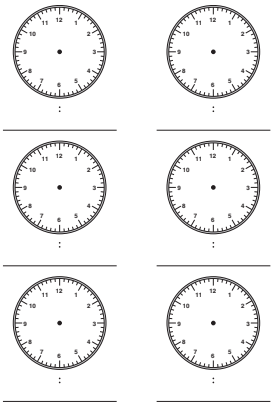


180 Degrees

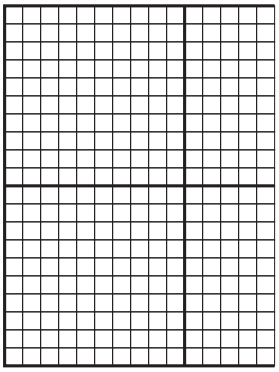


8 Wedges

Degrees and Wedges—30

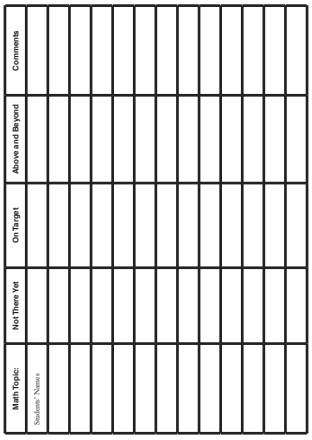


Clock Faces—31



1. Make two copies of this page. Cut out the grid from each copy.
2. Overlap the two grids, and tape onto a blank sheet to form a 20-by-25-cm grid with 4 complete hundreds squares and 2 rows of 5 tens each.
3. Use this as a master to make copies on card stock.

Base-Ten Materials—32



Observation Checklist—33

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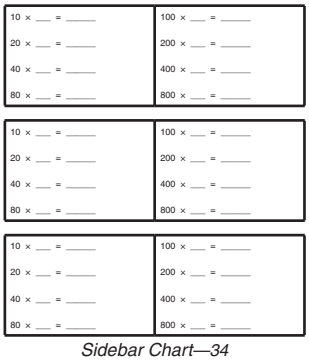

Sidebar Chart—34

Appendix D

Activities at a Glance

This table lists the named and numbered activities in Part II of the book. In addition to providing an easy way to find an activity, the table provides the main mathematics content and the related Common Core State Standards. Remember that this is a book about teaching mathematics and not a book of activities. It is extremely important not to take any activity as a suggestion for instruction without reading the full text in which it is embedded.

Chapter 7 Developing Early Number Concepts and Number Sense

Activity		Mathematics Content	CCSS-M	Page
7.1	Dot Plate Flash	Practice recognition of amounts without counting	K.CC.B.4a, b	135
7.2	Up and Back Counting	Practice skill of counting forward and counting backward	K.CC.A.1 K.CC.A.2	136
7.3	Story Book Counting	Count with a calculator	K.CC.A.1 K.CC.A.2	136
7.4	Fill the Tower	Develop one-to-one counting skills	K.CC.B.5	137
7.5	Counting with the Count	Counting using items and situations around the house	K.CC.A.1 K.CC.A.2	138
7.6	Number Necklaces	Practice number matching and develop basic addition	K.CC.A.3	138
7.7	Number Tubs	Develop counting skills and need for number names	K.CC.B.4 K.CC.B.5	139
7.8	Line Them Up!	Practice number sequence and the use of the number line	K.CC.B.4c K.CC.A.2	139
7.9	Counting On with Counters	Practice counting on	K.CC.A.2 K.CC.B.5	140
7.10	Real Counting On	Practice counting on to tell all	K.OA.A.1 1.OA.B.5	140
7.11	Counting Moose Tracks	Develop one-to-one correspondence and cardinality	K.CC.B.4a	141
7.12	Make Sets of More/Less/Same	Develop relational concept of more/less/same	K.CC.C.6	142
7.13	Find the Same Amount	Identify sets with more/less/same	K.CC.A.3 K.CC.C.6	142
7.14	More, Less, or the Same	Develop concept of 1-more, 2-more, 1-less, 2-less, zero	K.CC.A.3 K.CC.C.6 K.O.A.A.1	142
7.15	Make a Two-More-Than Set	Develop concept of 1-more, 2-more, 1-less, 2-less	K.OA.A.2 1.OA.C.5	144
7.16	A Two-More-Than Machine	Practice 1-more-, 2-more-, 1-less-, 2-less-than relationships	1.OA.C.5	144
7.17	Five-Frame Tell-About	Develop benchmark of 5 for numbers to 10	K.CC.C.7 K.OA.A.5	145
7.18	Number Medley	Develop the benchmark of 10 for numbers to 10	K.CC.C.6 K.CC.C.7	145

Activity		Mathematics Content	CCSS-M	Page
7.19	Little Ten-Frame Flash	Practice benchmarks of 5 and 10 for numbers to 10	K.CC.B.5	146
7.20	Build It in Parts	Practice part-whole concepts in a symbolic form	K.OA.A.3 1.OA.C.6	147
7.21	Ten Hot Dogs	Explore combinations of ten	K.CC.C.6 K.OA. A.3 1.OA.A.1	148
7.22	Covered Parts	Develop missing part concepts	K.OA.A.2	148
7.23	Missing-Part Cards	Practice missing part concepts	K.OA.A.4	149
7.24	I Wish I Had	Practice missing part concepts	K.OA.A.1 K.OA.A.2	149
7.25	Double War	Practice comparison of numbers to 20	K.CC.C.6	150
7.26	Find the Missing Dot Card	Practice missing parts concepts	K.OA.A.2	150
7.27	Number Sandwiches	Practice missing part concepts	K.OA.A.2 1.OA.C.6	151
7.28	Ten and Some More Stories	Develop concept of teen numbers	K.NBT.A.1	152
7.29	More and Less Extended	Extend one-more, two-more, one-less, two-less relationships to teen numbers	K.CC.C.6 K.NBT.A.1	153
7.30	Add a Unit to Your Number	Connect number to real-world measures	K.MD.A.1	155
7.31	Is It Reasonable?	Connect number to real-world measures	K.MD.A.1	155

Chapter 8 Developing Meanings for the Operations

Activity		Mathematics Content	CCSS-M	Page
8.1	Guinea Pigs in Cages	Develop strategies for exploring both addends unknown problems	K.OA.A.3 1.OA.A.1 1.OA.B.4	167
8.2	Missing Part Subtraction	Develop subtraction as a name for a missing part	K.OA.A.2 1.OA.A.1 1.OA.A.2 2.OA.A.1	168
8.3	Bears' Picnic	Practice one more and one less on a number path	K.CC.A.1 K.CC.A.2 1.OA.A.1	170
8.4	Up and Down the Number Line	Practice using units on a number line	K.OA.A.2 1.OA.A.1 1.OA.A.2 2.OA.A.1	170
8.5	More Than Two Addends	Practice adding more than two addends	1.OA.B.2	171
8.6	Boxes and Bags	Develop the concept of equal groups	3.OA.A.1 3.OA.A.3	177
8.7	Chocolate Chip Pancakes	Develop the concept of equal groups	3.OA.A.1 3.OA.A.3	178
8.8	It Stacks Up!	Develop of concepts of equal groups	3.OA.A.1 3.OA.A.3	178
8.9	Divide and Conquer	Develop measurement and partition concepts of division	3.OA.A.2	178

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Chapter 8 (*Continued*)

Activity		Mathematics Content	CCSS-M	Page
8.10	Factor Quest	Develop the connection between multiplication and division	4.OA.B.4	179
8.11	Factor Patterns	Develop patterns between arrays and factors	4.OA.B.4	180
8.12	Split it Up	Explore the distributive property with arrays	3.OA.B.5 5.OA.A.1	180
8.13	The Broken Division Key	Develop division as repeated subtraction and missing factor	4.NBT.B.6	181
8.14	Word Problem Sort	Classify multiplicative word problems	3.OA.A.1 3.OA.A.2	182

Chapter 9 Developing Basic Fact Fluency

Activity		Mathematics Content	CCSS-M	Page
9.1	Paired Story Problems	Use known facts to determine unknown facts	1.OA.A.1 1.OA.C.6 2.OA.B.2	192
9.2	How Many Feet in the Bed?	Practice facts for $+2$ and -2	1.OA.A.1 1.OA.C.6 2.OA.B.2	195
9.3	Game: First Bean to 19	Practice addition facts for $+1$ and $+2$	1.OA.C.5 1.OA.C.6 2.OA.B.2	195
9.4	Game: Dart to the Start	Practice subtraction facts for -1 and -2	1.OA.C.5 1.OA.C.6 2.OA.B.2	196
9.5	Zero Has an Identify	Practice addition and subtraction with 0	1.OA.C.5 1.OA.C.6 2.OA.B.2	196
9.6	Double the Ducks	Practice addition doubles facts	1.OA.C.5 1.OA.C.6 2.OA.B.2	196
9.7	Calculator Doubles	Practice addition doubles facts	1.OA.C.6 2.OA.B.2	197
9.8	How Many More to Equal 10?	Practice combinations of 10 facts	1.OA.B.4 1.OA.C.6 2.OA.B.2	197
9.9	Game: Go Fish for 10s	Practice combinations of 10 facts	1.OA.B.4 1.OA.C.6 2.OA.B.2	197
9.10	Game: Teen Time	Practice combinations of 10+ a one digit number	1.OA.B.4 1.OA.C.6 2.OA.B.2	198
9.11	Game: Flip and Find	Practice using the Near-Double strategy	1.OA.B.4 1.OA.C.6 2.OA.B.2	199
9.12	Move It, Move It	Develop Making 10 strategy	1.OA.B.3 1.OA.C.6 2.OA.B.2	200

Activity	Mathematics Content	CCSS-M	Page
9.13 What's Under my Thumb?	Connect addition and subtraction facts or multiplication and division facts	1.OA.B.4 1.OA.C.6 2.OA.B.2 3.OA.B.5 3.OA.B.6 3.OA.C.7	201
9.14 Apples in the Trees	Practice subtraction to 20	1.OA.B.4 1.OA.C.6 2.OA.B.2	202
9.15 Apples in Two Trees	Develop missing-value concept, relating addition to subtraction	1.OA.B.3 1.OA.B.4 1.OA.C.6 2.OA.B.2	202
9.16 Game: Lucky 13	Practice strategies for addition	1.OA.B.3 1.OA.C.6 2.OA.B.2	203
9.17 Game: Pick 2 to Connect 4	Practice strategies for addition	1.OA.B.3 1.OA.C.6 2.OA.B.2	203
9.18 Game: Take it Away	Practice strategies for subtraction	1.OA.B.4 1.OA.C.6 2.OA.B.2	204
9.19 Clock Facts	Develop minute intervals on the clock as a strategy for fives multiplication facts	3.OA.A.1 3.OA.C.7	206
9.20 Game: Squares Score	Practice square numbers	1.OA.B.4 1.OA.C.6 2.OA.B.2 3.OA.B.5 3.OA.B.6 3.OA.C.7	207
9.21 How Close without Going Over?	Looking for near-division facts	3.OA.A.2 3.OA.A.4 3.OA.B.6	210
9.22 Game: Strive to Derive	Use known facts to determine unknown facts	3.OA.A.1 3.OA.B.5 3.OA.C.7	211
9.23 Game: Bump Multiplication	Practice with finding multiples	3.OA.A.1 3.OA.B.5 3.OA.C.7	211
9.24 Game: Salute!	Identify the missing addend or factor	1.OA.B.4 1.OA.C.6 2.OA.B.2 3.OA.B.5 3.OA.B.6 3.OA.C.7	212
9.25 Game: Bowl-a-Fact	Practice creating equations in addition, subtraction, multiplication, and division	2.OA.B.2 3.OA.C.7 5.OA.A.1	212
9.26 If You Didn't Know	Use known facts to determine unknown facts	1.OA.C.6 2.OA.B.2 3.OA.B.5 3.OA.C.7	214

Chapter 10 Developing Whole-Number Place-Value

Activity		Mathematics Content	CCSS-M	Page
10.1	Counting in Groups	Develop concept of groups of ten as an efficient method of counting	K.NBT.A.1 1.NBT.B.2a	225
10.2	How Many in the Bag?	Develop concept of groups of ten as an efficient method of counting	K.NBT.A.1 1.NBT.B.2a 1.NBT.B.2b 1.NBT.B.2c	225
10.3	How Long Am I?	Develop attention to ten as a length unit	K.NBT.A.1 1.NBT.B.2a 1.NBT.B.2b	226
10.4	Too Many to Count?	Estimate and group quantities into hundreds, tens, and ones	2.NBT.A.1	227
10.5	Can You Make the Link?	Develop alternative groupings of ten to represent a number	1.NBT.B.2 1.NBT.C.5	228
10.6	Three Other Ways	Develop alternative groupings of tens and hundreds to represent a number	2.NBT.A.1 2.NBT.A.3 K.CC.B.5	228
10.7	Base-Ten Riddles	Develop alternative groupings of tens and hundreds to represent a number	1.NBT.A.1 2.NBT.A.1 2.NBT.A.3	229
10.8	Tens, Ones, and Fingers	Develop and connect three oral counting strategies	1.NBT.B.2a, b, c	229
10.9	Counting with Base-Ten Models	Develop and connect three oral counting strategies	1.NBT.B.2 1.NBT.C.5	230
10.10	Counting Rows of Ten	Develop and connect three oral counting strategies	1.NBT.B.2c	230
10.11	Building Numbers	Connect oral and symbolic names for numbers to physical representation	1.NBT.B.2 2.NBT.A.1	232
10.12	Say It/Press It	Connect oral and symbolic names for numbers to physical representation	2.NBT.A.1a, b 2.NBT.A.3	232
10.13	Digit Change	Apply place-value concepts to symbolic representations	1.NBT.B.2 1.NBT.C.5 1.NBT.C.6 2.NBT.A.1 2.NBT.A.3 2.NBT.B.5 2.NBT.B.8	233
10.14	Missing Numbers	Practice sequence of numbers to 100	K.CC.A.1 K.NBT.A.1 1.NBT.A.1 1.NBT.B.2 1.NBT.C.5	235
10.15	Finding Neighbors on the Hundred Chart	Explore patterns in numbers to 100	K.CC.A.1 K.NBT.A.1 1.NBT.A.1 1.NBT.B.2 1.NBT.C.5	235
10.16	Models with the Hundred Chart	Develop concepts of 1-more/less and 10-more/less for two-digit numbers	K.CC.A.1 K.NBT.A.1 1.NBT.A.1 1.NBT.B.2 1.NBT.C.5	236

Activity		Mathematics Content	CCSS-M	Page
10.17	Arrow around the Bottom-Up Hundred Chart	Develop concepts of 1-more/less and 10-more/less for two-digit numbers	1.NBT.C.4 1.NBT.C.5 1.NBT.C.6 2.NBT.B.5 2.NBT.B.6	236
10.18	A Thousand Chart	Extend patterns for 1 to 100 to patterns to 1000	2.NBT.A.1 2.NBT.A.2 2.NBT.A.3 2.NBT.B.8	236
10.19	Who Am I?	Develop relative magnitude of numbers to 100	1.NBT.A.1 1.NBT.B.2 1.NBT.B.3 2.NBT.A.1 2.NBT.A.2 2.NBT.A.4	237
10.20	Who Could They Be?	Develop relative magnitude of numbers to 100	1.NBT.A.1 1.NBT.B.2 1.NBT.B.3 2.NBT.A.1 2.NBT.A.2 2.NBT.A.4	237
10.21	Close, Far, and In Between	Explore relative differences between three-digit numbers	1.NBT.A.1 1.NBT.B.2 1.NBT.B.3 2.NBT.A.1 2.NBT.A.2 2.NBT.A.4	238
10.22	What Comes Next? (Concrete)	Develop the continuing pattern in the place-value system	2.NBT.A.1 2.NBT.A.3	240
10.23	What Comes Next? (Semi-Concrete)	Develop the continuing pattern in the place-value system	2.NBT.A.1 2.NBT.A.3	240
10.24	Collecting 10,000	Develop an understanding of the size of large numbers	2.NBT.A.2 2.NBT.A.3	241
10.25	Showing 10,000	Develop an understanding of the size of large numbers	2.NBT.A.2 2.NBT.A.3	242
10.26	How Long?/How Far?	Develop an understanding of the size of large numbers	2.NBT.A.1a 2.MD.A.3	242
10.27	A Long Time	Connect units of time to large numbers	3.MD.A.1	242
10.28	Really Large Quantities	Develop ability to estimate large quantities	2.MD.A.3 3.MD.A.1 3.MD.C.5	242

Chapter 11 Developing Strategies for Addition and Subtraction Computation

Activity		Mathematics Content	CCSS-M	Page
11.1	Calculator Challenge Counting	Develop mental addition strategies through skip counting	1.NBT.C.5	248
11.2	The Other Part of 100	Develop missing part strategies with a whole of 100	1.NBT.B.4 2.NBT.B.5	249
11.3	50 and Some More	Develop 50 as a part of numbers between 50 and 100	1.NBT.B.4	249

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Activity		Mathematics Content	CCSS-M	Page
11.4	Compatible Pairs Search	Explore addition combinations that make multiples of 10 or 100	2.NBT.B.5	249
11.5	Numbers, Squares, Lines, and Dots	Develop invented strategies for addition and subtraction	2.NBT.B.7	255
11.6	Little Ten-Frame Sums	Develop invented strategies for addition	2.NBT.B.5	256
11.7	Hundred Chart Addition	Practice adding two-or three-digit numbers	2.NBT.B.5	256
11.8	How Far to My Number?	Develop missing-part strategies for two-digit numbers	2.NBT.B.5 2.NBT.B.9	258
11.9	How Much Between?	Develop strategies to find the difference	2.NBT.B.5	259
11.10	Celebrating Book Birthdays	Use think addition and skip counting to find the difference	2.NBT.B.5 2.NBT.B.8	259
11.11	Just Adjust It	Practice using algebraic thinking to add related problems	2.NBT.B.5 2.NBT.B.9	262
11.12	Crossing a Decade	Extend the Making 10 strategy to two-digit numbers	1.NBT.C.4 2.OA.A.2	263
11.13	I Am, but Who Is?	Practice mental math with addition and subtraction of decade numbers	1.NBT.C.4 1.NBT.C.5 1.NBT.C.6	264
11.14	Exploring Subtraction Strategies	Practice solving word problems with two-digit numbers	2.OA.A.1	265
11.15	City Populations	Develop and solve word problems using invented strategies	2.NBT.B.5 2.NBT.B.6 2.NBT.B.7 3.NBT.B.2 4.NBT.B.2 4.NBT.B.4	266
11.16	Trading across Zero	Develop strategies to subtract with internal zeros in the minuend	2.NBT.B.7 2.NBT.B.9	270
11.17	Pick Your Strategy	Develop the ability to choose a strategy	1.NBT.C.4 2.NBT.B.5 2.NBT.B.6 2.NBT.B.7 2.NBT.B.9	271
11.18	Over or Under?	Develop strategies to make computational estimations	3.OA.C.8	273
11.19	Round Up?	Develop strategies to round numbers to the nearest ten or hundred	3.NBT.A.1	274
11.20	Box Math	Practice computation and estimation through problem solving	2.NBT.B.5	275

Chapter 12 Developing Strategies for Multiplication and Division Computation

Activity		Mathematics Content	CCSS-M	Page
12.1	Halve and Double	Develop the halve and double strategy	4.NBT.B.5	283
12.2	Make the Most of It	Strategize to produce the largest product	4.NBT.B.5	286
12.3	Carpet Tiles	Develop the multiplication concept through the area model	4.NBT.B.5	287
12.4	Build It and Break It	Develop the distributive property	4.NBT.B.5	288

Activity		Mathematics Content	CCSS-M	Page
12.5	Open Array Foldables	Develop partial products strategy through an open array model	4.NBT.B.5	289
12.6	Sharing Sea Shells	Develop the idea of sharing equal groups with multi-digit numbers	4.NBT.B.6	295
12.7	Left Overs	Develop the concept of remainders	4.NBT.B.6	296
12.8	Double, Double—No Toil or Trouble!	Develop a division strategy through doubling	5.NBT.B.6	299
12.9	High or Low?	Develop estimates of double digit multiplication	4.NBT.A.3 4.NBT.B.5	301
12.10	What Was Their Method?	Develop estimation strategies	4.NBT.A.3 4.NBT.B.5	303
12.11	Jump to It	Estimate division using a missing factor strategy	4.NBT.B.5 4.NBT.A.3	303
12.12	Hit the Target	Estimate for any of the four operations on a calculator	4.NBT.B.5	303

Chapter 13 Algebraic Thinking

Activity		Mathematics Content	CCSS-M	Page
13.1	Seesaw Students	Develop the concept of the equal sign as a balance	1.OA.D.7 2.NBT.A.4	310
13.2	Tilt or Balance?	Develop understanding of the equal sign and the less-than and greater-than symbols	2.OA.A.4 3.OA.A.4 5.NBT.A.3a, b 6.EE.A.4	310
13.3	True or False?	Explore the meaning of the equal sign	1.OA.B.3 1.OA.D.7 1.NBT.B.4 2.NBT.B.5 3.OA.B.5 4.NBT.B.5 5.NF.A.1	312
13.4	Make a Statement!	Create equivalent expressions	1.OA.D.7 1.OA.D.8 2.OA.A.1 3.OA.A.4 4.NF.B.3a 5.OA.A.2 6.EE.B.8	312
13.5	Ten and Some More Equations	Develop idea of equivalence while supporting basic fact development	1.OA.C.6 1.OA.D.7 2.OA.B.2	313
13.6	Border Tiles Expressions—Part 1	Explore a pattern to develop a generalization	5.OA.A.1 5.OA.A.2 6.EE.A.2b	314
13.7	Broken Calculator: Can You Fix It?	Explore properties of odd and even numbers	1.OA.B.3 2.OA.C.3 2.NBT.B.5 3.OA.D.9	315
13.8	Birds, Turtles, and Boats	Developing situations to explore the commutative property	1.NBT.C.4 2.NBT.B.9 3.OA.D.9	317

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Activity		Mathematics Content	CCSS-M	Page
13.9	Five Ways to Zero	Apply properties, such as identity for addition and/or multiplication, to creating equations	1.OA.B.3 1.OA.C.6 1.OA.D.7 2.OA.B.2 3.OA.D.9	320
13.10	Convince Me Conjectures	Make and test generalizations about whole number operations (and properties)	1.OA.B.3 2.NBT.B.9 3.OA.B.5 5.OA.A.1	322
13.11	Diagonal Sums	Explore place-value relationships and generalize patterns	1.NBT.C.4 2.NBT.B.9 3.OA.D.9	322
13.12	Border Tiles Expressions—Part 2	Write expressions to represent a situation; show variable expressions are equivalent	6.EE.A.1 6.EE.A.2a, b, c 6.EE.A.3 6.EE.A.4	323
13.13	Solving the Mystery	Explore properties and equivalent expressions	5.OA.A.2 6.EE.A.2a 7.EE.A.2	324
13.14	What's Missing?	Explore the meaning of the equal sign	1.OA.D.7 1.OA.D.8 2.OA.A.1 3.OA.A.4 5.OA.A.2 6.EE.A.3 6.EE.B.5	325
13.15	Ball Weights	Explore variables in context	6.EE.A.4 7.EE.A.2 8.EE.C.8b	326
13.16	Predict Down the Line	Explore the structure of repeating patterns analytically	2.OA.C.3 4.OA.A.3 4.OA.B.4	329
13.17	Hurricane Names	Explore the structure of repeating patterns analytically	4.OA.A.3 4.OA.B.4 5.OA.B.3 6.EE.B.2a	329
13.18	Predict How Many	Develop functional relationships in growing patterns	5.OA.B.3 6.EE.C.9 7.EE.B.4a 8.F.A.1	330
13.19	Perimeter Patterns	Generalize and graph geometric growing patterns (functions)	6.EE.C.9 7.EE.B.4a 8.F.A.1 8.F.A.2	333
13.20	The Grocery Store	Exploring contexts that represent linear relationships	5.MD.C.5b 6.EE.A.2c 6.G.A.2	333
13.21	G.E.T.S. Match-It	Exploring and comparing linear functions in abstract forms	7.EE.B.3 7.EE.B.4 8.FA.1, 8.FA.2	334
13.22	Is It Linear?	Explore graphical representations of functional relationships	8.FA.1, 8.FA.3	334
13.23	Designing the Largest Box	Explore volume and surface area algebraically	5.MD.C.5b 6.EE.A.2c 6.G.A.2	338

Chapter 14 Developing Fraction Concepts

Activity		Mathematics Content	CCSS-M	Page
14.1	Playground Fractions	Develop concepts of equal shares using an area model	1.G.A.3 2.G.A.3 3.NF.A.1	347
14.2	Who Is Winning?	Develop fractional concepts using a linear model	3.NF.A.2a, b 3.NF.A.3a, b, d	349
14.3	Class Fractions	Develop concepts of fractions using a set model	3.NF.A.1 3.NF.A.3b	350
14.4	Cut Me a Fair Share!	Explore a sharing task and connect to partitioning	1.G.A.3 2.G.A.3	352
14.5	How Much Cake to Take?	Explore equal shares.	1.G.A.3 2.G.A.3 3.NF.A.1	355
14.6	Partitioning: Fourths or Not Fourths?	Develop understanding of fractional parts	1.G.A.3 2.G.A.3 3.NF.A.1	355
14.7	Geoboard Finding Fourths	Exploring fractions of the same size but not the same shape	1.G.A.3 2.G.A.3 3.NF.A.1	356
14.8	Make Fraction Strips	Exploring partitioning with length models	3.NF.A.1 3.NF.A.2a, b	356
14.9	What Fraction Is Colored?	Determine shaded region on number strips where not all partitions are shown	3.NF.A.1 3.NF.A.2a, b	357
14.10	How Far Did Nicole Go?	Determine shaded region on number strips where not all partitions are shown	3.NF.A.1 3.NF.A.2a, b	358
14.11	Paper Plate Pieces	Iterate fraction parts using area models	3.NF.A.1 3.NF.A.3c 3.NF.A.3d	360
14.12	Pattern Block Creatures	Build a whole given a part	3.NF.A.1 3.NF.A.3c	361
14.13	Rows of Rods: More, Less, or Equal to One Whole	Develop understanding of fractional parts	3.NF.A.1 3.NF.A.2a, b	361
14.14	Counting Counters	Iterating fraction parts using set models	3.NF.A.1 3.NF.A.3c	362
14.15	Calculator Fraction Counting	Explore counting by a fraction unit	3.NF.A.1 3.NF.A.3a, c	363
14.16	A Whole Lot of Fun	Partition and iterate to determine lengths of paper strips	3.NF.A.1 3.NF.A.2a, b	363
14.17	Estimate and Iterate	Explore the connection between partitioning and iterating	3.NF.A.1 3.NF.A.2a, b	363
14.18	Fractions *on* the Number Line	Explore magnitude of fractions using a number line	3.NF.A.2a 3.NF.A.2b	364
14.19	Zero, One-Half, or One	Estimate relative magnitude of fractions related to benchmarks	3.NF.A.3d 4.NF.A.2	365
14.20	Making Stacks	Introduce to equivalent fractions using pattern blocks	3.NF.A.1 3.NF.3a, b, c	367
14.21	Dot Paper Equivalencies	Develop an understanding of the concept of equivalent fractions	3.NF.A.1 3.NF.3a, b, c	368

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14.22	LEGOLAND® : Building Options	Explore fraction equivalencies in model that can be area or set	2.G.A. 3 3.NF.A.1 3.NF.3a, b, c 4.NF.B.3a, b	369
14.23	Apples and Bananas	Explore equivalence using a set model	3.NF.A.1 3.NF.3a, b, c	369
14.24	Missing-Number Equivalencies	Use counters or rectangles to find equivalent fractions	3.NF.3a, b, c 4.NF.A.1	370
14.25	Garden Plots	Develop an equivalent fraction algorithm	3.NF.3b 4.NF.A.1	372
14.26	Stretching Number Lines	Explore equivalence of fractions and comparing fractions	3.NF.A.3a, b, d 4.NF.A.2	374
14.27	Visualizing and Ordering Unit Fractions	Compare and order unit fractions	3.NF.A.3d 4.NF.A.2	374
14.28	Which Fraction Is Greater?	Estimate size of fractions	4.a.NF.2	375

Chapter 15 Developing Fraction Operations

Activity		Mathematics Content	CCSS-M	Page
15.1	Gardening Together	Explore addition and subtraction of fractions in an area model	4.NF.B.3a, d 5.NF.A.1 5.NF.A.2	386
15.2	Jumps on the Ruler	Explore addition and subtraction of fractions in a linear model	4.NF.B.3a, d 5.NF.A.1 5.NF.A.2	387
15.3	Over or Under 1	Develop estimation of sums and differences of fractions	4.NF.B.3a 5.NF.A.2	388
15.4	Cups of Milk	Estimate addition and subtraction problems using a linear context (measuring cup)	4.NF.B.3a 5.NF.A.2	389
15.5	Can You Make It True?	Find missing values in addition and subtraction problems	4.NF.B.3a 5.NF.A.2	390
15.6	Game: How Low Can You Go?	Identify common multiples	4.OA.B.4 5.NF.A.1	391
15.7	Least-to-Greatest Fraction Sort	Estimate the magnitude of fractions	4.NF.B.3a 5.NF.A.2	393
15.8	Hexagon Wholes	Explore multiplication of a fraction by a whole number (e.g., $3 \times \frac{1}{4}$)	4.NF.B.4a, b	395
15.9	Jumps on the Ruler—Multiplication	Explore multiplication of fractions with a linear model	4.NF.B.4a, b	396
15.10	How Big Is the Banner?	Explore multiplication of a whole number by a fraction (e.g., $\frac{1}{2} \times 6$)	5.NF.B.4a, b	396
15.11	Counting Counters: Fraction of a Whole	Find parts of a whole using a set model	5.NF.B.4a, b	397
15.12	Solving Fraction Stories	Find a fraction of a fraction	5.NF.B.4a, b 5.NF.B.6	398
15.13	Quilting Pieces	Develop multiplication of fractions with an area model	5.NF.B4b 5.NF.B.5b	399

Activity		Mathematics Content	CCSS-M	Page
15.14	Part of a Part of a Lot	Explore multiplication of fractions and the commutative property	5.NF.B.4b 5.NF.B.6	400
15.15	Fraction Sentence Creation	Develop strategy fluency	5.NF.B.4b 5.NF.B.6	401
15.16	Can You See It?	Relate parts to wholes to explore multiplication of fractions greater than 1	5.NF.B.3 5.NF.B.5a, b	402
15.17	Plotting Your Garden Plots	Explore a fraction divided by a whole number	5.NF.B.7a, c	405
15.18	Sandwich Servings	Develop meaning for division by a fraction	5.NF.B.7b, c	406
15.19	Counting Counters: How Much in One Whole Set?	Explore whole number divided by a fraction	5.NF.B.7a, c	407
15.20	How Much for One?	Explore fraction divided by a fraction	6.NS.A.1	409
15.21	The Size Is Right: Division	Estimate division of fraction problems	5.NF.B.7a, b, c 6.NS.A.1	411

Chapter 16 Developing Decimal and Percent Concepts and Decimal Computation

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16.1	The Decimal Point Names the Unit	Develop an understanding of the purpose of the decimal point	4.NF.C.6 5.NBT.A.3a	417
16.2	Calculator Decimal Counting	Develop an understanding of the patterns in decimal notation	4.NF.C.6 5.NBT.A.1	418
16.3	Building Decimals	Develop an understanding of the patterns in decimal notation	4.NF.C.6 5.NBT.A.1 5.NBT.A.2 5.NBT.A.3a	421
16.4	Shifting Units	Practice shifting the unit in place value	4.NF.C.6 5.NBT.A.1 5.NBT.A.2 5.NBT.A.3a	421
16.5	Decimal Roll and Cover	Explore the size of decimals using an area model	4.NF.C.5 4.NF.C.6 5.NBT.A.1 5.NBT.A.3a	422
16.6	Place and Compare	Explore the magnitude of decimals using a length model	4.NF.C.6 4.NF.C.7	423
16.7	The Amazing Race	Explore the magnitude of decimals using a length model	4.NF.C.6 4.NF.C.7 5.NBT.A.3b	424
16.8	Show Me the Money!	Develop a connection between decimals, money and fraction notation	4.NF.C.6 5.NBT.A.3a	424
16.9	Build It, Name It	Practice connecting the decimal notation to a physical model	4.NF.C.6 5.NBT.A.1	425
16.10	Familiar Fractions to Decimals	Develop a conceptual connection between fractions and decimal notations	4.NF.C.7	426
16.11	Estimate, Then Verify	Develop a conceptual connection between fractions and decimal notations	4.NF.C.7	427
16.12	Decimals and Fractions on a Double Number Line	Develop a conceptual connection between fractions and decimal notations	4.NF.C.6	427

(*Continued*)

Chapter 16 (*Continued*)

Activity		Mathematics Content	CCSS-M	Page
16.13	Best Match	Practice estimation of decimal numbers with simple fractions	4.NF.C.6	428
16.14	Line 'Em Up	Develop an understanding of the way that decimal numbers are ordered	4.NF.C.6 4.NF.C.7 5.NBT.A.1 5.NBT.A.3a 5.NBT.A.3b	430
16.15	Close Decimals	Develop an understanding of the relative size of decimal numbers	5.NBT.A.3b	431
16.16	Zoom	Explore the density of decimals	5.NBT.C.3b 6.NS.C.6	431
16.17	Estimate Illustrate Situate	Build the connection between computation, models, and context	5.NBT.B.7	434
16.18	Hit the Target: Continuous Input	Estimate for any of the four operations	5.NBT.B.7 6.NS.B.3	436
16.19	Where Does the Decimal Go? Multiplication	Use estimation to place the decimal point in multiplication	5.NBT.B.7 6.NS.B.3	436
16.20	Where Does the Decimal Go? Division	Use estimation to place the decimal point in division	5.NBT.B.7 6.NS.B.3	438
16.21	Name the Shaded Region	Explore fractions, decimals and percents with various sized regions	6.RP.A.3c	440
16.22	Charged Up?	Estimating percents with a length model	6.RP.A.3c	440
16.23	Percent Necklaces	Estimating percents with circle graph data	6.RP.A.3c	441
16.24	Percent Memory Match	Explore the relationship between percents and circle graphs	6.RP.A.3c	441

Chapter 17 Ratios and Proportional Reasoning

Activity		Mathematics Content	CCSS-M	Page
17.1	Which Is It? Ratio, Rate, or Unit Rate?	Define ratios, rates and unit rates	6.RP.A.2	450
17.2	Fraction, Ratio, or Both?	Compare ratios to fractions	6.RP.A.1	450
17.3	Stocking the Pond	Solve real-world problems involving ratios	6.RP.A.1 6.RP.A.2 6.RP.A.3	451
17.4	Making Icing	Develop the concept of a composed unit and multiplicative comparison	6.RP.A.1 6.RP.A.2 6.RP.A.3	452
17.5	Which Has More?	Develop the distinction between additive and multiplicative comparisons	6.RP.A.1	454
17.6	Applying Reasoning to Explore Social Issues	Comparing additive and multiplicative reasoning	6.RP.A.1	454

Activity		Mathematics Content	CCSS-M	Page
17.7	Pencil-to-Pencil	Develop the distinction between additive and multiplicative comparison	6.RP.A.3a 7.RP.A.2b	457
17.8	Different Objects, Same Ratios	Explore ratio and proportion concepts in measurement	6.RP.A.1, 6.RP.A.2	458
17.9	Creating Paint Swatches	Create and compare ratios using the context of paint mixes.	6.RP.A.1 6.RP.A.3a, b 7.RP.A.1 7.RP.A.2a	460
17.10	Comparing Lemonade Recipes	Explore proportional situations and compare ratios	6.RP.A.1 6.RP.A.3a, b 7.RP.A.1 7.RP.A.2a	460
17.11	Which Camp Gets More Pizza?	Compare ratios using food portions.	6.RP.A.1 6.RP.A.3a, b 7.RP.1 7.RP.A.2a	461
17.12	Solving Problems Using Ratio Tables	Develop proportional reasoning through scaling	6.RP.A.3a, b 7.RP.A.1 7.RP.A.2b, c 7.RP.A.3	463
17.13	You and the Zoo	Use double number lines to solve situations involving proportions	6.RP.A.3a, d 7.RP.A.2b 7.RP.A.3	466
17.14	It's a Matter of Rates	Develop fluency with strategies using a scale factor	6.RP.A.3a, b 7.RP.1 7.RP.A.2b	467
17.15	Factory Ratios	Reasoning about missing value situations	7.RP.A.2a, b, c, d	468
17.16	Rates of Wind-Up Toys	Using ratio tables to solve problems	6.RP.A.3a 7.RP.A.1 7.RP.A.2b, d	468
17.17	Making Sense of Percents	Use double number lines to solve situations involving percents	6.RP.A.3c 7.RP.A.2c 7.RP.A.3	470
17.18	Look-Alike Rectangles	Develop the concepts of ratio and proportion in the context of similar rectangles	6.RP.A.1 6.RP.A.3a 7.RP.A.2a 7.G.A.1 8.G.A.4	470
17.19	Sketch and Grow	Develop the concepts of ratio and proportion in the context of similar two-dimensional figures	6.RP.A.1, 7.RP.A.3a	471
17.20	Interesting Scaling Investigations	Connect scale drawings to real life contexts	7.RP.A.2b 7.RP.A.2c 7.G.A.1	472
17.21	Rectangle Ratios: Graph It!	Explore proportions through graphs	7.RP.A.2a, b, c, d	473
17.22	Dripping Faucet Problem	Apply algebraic thinking to develop proportional reasoning	6.RP.A.1 6.RP.A.3a, b 7.RP.A.2b, c	473

Chapter 18 Developing Measurement Concepts

Activity		Mathematics Content	CCSS-M	Page
18.1	What's the Attribute?	Identify the attributes measured in real world examples	3.MD.A.2 3.MD.C.5 4.MD.A.1 5.MD.C.3	478
18.2	About One Unit	Develop familiarity with standard units (any attribute)	1.MD.A.2 2.MD.A.1 2.MD.A.3	480
18.3	Personal Benchmarks	Explore useful benchmarks using body lengths	2.MD.A.1 3.MD.B.4 4.MD.B.4	480
18.4	Measuring Favorite Things	Using standard units to measure	1.MD.A.2 2.MD.A.1 3.MD.B.4	481
18.5	Estimation Exploration	Practice estimating different attributes	K.MD.A.1 K.MD.A.2 1.MD.A.1 1.MD.A.2 2.MD.A.3 3.MD.A.2 5.MD.C.4 6.G.A.1 6.G.A.2 7.G.B.4 8.G.C.9	484
18.6	Estimation Scavenger Hunt	Practice measurement estimation in real contexts	1.MD.A.2 2.MD.A.1 2.MD.A.3 4.MD.A.1	484
18.7	Length (or Unit) Hunt	Develop the concept of length (or familiarity with a standard unit)	K.MD.A.2 1.MD.A.1 1.MD.A.2 2.MD.A.3	485
18.8	Crooked Paths	Develop concept of length along paths that are not straight	1.MD.A.1 2.MD.A.3 2.MD.A.4	485
18.9	How Long Is the Teacher?	Develop methods for measuring length	1.MD.A.2 2.MD.A.1 2.MD.A.2 2.MD.A.3	486
18.10	Estimate and Measure	Develop an understanding of length measurement	1.MD.A.2 2.MD.A.1 2.MD.A.3	486
18.11	Make Your Own Ruler	Develop understanding of rulers by making a ruler	1.MD.A.2 2.MD.A.1 2.MD.A.3	487
18.12	Broken Rulers	Develop understanding of measuring with a ruler	1.MD.A.2 2.MD.A.1 2.MD.A.3	488
18.13	Changing Units	Explore the inverse relationship between unit size and measure	2.MD.A.2 2.MD.A.3 4.MD.A.1 5.MD.A.1	488
18.14	Conversion Please	Develop the concept of converting from a larger unit to a smaller unit in the same system	4.MD.A.1 5.MD.A.1	489

Activity		Mathematics Content	CCSS-M	Page
18.15	Two-Piece Shapes	Develop an understanding of area; equivalent areas with different shapes	3.MD.C.5	490
18.16	Tangram Areas	Develop the concept of area	3.MD.C.5 3.MD.C.6	490
18.17	Two Rectangles, a Parallelogram, a Trapezoid, and a Blob	Develop strategies to compare areas	3.MD.C.5 3.MD.C.6 4.MD.A.3	491
18.18	Rectangle Comparison: Square Units	Develop an understanding of units to measure area; readiness for rectangle area formula	3.MD.C.5 3.MD.C.6 3.MD.C.7 5.NF.B.4b	492
18.19	What's the Rim?	Develop the concept of perimeter	3.MD.D.8	493
18.20	Fixed Perimeters and Fixed Areas	Explore the relationship between area and perimeter of rectangles when the perimeter is constant	3.MD.C.6 3.MD.C.7 3.MD.D.8 4.MD.A.3	494
18.21	Area of Parallelograms	Develop the area formula for parallelograms	6.G.A.1	496
18.22	Area of a Triangle	Develop the area formula for triangles	6.G.A.1	497
18.23	Making Boxes	Explore nets and surface area	6.G.A.4 7.G.B.6	498
18.24	Capacity Sort	Develop the concept of capacity	K.MD.A.2 3.MD.A.2 5.MD.C.3	500
18.25	Box Comparison: Cubic Units	Develop the concept of volume; readiness for volume formula for prisms	5.MD.C.3 5.MD.C.4 5.MD.C.5 6.G.A.2 7.G.B.6	500
18.26	That's Cool	Develop estimation with units of liquid capacity	3.MD.A.2	501
18.27	Squeeze Play	Develop estimation with units of liquid capacity	3.MD.A.2	501
18.28	Fixed Volume: Comparing Prisms	Explore the relationship between volume and surface area of prisms when the volume is constant	5.MD.C.3 5.MD.C.4 5.MD.C.5 6.C.A.2	502
18.29	Which Silo Holds More?	Explore volume of cylinders with nonstandard units	5.MD.C.3 6.G.A.4 7.G.B.7	503
18.30	Which Weighs More?	Estimate and compare the weights of objects	K.MD.A.2	503
18.31	A Unit Angle	Develop an understanding of how units are used to measure angle size	4.MD.C.5 7.G.B.5	504
18.32	Angle Maker	Estimate the size of benchmark angles	4.MD.C.5 7.G.B.5	505
18.33	Angle Relationships	Explore supplementary, complementary, vertical, and adjacent angles	4.MD.C.5 7.G.B.5	506
18.34	Ready for the Bell	Explore time as the duration of an event	1.MD.B.3	506
18.35	One-Handed Clocks	Develop an understanding of the hour hand in reading a clock	1.MD.B.3 2.MD.C.7	507
18.36	Money Skip-Counting	Develop skill in counting money	2.MD.C.8	509
18.37	Hundred Chart Money Count	Explore a strategy for counting money	2.MD.C.8	509

Chapter 19 Developing Geometric Thinking and Geometric Concepts

Activity		Mathematics Content	CCSS-M	Page
19.1	Tricky Shapes	Attend to relevant properties of shapes	K.G.A.2 K.G.B.4 1.G.A.1	517
19.2	Shape Sorts	Develop ways that two-dimensional shapes are alike and different	K.G.B.4 1.G.A.1 2.G.A.1 3.G.A.1	517
19.3	Property Lists for Quadrilaterals	Explore all properties attributable to special classes of quadrilaterals	2.G.A.1 3.G.A.1 4.G.A.2 4.G.A.3	518
19.4	Minimal Defining Lists	Develop logic and reasoning to minimally define shapes	4.G.A.2 5.G.B.3	519
19.5	Shape Show and Hunt	Explore attributes of two-dimensional shapes and identify their location in the environment	K.G.A.1 K.G.A.2 K.G.B.4	520
19.6	What's My Shape?	Develop oral descriptions of shapes	1.G.A.1 2.G.A.1 3.G.A.1 4.G.A.2 7.G.A.2	521
19.7	Can You Make It?	Practice representing shapes	1.G.A.1 2.G.A.1 3.G.A.1 4.G.A.2	522
19.8	Tangram Puzzles	Practice composing shapes	K.G.B.6 1.G.A.2 2.G.A.1	523
19.9	Mosaic Puzzle	Explore properties of two-dimensional shapes and compose and decompose shapes	K.G.B.6 1.G.A.2 2.G.A.1 3.G.A.1 4.G.A.2 7.G.A.2	523
19.10	Geoboard Copy	Practice representation of shapes	1.G.A.2 2.G.A.1 4.G.A.1 4.G.A.2 4.G.A.3	524
19.11	Decomposing Shapes	Practice decomposing shapes	1.G.A.2 1.G.A.3 2.G.A.1 2.G.A.3 3.G.A.2 3.NF.A.1	524
19.12	Shifting Shapes	Constructing and identifying two-dimensional shapes and their properties	K.G.A.1 K.G.B.4 1.G.A.1 1.G.A.2 2.G.A.1 3.G.A.1 4.G.A.2	526

Activity		Mathematics Content	CCSS-M	Page
19.13	Constructing Three-Dimensional Shapes	Explore the construction of three-dimensional shapes	K.G.A.3 K.G.B.5 1.G.A.2 2.G.A.1 6.G.A.4	528
19.14	Mystery Definition	Develop defining properties of special classes of shapes	1.G.A.1 2.G.A.1 3.G.A.2 4.G.A.2	529
19.15	Triangle Sort	Develop defining properties of triangles	2.G.A.1 3.G.A.2 4.G.A.2 5.G.B.3 5.G.B.4 7.G.A.2	529
19.16	Can You Build It?	Draw shapes based on descriptions and properties	4.G.A.2 4.G.A.3 5.G.B.3 5.G.B.4 7.G.A.2	530
19.17	Angle Sum in a Triangle	Draw conjectures for the sum of the angles in a triangle	5.G.B.3 7.G.A.2 8.G.A.5	530
19.18	Diagonals of Quadrilaterals	Explore the relationships between diagonals of classes of quadrilaterals	4.G.A.2 5.G.B.3 5.G.B.4 6.RP.A.1 7.G.A.2	531
19.19	Discovering Pi	Develop an understanding of pi as the ratio of circumference to diameter	7.G.B.4	532
19.20	True or False?	Explore informal deductive statements concerning properties of shapes	4.G.A.2 5.G.B.3 5.G.B.4 7.G.A.2	533
19.21	The Pythagorean Relationship	Explore the meaning of the Pythagorean relationship in geometric terms	8.G.B.6 8.G.B.7	533
19.22	Saber in a Box	Investigate the Pythagorean relationship	8.G.B.6 8.G.B.7	534
19.23	Finding Pythagorean Triples	Investigate triangles that are Pythagorean triples	8.G.B.6 8.G.B.7	534
19.24	Motion Man	Develop the concepts of slides, flips, and turns	8.G.A.1a, b, c	535
19.25	Pattern Block Mirror Symmetry	Develop the concept of line symmetry	4.G.A.3	536
19.26	Dot Grid Line Symmetry	Explore relationship between line symmetry and reflection	4.G.A.3	537
19.27	Are They Congruent?	Explore congruent triangles using translations on a grid	8.G.A.1 8.G.A.2	539
19.28	Polygon Dilations	Explore the result of multiplying coordinates by a constant	6.G.A.3 7.G.A.1 8.G.A.1 8.G.A.3	540
19.29	Hidden Positions	Develop a readiness for coordinates	5.G.A.1	541

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Chapter 19 (*Continued*)

Activity		Mathematics Content	CCSS-M	Page
19.30	Paths	Explore the concept of location using coordinates on a grid	5.G.A.1	541
19.31	Step Right Up	Write the ordered pair for points on a coordinate grid	5.G.A.1	542
19.32	Coordinate Slides	Develop the way coordinates are used to control translations	6.G.A.3 8.G.A.1 8.G.A.3	542
19.33	Coordinate Reflections	Explore the effect on coordinates when shapes are reflected about an axis	8.G.A.1 8.G.A.3	543
19.34	Finding Distance Using the Pythagorean Theorem	Develop the formula for the distance between two points	6.G.A.3 8.G.B.8	544
19.35	Can You Remember?	Develop early spatial visualization skills and visual memory	K.G.B.4 1.G.A.2 2.G.A.1	545
19.36	Pentominoes	Develop spatial visualization skills	1.G.A.2 2.G.A.1 3.G.A.1	545
19.37	Geometry Necklaces	Identifying mystery geometric shapes and terms using deduction	3.G.A.1 4.G.A.1 5.G.B.3	546
19.38	Face Matching	Explore solids in terms of their faces or sides	1.G.A.2 2.G.A.1 3.G.A.1 4.G.A.2	546
19.39	Building Views	Develop spatial visualization skills	6.G.A.4 7.G.A.1	547
19.40	Three-Dimensional Drawings	Develop spatial visualization skills	6.G.A.4 7.G.A.1	547
19.41	Slicing Solids	Explore the two dimensional figures that result from slicing three dimensional figures	7.G.A.3	548

Chapter 20 Developing Concepts of Data and Statistics

Activity		Mathematics Content	CCSS-M	Page
20.1	Who Is in Our Village?	Gather and analyze data about members of the class	2.MD.D.10 3.MD.B.3 6.SP.A.1 7.SP.B.3	558
20.2	What Can We Learn about Our Community?	Generate questions about the community, gather data, and analyze it	1.MD.C.4 2.MD.D.10 6.SP.A.1 6.SP.B.4 6.SP.B.5	558
20.3	How Do We Compare?	Develop an awareness of the importance of sampling	6.SP.A.1 6.SP.A.2 7.SP.A.1 7.SP.B.3	561
20.4	What about "Both"	Classify shapes by attributes on a Venn diagram	K.MD.3 1.MD.C.4	563
20.5	I.D. the Attribute	Make and test conjectures about how things are being classified	K.MD.3 1.MD.C.4	564

Activity		Mathematics Content	CCSS-M	Page
20.6	Hidden Labels	Make and test conjectures about how things are being classified	K.MD.3 1.MD.C.4	564
20.7	Picture Graphs to Bar Graphs	Analyze data using both bar and circle graphs	2.MD.D.4 3.MD.B.3	566
20.8	Stand by Me	Explore line plots in a concrete manner	1.MD.C.4 2.MD.D.10 3.MD.B.4	568
20.9	Storm Plotter	Explore line plots using real-life data	3.MD.B.4 4.MD.B.4 5.MD.B.2 6.SP.B.4 7.SP.B.3	569
20.10	Which Paper Airplane Design Is Best?	Create and analyze box plots	6.SP.B.4 7.SP.B.3	571
20.11	Is There a Relationship?	Explore situations involving bivariate data	8.SP.A.1	573
20.12	Spaghetti Lines	Develop the concept of finding a functional relationship to approximate trends in a data set	8.SP.A.1 8.SP.A.2	574
20.13	Mean Cost of Games	Explore a leveling interpretation of the mean	6.SP.A.2 6.SP.A.3 6.SP.B.5	576
20.14	The Mean Foot	Apply the leveling interpretation of the mean to develop the algorithm	6.SP.A.2 6.SP.A.3 6.SP.B.5	576
20.15	Balance Point Sticky Notes	Explore a balancing point interpretation of the mean	6.SP.A.2 6.SP.A.3 6.SP.B.5	577
20.16	Balancing Cubes on a Ruler	Explore a balancing point interpretation of the mean	6.SP.A.2 6.SP.A.3 6.SP.B.5	578
20.17	Average Cost of More Games	Explore how new data impact each measure of center	6.SP.A.2 6.SP.A.3 6.SP.B.5	579
20.18	Which Measure of Center Makes Sense?	Decide which measure of center, median or mean, makes more sense given various contexts	6.SP.A.2 6.SP.A.3 6.SP.B.5	579
20.19	You Be the Judge	Make decisions using your choice of statistical measures	6.SP.A.2 6.SP.A.3 6.SP.B.5	580
20.20	How Many Words on the Page?	Design strategy for gathering and analyzing data to solve a real-life problem	6.SP.A.1 6.SP.A.2 6.SP.B.5	583

Chapter 21 Exploring Concepts of Probability

Activity		Mathematics Content	CCSS-M	Page
21.1	Events in Lyrics: Possible or Not Possible?	Introduce concept of probability	7.SP.C.5	588
21.2	Is It Likely?	Introduce concept of probability	7.SP.C.5	588
21.3	1-2-3 How Likely?	Explore how likely particular outcomes are	7.SP.C.5 7.SP.C.6	589

(*Continued*)

Chapter 21 (*Continued*)

Activity		Mathematics Content	CCSS-M	Page
21.4	1-2-3 How Likely Are Sums?	Explore how likely particular outcomes are with two events	7.SP.C.5 7.SP.C.6 7.SP.C.8a	589
21.5	Race to Ten	Use area context for exploring probability	7.SP.C.6 7.SP.C.7a, b	589
21.6	The Likely Ladder	Build and label a probability ladder	7.SP.C.5	591
21.7	Design and Test Bags	Explore probability through experiments	7.SP.C.6	592
21.8	Mystery Bags	Explore probability through experiments	7.SP.C.6 7.SP.C.7a, b	592
21.9	Fair or Unfair Coin Games	Explore games theoretically and through experiments	7.G.B.4 7.SP.C.6 7.SP.C.7b	594
21.10	The Last Banana Contest	Explore game theoretically and through experiments	7.G.B.4 7.SP.C.6 7.SP.C.7b	595
21.11	Chance of Hitting the Target?	Explore probability in an area model	7.G.B.4 7.SP.C.6 7.SP.C.7b	596
21.12	Drop It!	Determine how likely outcomes are based on empirical data	7.SP.C.7b	597
21.13	Get All 6!	Explore impact of size of the trial	7.SP.C.6	598
21.14	What Are the Chances?	Develop the concept of randomness	7.SP.C.6	598
21.15	Are You a Spring Dog?	Use an area model to determine how likely independent events are	7.SP.C.8a, b	601
21.16	Best Chance of Purple	Explore compound events in a game context	7.SP.C.7a 7.SP.C.8a	603
21.17	Lu-Lu	Compare results of game (experiment) with sample space	7.SP.C.7a 7.SP.C.8a	604
21.18	Design a Winning Spinner	Find probability of independent events	7.SP.C.6 7.SP.C.7 7.SP.C.8a, b	605
21.19	Keys to a New Car	Find probability of dependent events	7.SP.C.8a, b	605
21.20	Probability of Getting Water	Design a simulation to determine probability of an event	7.SP.C.8b, c	607
21.21	Chance of Triplet Girls	Design a simulation to determine probability of an event	7.SP.C.8b, c	609
21.22	Money in Two Piggy Banks	Find probability of dependent events	7.SP.C.8a, b, c	609

Chapter 22 Exploring Exponents, Negative Numbers, and Real Numbers

Activity		Mathematics Content	CCSS-M	Page
22.1	Painted Cubes	Explore linear and exponential growth in a concrete task	5.MD.C.4 5.MD.C.5 6.EE.A.2c 6.GA.4 7.EE.B4	614
22.2	Stacks of Coins	Explore the order of operations conceptually	3.OA.D.8 6.EE.A.2c	615

Activity		Mathematics Content	CCSS-M	Page
22.3	I'm Thinking of a Number...	Explore the order of operations and using appropriate grouping symbols	6.EE.A.2a, b, c 6.EE.B.6	616
22.4	True or False Equations	Explore order of operations	3.OA.D.8 6.EE.A.3 6.EE.A.4	617
22.5	Entering Expressions	Explore expressions using expressions is the order of operations	6.EE.A.3 6.EE.A.4	617
22.6	Find the Error	Identify errors that involve order of operations	8.EE.A.1	619
22.7	How Far Away from the Sun?	Explore scientific notation and relative size of numbers	8.EE.A.4	620
22.8	At a Snail's Pace	Explore scientific notation of very small numbers in context	8.EE.A.4	621
22.9	Exploring Powers of 10	Develop an understanding of scientific notation and other expressions for powers of 10	8.EE.A.3	621
22.10	Football Statistics	Develop the concept of positive and negative numbers	6.NS.C.5 6.NS.C.6a	623
22.11	Greater, Less, Equal, or Don't Know?	Build an understanding for symbols (minus and parenthesis) used in expressions	6.NS.C.5 6.NS.C.6a	625
22.12	Height of Hot-Air Balloons	Add and subtract with positive and negative numbers	7.NS.A.1	627
22.13	Find the Zero	Develop strategy for adding with negative numbers	7.NS.1a, b, c	628
22.14	Conjectures for Operations with Negative Numbers	Estimate the result of operations involving negative numbers	7.NS.A.2a, b	632
22.15	How Many in Between?	Exploring real numbers	6.NS.C.5 7.NS.A.2d	633
22.16	Repeater or Terminator?	Develop the connection between repeating versus terminating decimal numbers	7.NS.A.2d	634
22.17	Edges of Squares and Cubes	Develop the concepts of square roots and cube roots	8.NS.A.1 8.EE.A.2	635

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